MAT 1600 : Probability I
Assignment 3, due October 7, 2020

1. *Chebyshev's inequality is and is not sharp (Durrett 1.6.2)*

   (a) For fixed $a > 0$, find a random variable X such

   $$\mathbb{P}(|X| \geq a) = \frac{E(X^2)}{a^2}\,.$$

   (b) On the other hand, if X is a random variable with $0 < E(X^2) < \infty$, then

   $$\lim_{a\to\infty} a^2 \mathbb{P}(|X| \geq a) = 0\,.$$

2. *The Hoeffding-Chernoff inequality for fluctuations of size $t \sim \sqrt{n}$ (Panchenko 2.1.2)*
   In the setting of Theorem 2.2 show that, for $s > 0$.

   $$\mathbb{P}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu) \geq s\right) \leq \exp\left(-\frac{s^2}{2\mu(1-\mu)} + O\big(n^{-\frac{1}{2}}\big)\right)\,.$$

3. *(Panchenko 2.1.5)* Suppose that the random variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are independent and, for all i, X_i and Y_i have the same distribution. Prove that

   $$\mathbb{P}\left(\sum_{i=1}^{n}(X_i - Y_i) \geq \sqrt{2t\sum_{i=1}^{n}(X_i - Y_i)^2}\right) \leq e^{-t}\,, \qquad (t > 0)\,.$$

   Hint: Think about a way to introduce Bernoulli $B(\frac{1}{2})$ variables ε_i into the problem and then apply Hoeffding's inequality.

4. *Continuous functions, cont'd from last week's Problem 4 (Panchenko 1.3.2)*
   Let $\mathcal{A}$ be the cylindrical σ-algebra in $\mathbb{R}^{[0,1]}$, and let $\Omega \subset [0,1]^{\mathbb{R}}$ be the subset of all continuous functions on $[0,1]$. Consider the σ-algebra

   $$\mathcal{B} := \{A \cap \Omega : A \in \mathcal{A}\}$$

   on Ω, consisting of the continuous functions in the cylindrical σ-algebra. Show that $\mathcal{B}$ contains the set

   $$C := \left\{\omega \in \Omega : \int_0^1 \omega(t)\,dt < 1\right\}\,.$$

   (*Remark*. I switched the roles of $\mathcal{A}$ and $\mathcal{B}$ — makes more sense to me this way.)

5. *Waiting for success*
Consider a sequence of independent tosses of a coin that shows Heads with probability p, and Tails with probability $q = 1 - p$. Let X_i be indicator that the ith toss comes up Heads. Let T_n be the number of the toss on which Heads appears for the n-th time.

   (a) *Geometric distribution* Find the distribution of T_1, and compute its expectation, variance, and moment-generating function.

   (b) *Negative binomial distribution.* Write the distribution $\mathbb{P}(T_n = t)$ in terms of suitable Binomial (n, p) random variables. (Do not try to simplify the formula.)

   (c) Show that $Y_1 = T_1$ and $Y_n = T_n - T_{n-1}$, $n = 2, 3, \ldots$ are all independent and identically distributed. Use this to compute the mean $\mathbb{E}T_n$, the variance $\mathrm{Var}\,(T_n)$, and the moment-generating function $\mathbb{E}e^{\lambda T_n}$.

6. Consider again the repeated coin toss described in the previous problem, and fix $n \geq 1$. Let

$$\sigma(T_n) = \{T_n^{-1}(A), A \subset \mathbb{N}\}$$

be the σ-algebra generated by the random variable T_k. For $j \geq 1$, find the conditional expectation $\mathbb{E}(X_j | T_n)$. (Express your answer as a random variable $f(T_n)$.)