APM 351: Differential Equations in Mathematical Physics
Assignment 19, due March 29, 2012

Summary:

- A **test function** on a domain $U \subset \mathbb{R}^d$ is a smooth function with compact support in D.

The space of test functions on $\mathbb{R}^d$ will be denoted by $\mathcal{D}$. We say that $\lim \phi_j = \phi$ in $\mathcal{D}$, if the functions ϕ_j converge uniformly to ϕ, and all their derivatives converge uniformly as well.

- A **distribution** is a linear transformation mapping test functions to $\mathbb{R}$ (or $\mathbb{C}$).

In other words, a distribution f assigns to each $\phi \in \mathcal{D}$ a scalar $f(\phi)$. We require that this transformation is **continuous** on $\mathcal{D}$, in the sense that

$$\lim \phi_j = \phi \quad \Rightarrow \quad \lim f(\phi_j) = f(\phi) \, .$$

We denote the space of distributions by $\mathcal{D}'$, and think of it as the dual space of $\mathcal{D}$.

- A sequence of distributions $\{f_j\}$ **converges weakly** to f, if $\lim f_j(\phi) = f(\phi)$ for all $\phi \in \mathcal{D}$.

Functions are important special cases of distributions. If f is a continuous function on $\mathbb{R}^d$, we can define the corresponding distribution by

$$f(\phi) = \int f(x)\phi(x) \, dx \, .$$

We often write distributions in this form, even when they are not given by a function. For example, the Dirac δ-distribution is defined by

$$\delta(\phi) = \int \phi(x)\delta(x) \, dx := \phi(0) \, .$$

The δ-distribution on $\mathbb{R}^d$ can be obtained as the weak limit of rescaled functions $j^d f(jx)$, where f is a nonnegative integrable function with $\int f(x) \, dx = 1$.

- **Distributional derivatives** of f are defined by $(D_i f)(\phi) = -f\left(\frac{\partial}{\partial x_i}\phi\right)$ for all test functions ϕ.

Distributional derivatives are also called *weak derivatives*. If f is given by a differentiable function, then its distributional derivatives are given by the classical derivatives of f. To give another example, the derivative of the δ-distribution is one dimension is defined by $\delta'(\phi) = -\phi'(0)$.

When solving a linear PDE $Lu = 0$, it is often useful to consider distributional solutions. For example, the fundamental solution of Laplace's equation $\Delta u = f$ on $\mathbb{R}^d$, given by

$$G_0(x) = C_d |x|^{2-d} \, ,$$

(where C_d is a specific dimension-dependent constant) solves

$$-\Delta u = \delta$$

in the sense of distributions. In this case, G_0 and ∇G_0 turn out to be functions. But note that distributional solutions make no sense for nonlinear equations, because a nonlinear function of a distribution is not a distribution. (For example, δ^2 has no meaning.)

Assignments:

Read Chapter 12 of Strauss.

1. Compute the first three distributional derivatives of the function

$$f(x) = \max\{0, 1 - x^2\}\,.$$

2. Let f be a distribution on $\mathbb{R}$ with $f' = 0$.

   (a) What does that mean?

   (b) Prove that $f(\phi) = 0$ for all test functions ϕ with $\inf_{\mathbb{R}} \phi(x)\, dx = 0$.

   (c) Conclude that f is given by a constant *function* $f(x) = c$, by showing that $f(\phi) = c \int_{\mathbb{R}} \phi(x)\, dx$ for all test function ϕ.

3. Consider **Burger's equation**

$$u_t + u u_x = 0\,, \quad u(x,0) = u_0(x) \tag{1}$$

   for $x \in \mathbb{R}$ and $t > 0$. A **integral solution** of the equation is a function u such that

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + \frac{1}{2} u^2 \phi_x\, dx dt + \int_{-\infty}^\infty u_0(x)\phi(0,x)\, dx \tag{2}$$

   holds for every smooth test function $\phi(x,t)$ with compact support in $\mathbb{R} \times [0, \infty)$. (Note that ϕ need not vanish on the line $t = 0$.)

   (a) Suppose u itself is smooth. Verify that then Eq. (2) and Eq. (1) are equivalent.

   (b) Let u be a smooth solution of Burger's equation. Assume that, for each $t \geq 0$, $u(\cdot, t)$ has compact support, and define its **mass** by

$$M(t) = \int_{-\infty}^\infty u(x,t)\, dx\,.$$

   Prove that mass is conserved, i.e., $M(t)$ is constant in time.
   (*Hint:* Compute $\frac{d}{dt} M(t)$.)

   (b) Suppose u is a continuous integral solution of Burger's equation, i.e., u satisfies Eq. (2). Assume furthermore that $u(\cdot, t)$ has compact support for each $t \geq 0$. Show that mass is conserved also in this case.
   (*Hint:* Use test functions of the form $\phi(x,t) = a(x)b(t)$.)