MAT 347 Factorization on polynomial rings January 28–30, 2015

Let us fix an integral domain R for this worksheet. We want to find out when R[X] is a UFD. Our strategy is as follows. Let F be the field of fractions of R. We know that F[X] is a Euclidean domain, hence a UFD. Let $f(X) \in R[X]$. We can factor f(X) uniquely as a product of irreducibles in F[X], but does this means we get a unique factorization in R[X]?

- 0. Let B be a UFD and let $A \subseteq B$ be a subdomain. Assume that
 - (a) B is a UFD.
 - (b) $A \cap B^{\times} = A^{\times}$. (This means that elements of A which are not units in A do not become units in B.)
 - (c) Every divisor of an element of A is in A.

Prove that A is then a UFD as well.

Explain how the hypotheses fail if we try to apply this to A = R[X] and B = F[X].

1. Prove that if R[X] is a UFD, then R is a UFD.

For the rest of this worksheet, we will assume that R is a UFD.

2. Let $I \leq R$. Denote by $(I) \leq R[X]$ the ideal of R[X] generated by the subset I. Notice that I consists of the polynomials in R[X] all of whose coefficients are in I. Prove that

$$R[X]/(I) \cong (R/I)[X]$$

This is mostly a matter of persuading yourself this is true.

3. Continue with the notation of question 2. Prove that $I \leq R$ is a prime ideal iff $(I) \leq R[X]$ is a prime ideal.

Hint: Use the characterization of prime ideals in terms of the quotient they generate.

4. Let $p \in R$. Prove that p is prime in R iff p is prime in R[X]. Hint: Use the characterization of prime element in terms of the ideal it generates. **Definitions.** Let R be a UFD. Let $f(X) \in R[X]$ be non-zero. We define the *content* of f(X), denoted C_f , as the GCD of all the coefficients of f(X). We could also interpret C_f to be the "greatest" divisor of f[X] among the elements in R. Notice that this is defined only up to associates. We say that f is *primitive* when its content is 1. Notice that every non-zero polynomial can be written as the product of its content and a primitive polynomial, and that this decomposition is unique up to multiplication by units.

- 5. Prove that the product of two primitive polynomials is a primitive polynomial. *Hint:* Assume that f(X), g(X) are primitive and that $d = C_{fg}$. Let p be an irreducible factor of d in R. Use question 4.
- 6. Prove that $C_{fg} = C_f C_g$ for every non-zero $f(X), g(X) \in R[X]$.
- 7. [GAUSS LEMMA.] Let $f(X) \in R[X]$. Prove that if f(X) is reducible in F[X], then it is reducible in R[X].

More specifically, assume that

$$f(X) = a(X)b(X)$$

with $a(X), b(X) \in F[X]$, deg $a(X) \ge 1$, deg $b(X) \ge 1$. Then show that we can find $r, s \in F^{\times}$ such that

$$f(X) = A(X)B(X)$$

with $A(X) = ra(X) \in R[X]$ and $B(X) = sb(X) \in R[X]$.

Give an example that shows that it is not possible to conclude that $a(X), b(X) \in R[X]$.

- 8. Let $f(X) \in R[X]$ be primitive. Show that f(X) is irreducible in R[X] if and only if it is irreducible in F[X].
- 9. Prove that R[X] is a UFD.