MAT 377: Probability
Midterm Assessment, Oct. 28 2020

(Four problems, 80 points in total. Time: 2 hours, plus 20 minutes for logistics. If you have questions, email or text me (416 262 5535).

1. (20pts) Let N_3 be the number of triangles in the Erdős-Rényi graph $G(n, p)$.
   Compute ...

   (a) ... the expectation $\mathbb{E}N_3$,

   (b) ... the variance $\mathrm{Var}(N_3)$.

   Please explain your method!

2. (20pts) Let X be a random variable with Poisson (μ) distribution.

   (a) Describe a situation that can be modeled by such a random variable. What are crucial assumptions? How should μ be chosen? What determines the error in the approximation? Please justify your answer in a sentence or two (using results we proved in class).

   (b) Set $\mu = 1$. Find the moment-generating function $M(\lambda) := \mathbb{E}e^{\lambda X}$.

   (c) Use Markov's inequality to derive a bound on $\mathbb{P}(X \geq t)$.
   (Remember to optimize over λ!)

3. (20pts) Consider a sequence of independent tosses of a coin that shows Heads with probability p, and Tails with probability $q = 1 - p$. Let X_i be the indicator that the i-th toss comes up Heads, and let T_k be the toss on which Heads appears for the k-th time.

   (a) *(Geometric distribution)*
   Find the distribution $\mathbb{P}(T_1 = t)$. What is its expectation?

   (b) Express $\mathbb{P}(T_k = t)$ in terms of suitable Binomial (n, p) random variables.
   (Do not try to simplify the result, but please explain briefly).

   (c) Let A be the event that $T_5 = 10$. Find the conditional probability $\mathbb{P}(X_i = 1 \,\big|\, A)$ for $i = 1, 2, \ldots$ (The answer depends on how large i is compared to 10).

4. (20pts) *(A maximal inequality)*
Suppose that the random variables $X_1, X_2, \ldots$ are i.i.d. and nonnegative.
Assuming that $\mathbb{E}X_1^2 < \infty$, show that

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\ldots,n} X_i \geq t\sqrt{n} \right) = 0$$

for every $t > 0$.

Hint: Use the union bound, and the improved version of Chebyshev's inequality

$$\mathbb{P}(X \geq x) \leq \frac{1}{x}\,\mathbb{E}XI_{X\geq x}\,, \quad (x > 0)\,.$$