# MAT 351: Partial Differential Equations
# February 2, 2018

Consider the wave equation $u_{tt} = c^2\Delta u$, with initial conditions $u(x,0) = \phi(x)$ and $u_y(x,u) = \psi(x)$. Two important properties of the wave equation in any dimension are:

- The **energy** $E(t) = \frac{1}{2}\int u_t(x,t)^2 + c^2|\nabla u(x,t)|^2\,dx$ is **conserved** (constant in time);
- **Causality:** The solution $u(x,t)$ depends on the initial conditions only on the ball $B_{c|t|}(x)$. In other words, the domain of dependence of $(x,t)$ is the solid backwards **light cone**

$$\left\{(y,s) \in \mathbb{R}^n \times \mathbb{R} \;\mid\; s \le t, |y-x|^2 \le c^2|t-s|^2\right\}.$$

We will restrict attention to the cases of two and three spatial dimensions appearing in classical Physics. In three dimensions, the solution of the wave equation is given by **Kirchhoff's formula**

$$u(x,t) = \frac{\partial}{\partial t}\left\{\frac{1}{4\pi c^2 t}\int_{|y-x|=ct}\phi(y)\,dS(y)\right\} + \frac{1}{4\pi c^2 t}\int_{|y-x|=ct}\psi(y)\,dS(y)\,.$$

Remarkably, the solution depends on the initial data only on the (surface of the) light cone, i.e., waves travel exactly at the speed of light. This is called **Huygens principle**. It is typical for solutions of the wave equation in all odd dimensions $n = 2k+1 \ge 3$.

In two dimensions, we have **Poisson's formula**

$$u(x,t) = \frac{\partial}{\partial t}\left\{\frac{1}{2\pi c}\int_{|y-x|<ct}\frac{\phi(y)}{(c^2t^2-|y-x|^2)^{1/2}}\,dy\right\} + \frac{1}{2\pi c}\int_{|y-x|<ct}\frac{\psi(y)}{(c^2t^2-|y-x|^2)^{1/2}}\,dy\,.$$

Note that Huygens' principle fails in two dimensions (and generally in even dimensions.)

The key to the proof of Kirchhoff's formula is the observation that the **spherical mean** of a solution, given by

$$\bar{u}(r,t;x) = \frac{1}{n\omega_n r^{n-1}}\int_{|y-x|=r} u(y,t)\,ds(y)$$

satisfies **Darboux' equation**

$$u_{tt} = c^2\left(u_{rr} + \tfrac{n-1}{r}u_r\right)\,.$$

(Here, the denominator $n\omega_n$ is the $n-1$-dimensional surface area of the $n$-ball. In $n=3$ dimensions, its value is $4\pi$.) Darboux's equation can be solved explicitly when $n$ is odd, and Kirchhof's formula follows by setting $u(x,t) = \bar{u}(0,t;x)$. From there, we obtain the solution in even dimensions by using **Hadamard's method of descent**.

**Read:** Sections 1-3 of Chapter 9.

**Assignment 12 (due Friday, February 9):**

(H1) A **plane wave** is a solution of the wave equation of the form $u(x,t) = f(k \cdot x - ct)$, where $f$ is a $C^2$-function. Find all the three-dimensional plane waves.

(H2) Thinking of space-time as $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, let $\Gamma$ be the diagonal $4 \times 4$ matrix with diagonal entries $1, 1, 1, -1$. A **Lorentz transformation** is an invertible matrix that satisfies $L^t \Gamma L = \Gamma$, or equivalently, $L^{-1} = \Gamma L^t \Gamma$.

(a) Prove that Lorentz transformations form a group, i.e., products and inverse of Lorentz transformations are again Lorentz transformations. What can you say about the determinant of $L$?

(b) Show that $L$ is Lorentz if and only if it preserves the quadratic form $m(x,t) = |x|^2 - t^2$, i.e., $m(L(v)) = m(v)$ for all $v = (x,t) \in \mathbb{R}^4$. The quadratic form $m$ is called the **Lorentz metric**.

(c) If $L$ is a Lorentz transformation, and $U(z) = u(L(z))$, show that

$$u_{tt} - \Delta u = U_{tt} - \Delta U \,,$$

i.e., if $u$ solves the wave equation, so does $U$.

(d) Explain the meaning of a Lorentz transformation in more geometrical terms, by drawing an analogy to the group of orthogonal matrices. How does $m$ relate to the light cone?

(H3) Consider the one-dimensional wave equation $u_{xx} = c^2 u_{xx}$ with initial values given on a surface $\mathcal{S} = \{(x,t) \mid t = \gamma(x)\}$, by

$$u((x, \gamma(x)) = \phi(x) \,, \quad \frac{\partial u}{\partial n} = \Psi(x) \,.$$

If $\mathcal{S}$ is space-like, i.e., $|\gamma'(x)| < \frac{1}{c}$, prove that the initial-value problem has a unique solution. (*Hint:* The solution can be written as $u(x,t) = F(x+ct) + G(x-ct)$.)

(H4) Verify that $u(x,t) = (c^2 t^2 - |x|^2)^{-1}$ satisfies the three-dimensional wave equation except on the light cone.

**For discussion and practice:**

1. Find a simple formula for the solution of the heat equation $u_t = \Delta u$ in three dimensions with initial values $u(x,y,z,0) = xy^2 z$.
(*Hint:* Differentiate the equation and the initial values with respect to the variables $x, y, z$.)