On the existence of maximal ideals
MAT347

The proof we know that every ring has a maximal ideal requires our ring to have a $1 \neq 0$. If the ring does not have an identity, the result is not necessarily true. These exercises allow you to build a counterexample.

1. Let A be any abelian group. Prove that if we define a product on A by $xy = 0$ for all $x, y \in A$, then A becomes a ring. Prove that subrings of $(A, +, \cdot)$ are the same as subgroups of $(A, +)$. Hence, to prove that there exists a ring without maximal ideals, it is enough to prove that there exists an abelian group without maximal subgroups.

2. An abelian group G is called *divisible* when for every $x \in G$ and for every positive integer n, there exists $y \in G$ such that $x = ny$. Prove that if G is a divisible abelian group, then it has no maximal subgroups.

   Hint: Assume that A is a maximal subgroup of G. Show that $G/A \cong Z_p$ for some prime p. Then prove that $px \in A$ for all $x \in G$. Then reach a contradiction.

3. Obtain one (non-zero) divisible abelian group, thus completing the proof.