MAT 1600 : Probability I
Assignment 5, due October 22, 2020

1. *(Panchenko 2.2.4)* If $\mathbb{E}|X| < \infty$ and $\lim \mathbb{P}(A_n) = 0$, show that $\lim \mathbb{E} X I_{A_n} = 0$.
   Hint: Use the Borel-Cantelli lemma over some subsequence.

2. We have recently proved *Jensen's inequality*: If g is a convex real-valued function on $\mathbb{R}$ and X a random variable with $\mathbb{E}|X| < \infty$, then $\mathbb{E}\, g(X) \geq g(\mathbb{E}X)$.

   (a) When is there equality in Jensen's inequality? Give a precise characterization in terms of g and the distribution of X.
   (Think about the special cases $g(x) = x^2$ and $g(x) = |x|$.) A sketch will help.)

   (b) Justify the statement that 'Under the hypotheses of Jensen's inequality, $\mathbb{E}g(X)$ is always well-defined, though it may take the value $+\infty$'.

3. *(Panchenko 2.2.7)* Suppose that $\{X_n\}_{n\geq 1}$ are independent random variables. Show that

$$\mathbb{P}\left(\sup_{n\geq 1} X_n < \infty\right) = 1 \quad \Longleftrightarrow \quad \sum_{n\geq 1} \mathbb{P}(X_n > M) < \infty \text{ for some} M > 0\,.$$

4. Let $\{X_n\}_{n\geq 1}$ be i.i.d., and $S_n = X_1 + \cdots + X_n$.

   (a) *(Panchenko 2.2.6)* If $S_n/n \to 0$ almost surely, show that $\mathbb{E}|X_1| < \infty$.
   (*Hint:* Use the idea in Eq. (2.2.2) and Borel-Cantelli).

   (b) *(Panchenko 2.2.8)* If, on the other hand, $X_i \geq 0$ and $\mathbb{E}X_1 = \infty$, show that $S_n/n \to \infty$ almost surely.

5. *(Durrett 2.2.5)* Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i > x) = \frac{e}{x \log x}$ for $x \geq e$. Construct a sequence of constants $\mu_n \to \infty$ such that $S_n/n - \mu_n \to 0$ in probability.
   Hint: To get μ_n, use the truncation $X I_{X_i \leq n}$ for $i = 1, \ldots, n$, and apply the union bound. (Remarkably, $\mathbb{E}X_1 = \infty$!).

6. *(Panchenko 2.2.10)* Let $\{X_n\}_{n\geq 1}$ be independent and exponentially distributed, i.e., with distribution function $F(x) = 1 - e^{-x}$ for $x \geq 0$. Show that

$$\mathbb{P}\left(\limsup \frac{X_n}{\log n} = 1\right) = 1\,.$$