MAT 1600 : Probability I
Assignment 4, due October 14, 2020

1. *Panchenko 2.2.1* Let $(X_n)_{n\geq 1}$ be i.i.d. random variables with $\mathbb{E}|X_i|^p < \infty$ for some $p > 0$. Show that $\max_{i\leq n} n^{-\frac{1}{p}}|X_i| \to 0$ in probability as $n \to \infty$.

2. *(Probabilistic proofs of the Borel-Cantelli Lemma (2.2.4)*
   Let $(A_n)_{n\geq a}$ be a sequence of events, and let N be the number of events that occur (i.e., $N(\omega) = \#\{n \geq 1 : \omega \in A_n\}$).

   (a) If A_n is a sequence of events with $\sum_{n\geq 1} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(N < \infty) = 1$. (Consider the expectation of N.)

   (b) If the events are independent and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(N < \infty) = 0$. (You can start, for example, by showing that $\mathbb{P}(N = 0) = 0$.)

3. *Weak LLN for U-statistics (Panchenko 2.2.2)*

   (a) Look up U-Statistic on Wikipedia.

   (b) If $(X_n)_{n\geq 0}$ are i.i.d. with mean μ and varianc e $\sigma^2 < \infty$, show that

   $$\binom{n}{2}^{-1} \sum_{1\leq i<j\leq n} X_i X_j \to \mu^2$$

   in probability as $n \to \infty$.

4. *(Panchenko 2.2.3)* If $u : [0,1]^n \to \mathbb{R}$ is continuous, prove that

   $$\sum_{0\leq j_1,\ldots,j_k\leq n} u\left(\frac{j_1}{n}, \ldots, \frac{j_k}{n}\right) \prod_{i\leq k} \binom{n}{j_i} x_i^{j_i}(1-x_i)^{n-j_i} \longrightarrow u(x_1, \ldots, x_n)$$

   as $n \to \infty$, uniformly on $[0,1]^k$.

5. *(Panchenko 2.1.7)* Suppose that the chances of winning the jackpot in a lottery is 1:139,000,000. Assuming that 100,000,000 people played independently of each other, estimate the probability that 3 of them will have to share the jackpot. Give a bound on the quality of your estimate.

6. *Monte Carlo integration (Durrett 2.2.3)* Let f be a Borel measurable, Lebesgue integrable function on the unit interval $[0,1]$. The objective is to construct a probabilistic method for computing the integral

$$I = \int_0^1 f(x)\,dx\,.$$

Let $U_1, U_2, \ldots$ be independent and uniformly distributed on $[0,1]$, and let

$$I_n = \frac{1}{n}\big(f(U_1) + \cdots + f(U_n)\big)$$

be average of the first n values.

(a) Show that $I_n \to I$ in probability.

(b) If, moreover, $\int |f(x)|^2\,dx < \infty$, use Chebyshev's inequality to estimate

$$P(|I_n - I|) > an^{1/2}\,, \quad \text{for } a > 0\,.$$

Remark. This method for computing integrals can be adapted (by change of variables) to numerically integrate functions of many variables over complicated regions.