FROM CLASSICAL TO DIFFERENTIAL GALOIS THEORY: A SUMMARY

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Disclaimer: This is only a rough summary (to complement the lecture on this topic in MAT347), and it may contain errors.

CLASSICAL GALOIS THEORY	DIFFERENTIAL GALOIS THEORY
Fields F	Differential fields F (fields with a derivation)
Extensions of fields K/F	Extensions of differential fields K/F
A polynomial is a formal expression $f(Y) = a_n Y^n + \ldots a_1 Y + a_0$ which we can think of as a map $f : F \to F$ Its roots are the solutions b to $f(b) = 0$	A DLO is a formal expression $T(y) = a_n y^{(n)} + \ldots a_1 y' + a_0 y$ which we can think of as a map $T : F \to F$ Its space of solution is $\ker_F(T) := \{b \in F \vert T(b) = 0\}$
If $\deg f(Y) = n$, then f has at most n roots	If $\deg T(y) = n$, then $\ker_F T$ is an F_0-vector space with dimension at most n
The splitting field of $f(Y)$ over F is a field K such that: • f has n roots in K, • K/F is generated by those roots	The Piccard-Vessiot extension of $T(y)$ over F is a differential field K such that: • $\ker_K T$ has dimension n • K/F is generated by $\ker_K T$
Thm: This extension exists and is unique up to isomorphism	**Thm:** This extension exists and is unique up to isomorphism
Galois group $\mathrm{Gal}(K/F)$	Galois group $\mathrm{Gal}(K/F)$
Galois correspondance between subfields of K/F and subgroups of G	Galois correspondance between closed subfields of K/F and closed subgroups of G
Let K be the sf of $f(Y)$ over F, then $\mathrm{Gal}(K/F) \leq S_n$	Let K be the P-V extension of $T(y)$ over F, then $\mathrm{Gal}(K/F) \leq \mathrm{GL}(n, F_0)$
An extension K/F is radical if there are fields $F = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_m = K$ such that M_i is obtained from M_{i_1} by adding one radical	An extension K/F is Liouvillian if there are differential fields $F = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_m = K$ such that M_i is obtained from M_{i_1} by adding one algebraic element, or an exponential, or an antiderivative
$f(Y) \in F[Y]$ solvable by radicals if its splitting field is contained in R such that R/F is radical	$T(y)$ is solvable by antiderivatives and elementary functions if its P-V extension is contained in R such that R/F is Liouvillian
Thm: Let K be the sf of $f(Y)$ over F. Let $G = \mathrm{Gal}(K/F)$. TFAE: 1) f is solvable by radicals 2) G is solvable	**Thm:** Let K be the P-V extensio of $T(y)$ over F. Let $G = \mathrm{Gal}(K/F)$. TFAE: 1) T is solvable by antiderivatives and elementary functions 2) G_0 (the connected component of the identity) is solvable