SOME EXTRA PROBLEMS ON RING THEORY
MAT347

1. Let X be any set. We denote its power set (that is, the set of subsets of X) by $\mathcal{P}(X)$. We define two operations in $\mathcal{P}(X)$; given $A, B \subseteq X$:

$$\begin{aligned} A \setminus B &= \{x \in A \mid x \notin B\} \\ A \triangle B &= (A \setminus B) \cup (B \setminus A) \end{aligned}$$

Notice that $(\mathcal{P}(X), \triangle, \cap)$ is an abelian ring with identity for any set X. (You do not need to prove this, but I recommend that you persuade yourself that this is true.)

   (a) Assume that X is finite. Then $(\mathcal{P}(X), \triangle)$ is a finite abelian group. By the Fundamental Theorem of Finite Abelian Groups, we know that it has to be isomorphic to a direct product of finite cyclic groups. Find positive integers $n_1, \ldots, n_r$ such that

$$(\mathcal{P}(X), \triangle) \cong Z_{n_1} \times \cdots \times Z_{n_r}$$

   and explicitly construct one such isomorphism.

   (b) Find all zero-divisors and all units of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (c) Describe all the principal ideals of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (d) Assume X is finite. Prove that every ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$ is principal.

   (e) Construct an explicit example of a set X and a non-principal ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (f) Assume X is finite. Find all prime ideals of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (g) Assume X is infinite. Prove that there exists a non-principal prime ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$.

2. Let R be a community (i.e. a commutative ring with $1 \neq 0$). Let $a, b \in R$. We say that a and b are *associates* when there exists $u \in R^x$ such that $b = ua$.

   (a) Assume R is an integral domain. Let $a, b \in R$. Prove that a and b are associates iff $(a) = (b)$.

   (b) Let $a, b \in R$ be associates. Prove that a is a unit iff b is a unit.

   (c) Let $a, b \in R$ be associates. Prove that a is a zero-divisor iff b is a zero-divisor.