MAT 1001 / 458 : Real Analysis II
Midterm Test, March 6, 2013

(Four problems; 20 points each. Time: 2 hours.)

Please be brief but justify your answers, citing relevant theorems.

1. (a) Let (u_n) be a sequence in a separable Hilbert space $\mathcal{H}$. What does it mean for (u_n) to be an *orthonormal basis*? Please give three equivalent conditions.

   (b) *(The Haar basis)*
   For $n \geq 0$ and $x \in [0,1]$, let
   $$v_n(x) = 1 - 2X_n \,,$$
   where X_n is the n-th digit in the binary expansion of x (i.e. $v_n(x) = 1$ when this digit is zero, and -1 otherwise. If x has several expansions, use the one that terminates in 0).
   Prove that (v_n) is an orthonormal basis for $L^2([0,1])$.

2. Let $(u_n)_{n\geq 1}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, and consider the linear transformation $T : \mathcal{H} \to \mathcal{H}$ given by $Tu_n = n^{-1}u_{n+1}$.

   (a) Show that T is compact, and compute its norm.

   (b) Show that T has no eigenvalues. Why does this not contradict the Spectral Theorem?

3. State the (Banach)-Alaoglu theorem (on a general Banach space). What does it imply about bounded sequences in $L^p(\mathbb{R})$? How? (Your answer will depend on the value of p.)

4. Show that
   $$Tf(x) = \frac{1}{\pi}\int_0^\infty \frac{f(y)}{x+y}\, dy$$
   defines a bounded linear operator on $L^2(0,\infty)$.

   Hint: Estimate $\langle Tf, f\rangle$, using
   $$|f(x)\bar{f}(y)| \leq \frac{1}{2}\left(\left(\frac{x}{y}\right)^{\frac{1}{2}}|f(x)|^2 + \left(\frac{y}{x}\right)^{\frac{1}{2}}|f(y)|^2\right) .$$