MAT 347
Death by a thousand cuts
(Classification of non-abelian groups of order 6)
October 1, 2014

1. As a warm up, prove that every group of order prime is cyclic.

   Hint: Use Lagrange's theorem.

2. Let G be a group such that $x^2 = 1$ for all $x \in G$. Prove that G is abelian.

3. Let G be a group. Assume that $|G|$ is even. Prove that G has an element of order 2.

   Hint: Pair up each element with its inverse.

4. Let G be a group. Let $x, a \in G$. What is the relation between $|a|$ and $|xax^{-1}|$?

5. Let G be a group. Let $a \in G$. Assume that a is the only element of G of order 2. Prove that $a \in Z(G)$.

   Hint: Use question 4.

6. Let G be a group generated by the subset S. Assume that $ab = ba$ for all $a, b \in S$. Prove that G is abelian.

7. Let G be a group of order 6. Let $a, b \in G$. Assume that a has order 2, that $b \neq a$ and that $b \neq 1$. Prove that $G =< a, b >$.

   Hint: What values could $| < a, b > |$ have? Use Lagrange's Theorem.

8. Let G be a group of order 6. Assume that G has a unique element of order 2. Prove that G must be abelian.

   Hint: Use questions 5, 6, 7.

9. Let G be a non-abelian group of order 6. How many elements of each order does G have?

   Hint: There is a single possible answer! Use Lagrange's Theorem and questions 3 and 8.

10. Prove that every non-abelian group G of order 6 is isomorphic to S_3

    Hint: Consider the action of G by conjugation on the set of elements of G of order 2. This action, like every action, produces a group homomorphism. What do you know about it?

Bonus question: Prove that every abelian group of order 6 is isomorphic to C_6.