MAT 347
Quotient groups
October 3, 2014

Let G be a group and let $S \subseteq G$. We want to define an equivalence relation in G that will identify all the elements in S, and we want to maintain the group operation. Given $a, b \in G$, we say that $a \sim b$ when there exists $x \in S$ such that $b = ax$. In general, this relation will not be an equivalence relation.

1. Find necessary and sufficient conditions for $\sim$ to be an equivalence relation.

For the rest of this worksheet, let G be a group and let $H \leq G$. We will consider the equivalence relation defined above with $S = H$. Given $a \in G$, the *left coset* of a is the equivalence class of this relation, and we denote it aH. (Why do we use this notation?) The *quotient set* G/H is the set of all equivalence classes. The *index* of H on G, written $|G : H|$ is the number of equivalence classes.

2. Prove that $aH = bH$ iff [there exists $x \in H$ such that $b = ax$] iff $a^{-1}b \in H$
3. In general, what is the cardinality of each coset aH? What is the relation between $|G|$, $|H|$, and $|G : H|$?
4. Consider the group $G = D_8$ and consider the two subgroups $H_1 :=< s >$ and $H_2 :=< r >$. For each of them, write the complete list of cosets, and list which elements are in each coset.

Next, we want to try to use the operation on G to define an operation on the set G/H. Given $aH, bH \in G/H$, we can try to define their product by

$$(aH) \star (bH) = (ab)H$$

[*Note:* I am using $\star$ to emphasize that I am defining a new operation. As soon as we make sure this operation works and there is no ambiguity, we will drop the $\star$.]

5. In general, the operation $\star$ is not well-defined. Going back to the example in Question 4, show that with one of those subgroups, the operation is well-defined, but with the other subgroup, the operation is not well-defined.

Definition: We say that the subgroup H is a *normal subgroup* of G when the operation $\star$ in G/H is well-defined. We write it $H \trianglelefteq G$

6. Assume $H \trianglelefteq G$. In this case we know the operation in G/H is well-defined. What other conditions do we need to impose so that G/H is a group with this operation?

The big theorem about normal subgroups

Notation: Let G be a group. Given subsets A, B and elements x, y we will use the following notation:

$$xA := \{xa \mid a \in A\}$$
$$xAy := \{xay \mid a \in A\}$$
$$AB := \{ab \mid a \in A, b \in B\}$$

et cetera

7. Let G be a group and let $H \subseteq G$. Explore the relation between the following statements (which ones imply which ones)?
   (a) $H \trianglelefteq G$
   (b) $aH = Ha$ for all $a \in G$
   (Notice that this does not mean that a commutes with the elements of H. It only means that the sets aH and Ha are the same set.)
   (c) $aHa^{-1} = H$ for all $a \in G$
   (d) $aHa^{-1} \subseteq H$ for all $a \in G$
   (e) There exists some group L and some group homomorphism $f : G \to L$ such that $H = \ker f$.

Joins

Let G be a group and let H, K be subgroups. We have already explained above how to define the set HK. We call this the *product* of H and K. In general, this is not a subgroup! It is just a subset of G. Do not confuse this product with the abstract construction of the direct product. We define the *join* of H and K as the smallest subgroup of G containing both H and K. In other words, the join of H and K is $< H \cup K >$.

8. Show that $HK \subseteq < H \cup K >$.

9. Explore the relation between the following statements (which ones imply which ones)?
   (a) $HK = < H \cup K >$.
   (b) $HK \leq G$.
   (c) $HK = KH$.
   (Notice that this does not mean that the elements of H and the elements of K commute with each other! It only means that HK and KH are the same set.)
   (d) $H \trianglelefteq G$.