MAT 377: Probability
Final Assessment, December 14/15 2020

Six problems, 120 points in total. Time: 3 hours, plus 30 minutes for logistics.

If you have questions, email: almut@math.toronto.edu or text/call: 416 262 5535. I will be unreachable from 9-10am on Monday, but online otherwise.

Please be brief but explain your reasoning. Refer to known results as needed. Good luck!

1. *(Exercise 1.3.3, with slightly different numbers.)* An emperor orders preparation for a banquet for all of his 250 nobles. The Chancellor prepares 245 seats, knowing that the probability that a noble will not come is $p = .04$.

   (a) Find the mean and variance of the number of no-shows. What assumptions are you making?

   (b) Use the Poisson approximation to compute the probability that there will be enough seats. (You need not evaluate numerically).

   (c) Please provide an explicit error estimate for your calculation!

2. Let X be a random variable. Give short proofs (1-2 lines each) of the following two inequalities:

   (a) If $\mathbb{E}e^X < \infty$, then $e^{sX} < \infty$ for all $s \in (0, t)$.

   (b) If $\phi : \mathbb{R} \to \mathbb{R}$ is positive and increasing, with $\mathbb{E}\phi(X) < \infty$, then

$$\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}\phi(X)}{\phi(x)}, \qquad x > 0.$$

3. *(Exercise 3.5.4.)* Suppose we throw n balls into m boxes at random, and let N be the number of boxes with at least two balls in them.

   (a) Compute $\mathbb{E}N$.

   (b) What is $\lim_{n\to\infty} \mathbb{E}N/n$ when $m = \alpha n$ for some fixed $\alpha > 0$?

   (c) What does Azuma's inequality say about $|N - \mathbb{E}N|$? Please explain.

4. *The emperor's banquet, revisited.* Consider again the situation in Problem 1.

   (a) Use the Central Limit Theorem to estimate the probability that there will be enough seats. (Again, you are not asked to evaluate numerically. The result is close to .96)

   (b) Briefly comment on the relative merits of the two approximations (Poisson and CLT). Which one would you use, and why? How would your answer change if the number of guests ($n = 250$) or the no-show probability ($p = .04$) were changed?

5. *Min and max.* Suppose X, Y are independent and exponentially distributed, with means α^{-1} and β^{-1}, respectively. (That is, their probability densities are $p(x) = \alpha e^{-\alpha x}$ and $q(x) = \beta e^{-\beta x}$ for $x > 0$).

   (a) Find the probability densities of $\min(X, Y)$ and $\max(X, Y)$.

   (b) Also compute their means, $\mathbb{E}\min(X, Y)$ and $\mathbb{E}\max(X, Y)$.
   (*Useful observation:* $\min(X, Y) + \max(X, Y) = X + Y$.)

6. *Uniform distribution on the sphere.* Let $g_1, \dots g_n$ be i.i.d. standard normal random variables. Define a random unit vector on $\mathbb{R}^n$ by

$$U_i := \frac{g_i}{\sqrt{g_1^2 + \ldots + g_n^2}}, \quad i = 1, \ldots, n.$$

   (a) Argue that and $\mathbb{E}U_i = 0$ for $i = 1, \ldots, n$.

   (b) Moreover, $\mathrm{Cov}(U) = \frac{1}{n}I$. (*Hint:* Write down the relevant Gaussian integrals. You may find it helpful to consider $\mathrm{Var}(U_1) + \cdots + \mathrm{Var}(U_n)$.)

   (c) Are the U_i independent? Pairwise independent? Why/why not?
   (Look for geometric reasons, avoiding calculation.)

   (d) Prove that the distribution of U is rotation invariant: If Q is an orthogonal matrix, then

$$\mathbb{P}(U \in A) = \mathbb{P}(QU \in A), \quad \text{for all events } A \subset S^{n-1}.$$

   (See Example 4.2.4).

Remark. It follows form Part (d) that U is uniformly distributed on S^{n-1}, that is,

$$\mathbb{P}(U \in A) = \frac{\text{surface area of } A}{\text{surface area of } S^{n-1}}, \quad \text{for all events } A \subset S^{n-1}.$$

(You are not asked to prove this.)