MAT 1000 / 457 : Real Analysis I
Midterm Test, November 7, 2012

(Four problems; 20 points each. Time: 2 hours.)

Please be brief but justify your answers, citing relevant theorems.

1. Prove that the set of all $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3}$$

is a set of meaure zero. (*Hint:* Use the Borel-Cantelli lemma.)

2. Give an example of ...

(a) a measure space that is not σ-finite but contains sets of every finite positive measure;

(b) an example where the conclusion of the Fubini-Tonelli theorem fails.

Remark: You need not prove that your constructions work, but please describe them clearly. Sometimes a sketch can help ...

3. Consider a sequence $\{f_n\}_{n\geq 1}$ of integrable functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0\,.$$

(a) Assuming that the functions f_n are nonnegative, compute

$$\lim_{n\to\infty} \int_0^1 e^{-f_n(x)}\,dx\,.$$

(b) What can you cay about this limit if the f_n's change sign?

4. (a) Given two functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, with $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, consider their convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy\,.$$

Prove that the integral is well-defined for each $x \in \mathbb{R}^n$, and that $f * g$ is bounded.

(b) Furthermore, $f * g$ is continuous, and $\lim_{|x|\to\infty} f * g(x) = 0$.