MAT 377: Probability
Midterm Assessment, Oct. 18 2021

(Four problems, 80 points in total. Time: 1 hour 50 minutes.

1. (20pts) A standard deck of cards contains 54 different cards, 4 of which are Aces (namely A♣, A♠, A♡, A♢).

   You are dealt 5 cards from a well-mixed deck. Let N be the number of Aces you received. Compute ...

   (a) ... the expectation $\mathbb{E}N$,

   (b) ... the variance $\mathrm{Var}(N)$.

2. (20pts) *(Waiting for success.)*
   Let $X_1, X_2, \ldots$ a sequence of i.i.d. random variables with Bernoulli $B(p)$ distribution. Interpet X_i as the indicator that the i-th toss of a coin comes up Heads.

   For $k \geq 1$, let T_k be the toss on which Heads appears for the k-th time.

   (a) What is the distribution of T_1? (Name and formula)

   (b) Prove that $W_1 := X_1$ and $W_2 := X_2$ are independent and identically distributed.

   It turns out that all the random variables $W_1 := T_1$ and $W_k := T_k - T_{k-1}$, $k \geq 2$ are i.i.d. (You are not asked to prove this.)

   (c) What does the Law of Large Numbers say about $\frac{1}{n}T_n$?
   (Remember to verify its assumptions!)

3. (20pts) *(Weighted averages)*
Let X be a random variable on a probability space Ω (with probability measure $\mathbb{P}$). Define, for $A \subset \Omega$,

$$\mathbb{P}'(A) = \mathbb{E}(X I_A)\,.$$

(a) Under what assumptions on X is $\mathbb{P}'$ a probability measure?

In the following let $\mathbb{P}'$ be a probability measure defined by (a).

(b) Let Y be another random variable on Ω.
Find a formula for the expected value $\mathbb{E}'Y$ (with respect to $\mathbb{P}'$).
Under what assumptions on Y is this expected value well defined?

(c) Specifically, compute $\mathbb{E}'X$ in terms of the variance of X.

(d) Prove that $\mathbb{E}'X^m \geq (\mathbb{E}X^m)^{\frac{m+1}{m}}$ for all $m \geq 1$.

Please justify your claims!

4. (20pts) The *moment-generating function* of a random variable X is defined by

$$M(s) := \mathbb{E}(e^{sX})\,, \quad s \in \mathbb{R}\,.$$

Note that this is well-defined (but may take the value $+\infty$).

(a) If X has geometric(p) distribution, compute its moment-generating function.
For what values of s is it finite?

(b) Prove the *Chernoff bound*

$$\mathbb{P}(X \geq x) \leq \inf_{s \geq 0} \left\{ e^{-sx} M(s) \right\}\,.$$

(c) If $X_1, \ldots, X_n$ are i.i.d., with the same distribution as X, prove that the moment-generating function of the sum is given by $\mathbb{E}(e^{s(X_1+\cdots+X_n)}) = \big(M(s)\big)^n$.
Conclude that

$$\mathbb{P}(X_1 + \cdots + X_n \geq nx) \leq \left(\inf_{s \geq 0} \left\{ e^{-sx} M(s) \right\} \right)^n\,.$$

Remark: This is useful, for example, for estimating the random variables T_k from Problem 2.