SOLITON DYNAMICS AND SYMPLECTIC GEOMETRY

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Abstract: A bright soliton in dimension one is obtained by solving

$$-\frac{1}{2}\eta'' - \eta^3 + \frac{\mu^2}{2}\eta = 0\,,$$

which gives $\eta(x) = \mu\mathrm{sech}(\mu x)$. Consequently, $u(x,t) = e^{i\gamma(t)}e^{iv(x-a(t))}\eta(x-a(t))$, with $a(t) = a_0 + vt$ and a suitably chosen $\gamma(t)$, solves the nonlinear Schrödinger equation,

$$iu_t = -\frac{1}{2}u_{xx} - |u|^2 u\,,$$

providing a perfect profile moving with the velocity v and exhibiting a lot of stability.

About 20 years ago, Andreas Floer and Alan Weinstein showed that the semiclassical Schrödinger equation (with the semiclassical parameter, h, controlling the slow variation of the potential),

$$-\frac{1}{2}\eta'' + W(hx)\eta - \eta^3 + \frac{\mu^2}{2}\eta = 0\,,$$

has soliton like solutions concentrating at non-degenerate extrema of the potential W. That lead to interesting new solutions of the Gross-Pitaevskii equation important in the study of Bose-Einstein condensates,

$$iu_t = -\frac{1}{2}u_{xx} + W(hx)u - |u|^2 u\,, \tag{1}$$

and motivated a lot of further work on semiclassical nonlinear Schrödinger equations.

The symplectic structure of these equations has been exploited in the study of soliton stability since the work of Michael Weinstein, contemporary with the work of Floer-Weinstein.

In this talk I will show how additional insights from elementary symplectic geometry improve recent results of Jürg Fröhlich, I. Michael Sigal and their collaborators. We prove that the solution to (1) with initial condition given by a soliton stays within $\mathcal{O}(h^2)$ to a soliton evolving according to natural classical equations in the parameters γ, a, v, and μ (a four dimensional symplectic manifold). What is particularly appealing is the striking numerical improvement corresponding to the theoretical improvement in semiclassical errors, $h \to h^2$.

This is joint work with Justin Holmer.