MAT 347
Classification of finite abelian groups
November 14, 2014

We want to prove two results:

1. Every finite abelian group is isomorphic to a direct product of cyclic groups.
2. Since different direct products of cylic groups are sometimes isomorphic, we want an easy way to obtain a list of all the abelian groups of order n, without repeats.

In a way, think of Part 1 as an "existence" result, and Part 2 as a "uniqueness" result.

Part 1

1. *(Cauchy's Theorem)*. Let G be a group. Let p be a prime. Assume that p divides $|G|$. Prove that G has a subgroup of order p.

   Hint: Use Sylow's Theorems.

2. Prove that every finite abelian group is isomorphic to the direct product of its Sylow subgroups. (Why does the proof not work for non-abelian groups?) Conclude that it is enough to prove Part 1 for p-groups.

3. Let G be an abelian p-group. (Recall: this means that the order of G is a power of the prime p.) Prove that G has a unique subgroup of order p if and only if G is cyclic.

   Hint: For the difficult direction, consider the map $\psi : G \to G$ defined by $\psi(x) = x^p$ for all $x \in G$ and use induction on $|G|$.

4. Let G be a finite, abelian p group. Let A be a maximal cyclic subgroup of G. Prove that A has a complement. This means that there exists another subgroup $B \leq G$ such that $A \cap B = 1$ and $AB = G$.

   Hint: Use induction on $|G|$ and Problem 3.

5. Use Problem 4 to prove Part 1.

Part 2

6. As a warm-up, complete and prove the following claim:

   Let a, b be positive integers. Then $Z_a \times Z_b \cong Z_{ab}$ iff ...

7. Still as warm-up, show that $Z_{20} \times Z_6 \cong Z_{12} \times Z_{10}$ and that none of them is isomorphic to Z_{120}.

8. Solve Part 2. There are two standard ways to do it. Given a positive integer n we can obtain a list of all abelian groups of order n...

   - ... by writing each one of them as product of cyclic groups as small as possible, or
   - ... by writing each one of them as product of cyclic groups as large as possible, in some canonical way.

   Either way, you have to prove that every abelian group of order n is isomorphic to one on your list, and that no two different groups on your list are isomorphic to each other.

Challenge question

9. *[Putnam 2009 - A5]* Is there a finite abelian group such that the product of the orders of all its elements is 2^{2009}?