MAT 347
Polynomials
January 23, 2015

Let R be a community. We denote the set of formal polynomials with coefficients in R as $R[X]$. By "formal" we mean that we think of a polynomial as a set of symbols, and X is a formal variable, not the input of a function. $R[X]$ is always a community.

I assume you are familiar with the definitions of *degree* of a polynomial, *leading coefficient* of a polynomial, and *monic polynomial*. Remember that a non-zero, constant polynomial has degree 0, but the 0 polynomial has no degree.

1. $R[X]$ is an integral domain iff R ...

2. Find an example of a polynomial with degree greater than 0 which is a unit in $R[X]$.

3. Let $f(X), g(X) \in R[X]$.

   (a) What is the relation between $\deg f(x)$, $\deg g(x)$, and $\deg(f(x)+g(x))$?

   (b) What if R is an integral domain?

   (c) What is the relation between $\deg f(x)$, $\deg g(x)$, and $\deg(f(x)g(x))$?

   (d) What if R is an integral domain?

4. Let $\mathcal{F}_R$ be the set of map from R to R. (We are including *any* maps.) Notice that $\mathcal{F}_R$ is a ring with pointwise addition and multiplication of maps. There is a natural map
$$T : R[X] \to \mathcal{F}_R$$

   (a) Is T a ring homomorphism? If the answer in general is "no", are there some cases of R for which the answer is "yes"?

   (b) Is T injective? If the answer in general is "no", are there some cases of R for which the answer is "yes"?

   (c) Is T surjective? If the answer in general is "no", are there some cases of R for which the answer is "yes"?

5. Let $I = (2, X) \trianglelefteq \mathbb{Z}[X]$. Prove that I is not a principal ideal. Prove that I is a maximal ideal. Do 2 and X have a GCD on $\mathbb{Z}[X]$?

6. Let $I \trianglelefteq R$. Let us denote by $(I) := I[X]$ the ideal generated by I in $R[X]$. Prove that $I \trianglelefteq R$ is a prime ideal iff $(I) \trianglelefteq R[X]$ is a prime ideal.

   Note: Remember the characterization of prime ideal in terms of the quotient it produces.

7. Let R be an integral domain. Prove that the following three conditions are equivalent:

   (a) R is a field.

   (b) $R[X]$ is a Euclidean domain.

   (c) $R[X]$ is a PID.

   Hint: Notice that X is irreducible in $R[X]$. Remember the characterization of prime and irreducible elements in terms of the ideals they generate. Remember the characterization of maximal and prime ideal in terms of the quotient they produce.