MAT 347
Counting, group actions, and the Orbit-Stabilizer Lemma
September 19, 2014

Actions

Definition. Let G be a group and let A be a set. An *action* of G on A is a map

$$\begin{aligned} G \times A &\longrightarrow A \\ (g, a) &\mapsto g \cdot a \end{aligned}$$

that satisfies the following two properties:

- $g \cdot (h \cdot a) = (gh) \cdot a$ for all $a \in A$, $g, h \in G$.
- $1 \cdot a = a$ for all $a \in A$,

1. Show that the second property in the definition of action is necessary (in other words, the first property does not imply the second).
2. Which ones of the following are actions?
   (a) For any set A, $G = S_A$ and the map is the natural one.
   (b) $G = D_{2n}$, A is the set of vertices of a regular n-gon, and the map is the natural one.
   (c) $G = D_{2n}$, A is the set of diagonals of a regular n-gon, and the map is the natural one.
   (d) G is any group, $A = G$ as a set, and the map is $g \cdot a := ga$.
   (e) G is any group, $A = G$ as a set, and the map is $g \cdot a := ag$.
   (f) G is any group, $A = G$ as a set, and the map is $g \cdot a := gag^{-1}$.
   (g) G is any group, A is the set of subsets of G, and the map is ...
3. Assume we have an action of the group G on the set A. For each $g \in G$, let us define a map $\phi_g : A \to A$ by the equation $\phi_g(a) := g \cdot a$. Show that ϕ_g is a bijection. This defines a map $\phi : G \to S_A$ by the equation $\phi(g) := \phi_g$. Show that ϕ is a group homomorphism.
4. Reciprocally, show that every group homomorphism $G \to S_A$ comes from an action of G on A. In other words, there is a natural one-to-one correspondence between actions of G on A and group homomorphisms from G to S_A. This is why some authors define an action as a group homomorphism $G \to S_A$ instead.

The Orbit-Stabilizer Lemma

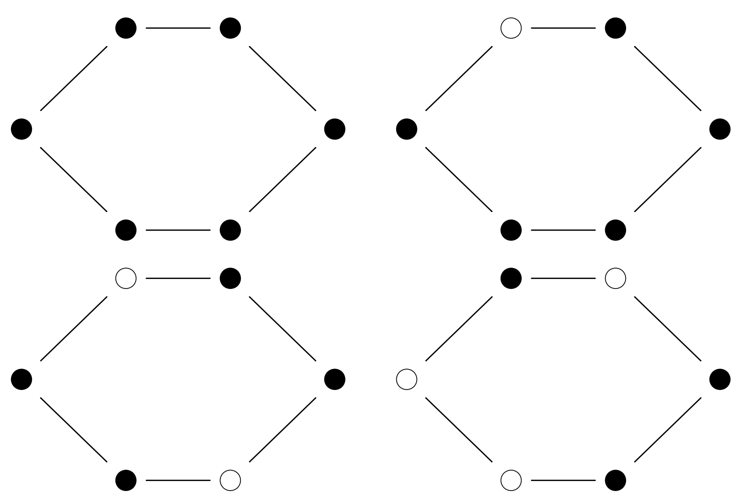
Definitions. Let G be a group acting on a set A.

- Given $g \in G$, we define the *fixed set* of g as the set
$$\text{Fix}(g) := \{a \in A \mid g \cdot a = a\} \subseteq A$$
- Given $a \in A$, we define the *stabilizer* of a as the set
$$\text{Stab}(a) := \{g \in G \mid g \cdot a = a\} \subseteq G$$
- Given $a \in A$ we define the *orbit* of a as the set
$$\Omega_a := \{g \cdot a \mid g \in G\} \subseteq A$$

5. Say we want to count how many *different* necklaces we can build with 6 stones each, if we have stones of two different colours. Define a *diagram* to be any way of colouring each of the six vertices of a hexagon with black or white. Notice that $|A| = 64$. Show that D_{12} acts on A, and that the number of orbits of this action equals the number of different necklaces.

   Note: This shows that the problem of counting the number of orbits of an action is a interesting problem in combinatorics.

6. Regarding the previous question, consider the following diagrams:



   For each one of them, compute the size of its orbit and the size of its stabilizer. Make a conjecture or a formula that relates these two numbers for an arbitrary element in an arbitrary action. Then prove it.