MAT 347
Differential Galois Theory
March 30, 2015

Differential fields

A differential field is a field F together with a linear map $D : F \to F$ that satisfies

$$D(\alpha\beta) = D(\alpha)\beta + \alpha D(\beta) \text{ for all } \alpha, \beta \in F$$

For every $\alpha \in K$, we define

$$\alpha' = D(\alpha), \quad \alpha'' = D(\alpha'), \quad \ldots, \quad \alpha^{(n)} = D(\alpha^{(n-1)}).$$

The main example is the field of rational functions $\mathbb{C}(X)$. We can think of X as a formal variable, or we can think of the elements in $\mathbb{C}(X)$ as actual function.

1. Guess the definition of "*extension of differential fields*". It is a bit more than merely saying "it is a field extension K/F such that both K and F are differential fields".
2. Let K/F be an extension of differential fields. Let $S \subseteq K$ be a subset. Define $F\{S\}$: the differential field generated by S over F. Describe the following:
   (a) $\mathbb{C}(X)\{e^X\}$
   (b) $\mathbb{C}(X)\{\sin X\}$

   I am abusing notation here, because I am not saying what K is.
3. Define Galois group of an extension of differential fields.

Differential operators

A differential linear operator with coefficients in F is a formal expression of the form

$$L(\xi) := a_n\xi^{(n)} + \ldots + a_2\xi'' + a_1\xi' + a_0\xi$$

where $a_i \in F$, and ξ is a formal variable.. We denote by $\mathcal{L}(F)$ the set of all such differential operators.

Notice that we can interpret $L(\xi)$ as a map from F to F, namely as the map

$$a_nD^n + \ldots + a_2D^2 + a_1D + a_0I,$$

where I is the identity map. As we do with polynomials, we will use both interpretations.

The *set of solutions of* $L(\xi)$ in K is the set of $\beta \in K$ such that $L(\beta) = 0$. There is a result about existence and uniqueness of solutions of a differential operator in a bigger differential field which is akin to existence and solutions to splitting fields of polynomials.

4. Find the set of solutions of the following linear differential operators
   (a) $\xi' - \xi$
   (b) $\xi'' + \xi$
5. Let K/F be an extension of differential fields. Let $G = \mathrm{Gal}(K/F)$. Let $L(\xi) \in \mathcal{L}(F)$. Prove that G acts on the set of solutions of $L(\xi)$ on K.
6. Compute the Galois group of the following extensions K/F, where $F = \mathbb{C}(X)$:
   (a) $K = F(e^X)$
   (b) $K = F(\sin X)$

   Note: These are not finite groups. They are Lie groups (or matrix groups).

How we attacked solvability of polynomials via radicals

Recall the following ideas. Let K/F be an algebraic field extension (not differential fields, just plain vanilla fields).

- We said that K is obtained by adding one root from F when there exists $\alpha \in K$, $n \in \mathbb{N}$ such that $K = F(\alpha)$ and $\alpha^n = K$.
- We said that K/F is radical when there is a tower of subextensions $F = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m = K$ such that K_i is obtained from K_{i-1} by adding one root.
- We proved that K is obtained from F by adding one root if and only if $\mathrm{Gal}(K/F)$ is cyclic – assuming some conditions.
- Then we proved that K/F is radical (i.e. it has a tower of sub extensions such that...) iff $\mathrm{Gal}(K/F)$ is solvable (i.e it has a cyclic tower)– assuming some conditions.

Finally, meet our prey

7. Now let K/F be an extension of differential fields. Define the following:

   (a) K is obtained from F by adding one antiderivative.

   (b) K is obtained from F by adding the exponential of one element of F.
   Hint: Think of the differential equation satisfied by $e^{f(X)}$.

   (c) K is obtained from F by adding one logarithm of one elements of F.

8. An *elementary function* is a function that we can obtain with complex numbers, polynomials, exponentials, trig functions, logarithm, inverse trig functions, roots, the field operations, and their compositions. Give a better definition of "elementary functions".

   Hint: A lot of the elements in the previous list are redundant.

9. One kinky theorem is that the function $F(X) = \displaystyle\int_0^X e^{-t^2} dt$ is not an elementary function. How would you go about formalizing this problem in terms of differential Galois theory, and how would the structure of a proof go?

10. One important problem is which differential equations can be solved using elementary functions and their antiderivatives. How would you state it in terms of Galois theory?