THE BANACH-TARSKI THEOREM

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Notation:

- Given sets X, Y, Z, when we write $X = Y \uplus Z$, we mean that X is the *disjoint union* of X and Y. In other words, $X = Y \cup Z$ and $Y \cap Z = \emptyset$.
- The notation $\mathrm{Isom}(\mathbb{R}^3)$ refers to the group of isometries of $\mathbb{R}^3$, i.e. the set of rotations and translations in $\mathbb{R}^3$ and their compositions.
- We denote by $SO(3)$ the group of rotations of $\mathbb{R}^3$ around any axis that goes through the origin. Notice that this is a group because the compositions of two rotations in $\mathbb{R}^3$ is another rotation. Notice also that $SO(3)$ is a subgroup of $\mathrm{Isom}(\mathbb{R}^3)$.
- Given $g \in \mathrm{Isom}(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$ we write $g \cdot x$ to mean $g(x)$. We may also write gx instead of $g \cdot x$ if there is no ambiguity. Moreover, given $X \subseteq \mathbb{R}^3$, the notation $g \cdot X$ represents

$$g \cdot X := \{g \cdot x \mid x \in X\}$$

1 Equidecomposable sets

Definition 1.1. Let $A, B \subseteq \mathbb{R}^3$. We say that A and B are *equidecomposable* when we can find a natural number $n \in \mathbb{N}$, subsets $A_1, \ldots, A_n \subseteq \mathbb{R}^3$, subsets $B_1, \ldots, B_n \subseteq \mathbb{R}^3$, and isometries $g_1, \ldots g_n \in \mathrm{Isom}(\mathbb{R}^3)$, such that:

- $A = A_1 \uplus \ldots \uplus A_n$,
- $B = B_1 \uplus \ldots \uplus B_n$, and
- $B_j = g_j \cdot A_j$ for all j.

We write this as $A \sim B$.

Example 1.2. Let A be a segment of length 2 and Let B be a right angle whose sides have length 1, as in the figure below. They are equidecomposable.

Exercise 1.3. Show that being equidecomposable is an equivalence relation among subsets of $\mathbb{R}^3$.

Theorem 1.4. A unit ball in $\mathbb{R}^3$ is equidecomposable with two copies of itself. This is the statement of the Banach-Tarski theorem. The rest of this handout is devoted to proving it.

2 Preliminary examples

Example 2.1. Let C be a circle and let x be a point in the circle. Then $C \sim C \setminus \{x\}$. To see this, let θ be an angle such that $\theta/2\pi$ is not a rational number. Let R be a rotation around the centre of the circle with angle θ. Let $H := \{R^n(x) \mid n = 0, 1, 2, \ldots\}$. Then:

- $C = (C \setminus H) \cup H$,
- $C \setminus \{x\} = (C \setminus H) \cup (H \setminus \{x\})$, and
- $H \setminus \{x\} = R \cdot H$

Example 2.2. Let C be a circle and let x and y be two distinct points in C. We are going to try to prove that $C \sim C \setminus \{x, y\}$. We choose an angle θ, to be determined later, and we let R be a rotation around the centre of the circle with angle θ. We let $H := \{R^n(x) \mid n = 0, 1, 2, \ldots\} \cup \{R^n(y) \mid n = 0, 1, 2, \ldots\}$. Then, we would like to claim that:

- $C = (C \setminus H) \cup H$,
- $C \setminus \{x, y\} = (C \setminus H) \cup (H \setminus \{x, y\})$, and
- $H \setminus \{x, y\} = R \cdot H$

For this to work, we need various conditions on the angle θ. Specifically, we need $R^n x \neq x$, $R^n x \neq y$, $R^n y \neq x$, and $R^n y \neq y$, for every $n = 1, 2, \ldots$. These conditions, all together, ban a countable number of angles that won't work. Since $[0, 2\pi]$ is uncountable, we can always choose a value of θ that will work.

Example 2.3. Let S^2 be the unit sphere in $\mathbb{R}^3$. Let E be any countable subset of S^2 such that, if $x \in E$, then $-x \in E$. We are going to prove that $S^2 \sim S^2 \setminus E$. First, choose an axis that goes through the centre of the sphere and which does not intersect any point in E (this is possible, since E is countable). Next, choose an angle θ such that $R^n x \neq y$ for all $n > 0$ and for all $x, y \in E$, where R is rotation by an angle θ around the chosen axis. (This is possible because these conditions only ban a countable number of angles θ). Let $H := \{R^n x \mid x \in E;\ n = 0, 1, 2, \ldots\}$. Finally, notice that:

- $S^2 := (S^2 \setminus H) \cup H$,
- $S^2 \setminus E := (S^2 \setminus H) \cup (H \setminus E)$, and
- $H \setminus E = R \cdot H$,

which completes the proof.

3 Interlude: an excursion in group theory

Definition 3.1. Consider a finite set $S = \{a_1, \ldots, a_n\}$. A *word* in S is any finite string of symbols, where every symbol is one of $\{a_1, \ldots, a_n\}$ or of $\{a_1^{-1}, \ldots, a_n^{-1}\}$, with the understanding that if the symbols a_j and a_j^{-1} are consecutive for any j, they can be cancelled. We denote the set of all such words $F(S)$. The operation "juxtaposition" on $F(S)$ is well-defined, and makes $F(S)$ into a group, where the identity is the empty word (which we denote 1). We call $F(S)$ the *free group generated by* S. We will also write F_2 to mean $F(\{a, b\})$.

Theorem 3.2. We can decompose the group F_2 as disjoint union of four pieces $F_2 = A \uplus B \uplus C \uplus D$ such that $F_2 = A \uplus aB = C \uplus bD$.

Proof. If α is any of the symbols a, b, a^{-1}, or b^{-1}, let us denote by $S(\alpha)$ the set of all words which, after applying any legal cancelation, start with the symbol α. Notice that $F_2 = \{1\} \uplus S(a) \uplus S(b) \uplus S(a^{-1}) \uplus S(b^{-1})$. Let $M := \{a^{-1}, a^{-2}, a^{-3}, \ldots\}$. Then the following partition of F_2 works:

- $A = S(a) \cup \{1\} \cup M$,
- $B = S(a^{-1}) \setminus M$,
- $C = S(b)$,
- $D = S(b^{-1})$.

$\square$

Lemma 3.3. $SO(3)$ has a subgroup isomorphic to F_2. This means that there are elements $a, b \in SO(3)$ such that any non-trivial word on the symbols $\{a, b\}$ is not the identity when we think of it as a product of powers of the elements a and b. For example, for most values of the angle θ, if a is a rotation by angle θ around the x-axis and b is a rotation by angle θ around the y-axis, the result is true.

4 The proof of the Banach-Tarski Theorem

Notation 4.1. We denote the 2-dimensional sphere in $\mathbb{R}^3$ by S^2. We denote the (closed) unit-ball by B^3. We also write $B^{3\star} := B^3 \setminus \{0\}$. Remember that our goal is to prove that B^3 is equidecomposable to two copies of B^3.

Theorem 4.2. S^2 is equidecomposable to two copies of S^2.

Proof. From the results in Section 3 we know that there are elements $a, b \in SO(3)$ that generate a free group isomorphic to F_2. Let us call this group G. We also know that we can partition G as disjoint union of four pieces with the following properties:

- $G = A \uplus B \uplus C \uplus D$
- $G = A \uplus aB$
- $G = C \uplus bD$

Next, notice that the group G acts on the set S^2. Let us define an equivalence relation in S^2 as follows. Given $x, y \in S^2$, we say that $x \approx y$ when $y = g \cdot x$ for some $g \in G$. (In the group-theoretic language, this means that x and y are in the same orbit of this action.) Let M be a subset of S^2 that contains exactly one element of each equivalence class. Notice that every $y \in S^2$ can be written as $y = g \cdot x$ for some $g \in G$ and $x \in M$. This way of writing y is not unique, however.

Let K be the set of points $y = S^2$ that can be written in more than one way as $y = g \cdot x$ with $g \in G$ and $x \in M$. We claim that K is countable. (Specifically, K is the set of points that are the intersection of S^2 with the axis of one of the rotations in G). In addition, $x \in K$ iff $-x \in K$. Hence, using Example 2.3, we know that $S^2 \sim S^2 \setminus K$. Moreover, K is invariant (as a set) under G.

Now, let us define

- $\widehat{A} := \{g \cdot x \,|g \in A, x \in M, x \notin K\}$
- $\widehat{B} := \{g \cdot x \,|g \in B, x \in M, x \notin K\}$
- $\widehat{C} := \{g \cdot x \,|g \in C, x \in M, x \notin K\}$
- $\widehat{D} := \{g \cdot x \,|g \in D, x \in M, x \notin K\}$

We get the following decompositions:

- $S^2 \setminus K = \widehat{A} \uplus \widehat{B} \uplus \widehat{C} \uplus \widehat{D}$,
- $S^2 \setminus K = \widehat{A} \uplus a \cdot \widehat{B}$,
- $S^2 \setminus K = \widehat{C} \uplus b \cdot \widehat{D}$

This proves that $S^2 \setminus K$ is equidecomposable to two copies of $S^2 \setminus K$. Hence S^2 is equidecomposable to two copies of S^2. □

Corollary 4.3. $B^{3\star}$ is equidecomposable to two copies of $B^{3\star}$.

Proof. Take the partitions of S^2 that allowed us to prove Theorem 4.2 and use rays from the origin to produce partitions of $B^{3\star}$. □

Corollary 4.4. B^3 is equidecomposable to two copies of B^3.

Proof. Let C be a circle inside of B^3 that contains the origin 0 as one of its points. We know the following:

- $B^3 = (B^3 \setminus C) \uplus C$ and $B^{3\star} = (B^3 \setminus C) \uplus (C \setminus \{0\})$.
- $C \sim C \setminus \{0\}$ from Example 2.1.
- Hence $B^3 \sim B^{3\star}$.
- $B^{3\star}$ is equidecomposable to two copies of $B^{3\star}$ from Corollary 4.3.
- Hence B^3 is equidecomposable to two copies of B^3.

□