MULTIPLIERS AND A PALEY-WIENER THEOREM FOR REAL REDUCTIVE GROUPS*

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The classical Paley-Wiener theorem is a description of the image of $C_c^\infty(\mathbb{R})$ under Fourier transform. The Fourier transform

$$\hat{f}(\Lambda) = \int_{-\infty}^{\infty} f(x)e^{\Lambda x}\, dx$$

is defined a priori for purely imaginary numbers Λ, but if f has compact support $\hat{f}$ will extend to an entire function on the complex plane. The image of $C_c^\infty(\mathbb{R})$ under this map is the space of entire functions F with the following property - there exists a constant N such that

$$\sup_{\Lambda\in\mathbb{C}}\left(|F(\Lambda)\ e^{-N|\mathrm{Re}\ \Lambda|}(1 + |\mathrm{Im}\ \Lambda|)^n\right) < \infty$$

for every integer n. (There is a similar theorem which characterizes the image of the space of compactly supported distributions.)

Our purpose is to describe an analogous result for a reductive Lie group. We shall also discuss a closely related theorem on multipliers, a result whose statement is especially simple. Both results were proved in detail in the paper [1]. We will be content here to just describe some of the main ideas. In the case of groups of real rank 1, the theorems were proved by Campoli [2]. The new ingredients for higher rank are (a) a scheme for keeping track of multi-dimensional residues, reminiscent of Langlands' work on Eisenstein series [9(b), Chapter 7], [9(a), §10], and (b) a theorem of Casselman on partial matrix coefficients of induced representations.

A number of mathematicians have proved Paley-Wiener theorems for particular classes of groups. We mention the papers of Ephrenpreis and Mautner ([4(a)], [4(b)]), Helgason ([7(a)], [7(b)], [7(c)], [7(d)]), Gangolli ([5]), Zelobenko ([10]), Delorme ([3]), and Kawazoe ([8(a)], [8(b)]) in addition to the thesis of Campoli cited above.

1. A MULTIPLIER THEOREM

Let G be a reductive Lie group, with Iwasawa decomposition

$$G = N_0 A_0 K.$$

We shall assume that G satisfies the general axioms of Harish-Chandra in [6(a)]. We shall denote the Lie algebras of Lie groups by lower case German script letters, and we will add a subscript $\mathbb{C}$ to denote complexification. Thus,

$$\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}.$$

Let

$$\mathcal{H} = C_c^\infty(G,K)$$

be the Hecke algebra. It is the space of functions in $C_c^\infty(G)$ whose left and right translates by K span a finite dimensional space; it becomes an algebra under convolution. We are interested in multipliers of $\mathcal{H}$. By this, we mean linear maps

$$C\colon \mathcal{H} \to \mathcal{H}$$

such that

$$C(f * g) = C(f) * g = f * C(g),$$

for every f and g in $\mathcal{H}$. (This condition is equivalent to saying that C commutes with the left and right action on $\mathcal{H}$ of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.)

Suppose that C is such a multiplier and that π belongs to $\Pi(G)$, the set of irreducible admissible representations of G. Then

$$\pi(C(f)) = C_\pi \pi(f), \qquad f \in \mathcal{H},$$

for a complex number C_π which is independent of f. The multiplier will be completely determined by the map

$$\pi \to C_\pi .$$

Because of Harish-Chandra's subquotient theorem, we can actually restrict our attention to the principal series. Recall that if M_0 is the centralizer of A_0 in K, the principal series

$$I(\sigma,\Lambda), \qquad \sigma \in \Pi(M_0), \qquad \Lambda \in \mathfrak{a}^*_{0,\mathbb{C}},$$

can be defined to act on a Hilbert space U_σ which is independent of Λ. It is irreducible for almost all Λ, so that $C_{I(\sigma,\Lambda)}$ is defined. As a function of Λ, $C_{I(\sigma,\Lambda)}$ is analytic and extends to an entire function on $\mathfrak{a}^*_{0,\mathbb{C}}$. If π is equivalent to a subquotient of $I(\sigma,\Lambda)$,

$$C_\pi = C_{I(\sigma,\Lambda)} .$$

Thus,

$$(\sigma,\Lambda) \to C_{I(\sigma,\Lambda)}, \qquad \sigma \in \Pi(M_0), \quad \Lambda \in \mathfrak{a}^*_{0,\mathbb{C}},$$

is an entire function in Λ which completely determines C. It provides a very concrete way to realize any multiplier.

As an example, consider the center $\mathfrak{z}$ of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. If $z \in \mathfrak{z}$, then

$$C(f) = zf, \qquad f \in \mathcal{H},$$

is a multiplier of $\mathcal{H}$. To represent it as above, let $\mathfrak{a}_K$ be a Cartan subalgebra of $\mathfrak{m}_0$. Then

$$\mathfrak{h} = i\mathfrak{a}_K \oplus \mathfrak{a}_0$$

is a real vector space, and is a Cartan subalgebra of the split real form of $\mathfrak{g}_\mathbb{C}$. Its interest comes from the fact that it is invariant under the complex Weyl group W of $(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$. Let γ_z be the W-invariant differential operator on $\mathfrak{h}$ obtained from z by the

Harish-Chandra map. We shall regard γ_z as a W-invariant distribution on $\mathfrak{h}$ which is compactly supported. (It is in fact supported at the origin.) Its Fourier transform

$$\hat{\gamma}_z(\nu), \qquad \nu \in \mathfrak{h}^*_{\mathbb{C}},$$

is a W-invariant polynomial on $\mathfrak{h}^*_{\mathbb{C}}$. If μ_σ is the linear functional in $i\mathfrak{a}^*_K$ which defines the infinitesimal character of a representation $\sigma \in \Pi(M_0)$,

$$C_{I(\sigma,\Lambda)} = \hat{\gamma}_z(\mu_\sigma + \Lambda), \qquad \Lambda \in \mathfrak{a}^*_{0,\mathbb{C}}.$$

The multipliers from this last example are of course well known. They extend to the full convolution algebra $C_c^\infty(G)$. Since they are defined directly for any function f, it is not really necessary to look at the function $C_{I(\sigma,\Lambda)}$. It turns out, however, that there is a richer family of multipliers for H which do not in general extend to $C_c^\infty(G)$. These multipliers are intrinsically more algebraic, and can only be described by the functions $C_{I(\sigma,\Lambda)}$.

THEOREM 1. *Let γ be any compactly supported, W-invariant distribution on $\mathfrak{h}$. Then there is a unique multiplier C of H such that*

$$C_{I(\sigma,\Lambda)} = \hat{\gamma}(\mu_\sigma + \Lambda)$$

for all $\sigma \in \Pi(M_0)$ and $\Lambda \in \mathfrak{a}^*_{0,\mathbb{C}}$.

2. THE PALEY-WIENER THEOREM

Theorem 1 describes multipliers in terms of the Fourier transform on H. In order to prove it, we must characterize the image of H under Fourier transform. For any $f \in H$, set

$$\hat{f}(\sigma,\Lambda) = I(\sigma,\Lambda,f) = \int_G f(x)I(\sigma,\Lambda,x)dx,$$

with $\sigma \in \Pi(M_0)$ and $\Lambda \in \mathfrak{a}^*_{0,\mathbb{C}}$. Then $\hat{f}(\sigma,\Lambda)$ is an entire function of Λ which, for any σ, takes values in the space of operators on

U_σ. It is K-finite, in the sense that the space spanned by the functions

$$(\sigma,\Lambda) \to I(\sigma,\Lambda,k_1)\hat{f}(\sigma,\Lambda)I(\sigma,\Lambda,k_2),$$

indexed by k_1 and k_2 in K, is finite dimensional. There is a constant N, which depends in a simple way on the support of f, such that for every n,

$$\sup_{(\sigma,\Lambda)} (\|\hat{f}(\sigma,\Lambda)\| e^{-N\|\mathrm{Re}\,\Lambda\|}(1 + \|\mathrm{Im}\,\Lambda\|)^n) < \infty.$$

The function has another property, which comes from the various intertwining maps between principal series. Suppose there is a relation

$$\sum_{k=1}^m D_k(I(\sigma_k, \Lambda_k, x)u_k, v_k)) = 0 \ , \tag{2.1}$$

valid for all $x \in G$, in which each D_k is a differential operator on $\mathfrak{a}^*_{0,\mathbb{C}}$ acting through Λ_k, and u_k, v_k are vectors in U_{σ_k}. Integrating this against the function $f(x)$, we see that

$$\sum_{k=1}^m D_k(\hat{f}(\sigma_k, \Lambda_k)u_k, v_k) = 0.$$

Relations of this form are common, but are not easy to characterize explicitly. For example, there will be such a relation any time an irreducible representation occurs in two different ways as a composition factor of the principal series.

Let PW(G,K) be the space of functions

$$F: (\sigma,\Lambda) \to \mathrm{End}(U_\sigma), \qquad \sigma \in \Pi(M_0), \qquad \lambda \in \mathfrak{a}^*_{0,\mathbb{C}},$$

such that

(i) $F(\sigma,\Lambda)$ is entire in Λ.

(ii) F is K finite.

(iii) There is a constant N such that for any n,

$$\sup_{(\sigma,\Lambda)} (\|F(\sigma,\Lambda)\| e^{-N\|\mathrm{Re}\,\Lambda\|}(1 + \|\mathrm{Im}\,\Lambda\|)^n) < \infty.$$

(iv) Whenever a relation of the form (2.1) holds, we have

$$\sum_{k=1}^{m} D_k(F(\sigma_k, \Lambda_k)u_k, v_k) = 0. \tag{2.2}$$

There are natural topologies which turn both H and PW(G,K) into Frechet spaces. Our Paley-Wiener theorem is

THEOREM 2. *The map*

$$f \to \hat{f}$$

is a topological isomorphism for H *onto* PW(G,K).

As we shall see in the next section, Theorem 1 is an easy consequence of Theorem 2. However, the proof of Theorem 2 is considerably harder.

3. PROOF OF THEOREM 1

Following an argument of Campoli, we shall derive the multiplier theorem as a corollary of Theorem 2. Suppose that γ is a compactly supported, W-invariant distribution on $\mathfrak{h}$, and that f belongs to H. Theorem 1 amounts to showing that

$$(\sigma,\Lambda) \to \hat{\gamma}(\mu_\sigma + \Lambda)\hat{f}(\sigma,\Lambda) \tag{3.1}$$

is the Fourier transform of some other function in H. By Theorem 2, we need only show that this function belongs to PW(G,K). The first three conditions in the definition of PW(G,K) clearly hold. We must establish the less obvious fourth condition.

Now $\hat{\gamma}$ is an entire, W-invariant function on $\mathfrak{h}_{\mathbb{C}}^*$. Its Taylor series converges everywhere, and consists of polynomials on $\mathfrak{h}_{\mathbb{C}}^*$ which are W-invariant. It follows that

$$\hat{\gamma}(\nu) = \sum_{j=1}^{\infty} \hat{\gamma}_{z_j}(\nu), \qquad \nu \in \mathfrak{h}_{\mathbb{C}}^*,$$

for a sequence $\{z_j\}$ of elements in $\mathfrak{Z}$. But

$$\hat{\gamma}_{z_j}(\mu_\sigma + \Lambda)\hat{f}(\sigma,\Lambda) = (z_j f)^\wedge(\sigma,\Lambda).$$

If we have a relation (2.1), we can integrate it against $(z_j f)(x)$ to obtain

$$\sum_{k=1}^m D_k(\hat{\gamma}_{z_j}(\mu_{\sigma_k} + \lambda_k)\hat{f}(\sigma_k, \Lambda_k)u_k, v_k) = 0.$$

Since a convergent Taylor series can be differentiated term by term, the relation holds also for the function (3.1). So the function does belong to $PW(G,K)$ and γ does define a multiplier.

4. EISENSTEIN INTEGRALS

In the rest of this paper we shall try to give an idea of the proof of Theorem 2. It is almost immediate that the Fourier transform maps H into $PW(G,K)$. The problem is to show that the map is surjective. This amounts to being able to construct the inverse map from $PW(G,K)$ to H.

It is convenient to work within Harish Chandra's framework of Eisenstein integrals. Let τ be a unitary two-sided representation of K on a finite dimensional Hilbert space V_τ. Theorem 2 has an equivalent formulation in this context. The Hecke algebra is replaced by the space $C_c^\infty(G,\tau)$ of smooth, compactly supported functions from G to V_τ which are τ spherical. The original Paley-Wiener space is replaced by a space $PW(G,\tau)$ of entire functions from $\mathfrak{a}_{0,\mathbb{C}}^*$ to the finite dimensional vector space

$$A_0 = C_c^\infty(M_0,\tau).$$

If $f \in C_c^\infty(G,\tau)$, let $\hat{f}$ be the function in $PW(G,\tau)$ such that

$$(\hat{f}(\Lambda), \phi) = \int_G (f(x), E_{P_0}(x,\phi, -\bar{\Lambda}))dx,$$

for any $\Lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$ and $\phi \in A_0$. Here, $E_{P_0}(\cdot,\cdot,\cdot)$ is the Eisenstein integral associated to the minimal parabolic subgroup

$$P_0 = N_0A_0M_0.$$

It is essentially a matrix coefficient of the representation $I(\sigma,\Lambda)$.

Theorem 2 is equivalent to the assertion that for any τ,

$$f \to \hat{f}$$

is a topological isomorphism from $C_c^\infty(G,\tau)$ onto $PW(G,\tau)$.

Suppose that for $j = 0, 1, \ldots, n$, Λ_j is a point in $\mathfrak{a}_{0,\mathbb{C}}^*$ and S_j is a finite dimensional subspace of the symmetric algebra on $\mathfrak{a}_{0,\mathbb{C}}^*$. Suppose that F is a function in $PW(G,\tau)$. For any j, let

$$d_{S_j} F(\Lambda_j)$$

be the vector in

$$\mathrm{Hom}(S_j, A_0)$$

whose value at any $p \in S_j$ is the derivative

$$\partial(p)F(\Lambda_j).$$

After a little thought the reader will believe the following.

LEMMA 1. *There is a function* $g \in C_c^\infty(G,\tau)$ *such that*

$$d_{S_j} F(\Lambda_j) = d_{S_j} \hat{g}(\Lambda_j)$$

for $j = 0, 1, \ldots, n$.

See Lemma III.2.1 of [1].

This lemma is actually equivalent to the analogue of the condition (2.2) for $PW(G,\tau)$. It asserts that a function F in $PW(G,\tau)$ is locally the Fourier transform of a function in $C_c^\infty(G,\tau)$. The result we are trying to prove is that F is a Fourier transform globally (i.e., for all Λ).

5. CHANGE OF CONTOUR

The function $F \in PW(G,\tau)$ will be fixed from now on. We are attempting to construct a function $f \in C_c^\infty(G,\tau)$ whose Fourier transform is F. From Harish Chandra's Plancherel formula ([6(b)]) we know that any such f can be written uniquely as a sum of τ spherical Schwartz functions, indexed by the associativity classes of cuspidal parabolic subgroups. The only one of these functions which we can write down at the moment is the one which corresponds to the minimal parabolic subgroup. It equals

$$F_{P_0}^{\vee}(x) = \int_{i\mathfrak{a}_0^*} E_{P_0}(x, \mu(\Lambda)F(\Lambda),\Lambda)d\Lambda,$$

where $\mu(\Lambda)$ is the Plancherel density. We must somehow obtain from this a function of compact support.

Let a be a point in

$$A_0^+ = \exp \mathfrak{a}_0^+,$$

the positive chamber in A_0. Then

$$E_{P_0}(a, \mu(\Lambda)F(\Lambda), \Lambda) = \sum_{s\in W_0} \Phi(a, \mu(s\Lambda)F(s\Lambda), s\Lambda), \qquad (5.1)$$

where W_0 is the restricted Weyl group, and $\Phi(a, \mu(\Lambda)F(\Lambda), \Lambda)$ is a function defined by a convergent asymptotic series whose leading term is

$$(\mu(\Lambda)F(\Lambda))(1)e^{(\Lambda-\rho)(\log a)}.$$

As a function of Λ, $\Phi(a, \mu(\Lambda)F(\Lambda), \Lambda)$ is meromorphic. Its poles can be shown to lie along hyperplanes of the form

$$\langle\beta,\Lambda\rangle = r, \qquad r \in \mathbb{R},$$

for roots β of $(\mathfrak{g},\mathfrak{a}_0)$. Only finitely many of these singular hyperplanes intersect the negative chamber $-(\mathfrak{a}_0^*)^+$ in $\mathfrak{a}_0^*$.

Thus, for fixed $a \in A_0^+$, $F_{P_0}^{\vee}(a)$ is given by the integral over $\Lambda \in i\mathfrak{a}_0^*$ of a function which is asymptotic to

$$\sum_{s\in W_0} (\mu(s\Lambda)F(s\Lambda))(1)e^{(s\Lambda-\rho)(\log a)}.$$

The proof of the classical Paley-Wiener theorem suggests that we should change the contour of integration to $X + i\mathfrak{a}_0^*$, where X is some large vector in $\mathfrak{a}_0^*$. However, there is an immediate complication. While some of the terms in the integrand will be seen to be small after such a change of the contour, other terms will only blow up. It is necessary to first change variables. Let ε be a very small vector in $-(\mathfrak{a}_0^*)^+$. Then

$$F_{P_0}^{\vee}(a) = \int_{i\mathfrak{a}_0^*} E_{P_0}(a, \mu(\Lambda)F(\Lambda), \Lambda)d\Lambda$$

$$= \int_{\varepsilon+i\mathfrak{a}_0^*} \Big(\sum_{s\in W_0} \Phi(a, \mu(s\Lambda)F(s\Lambda), s\Lambda)\Big)d\Lambda.$$

With a change of variables we then see that $F_{P_0}^{\vee}(a)$ equals

$$\sum_{s\in W_0} \int_{s\varepsilon+i\mathfrak{a}_0^*} \Phi(a, \mu(\Lambda)F(\Lambda), \Lambda)d\Lambda. \tag{5.2}$$

Now, each integrand will be asymptotic to

$$(\mu(\Lambda)F(\Lambda))(1)e^{(\Lambda-\rho)(\log a)}.$$

If each contour of integration is replaced by $X + i\mathfrak{a}_0^*$, where X is a point in the negative chamber $-(\mathfrak{a}_0^*)^+$ which is far from the walls, we might expect the result to vanish for large $a \in A_0^+$. Incidentally, ε was introduced because the summands on the right of (5.1) could have singularities which meet $i\mathfrak{a}_0^*$, even though their sum is regular on $i\mathfrak{a}_0^*$.

If X is any point in $-(\mathfrak{a}_0^*)^+$ which is far from the walls, the integrand in (5.2) is analytic on $X + i\mathfrak{a}_0^*$. Define

$$F^{\vee}(a) = |W_0| \int_{X+i\mathfrak{a}_0^*} \Phi(a, \mu(\Lambda)F(\Lambda), \Lambda), \qquad a \in A_0^+.$$

LEMMA 2. *There is a number* N *such that* $F^{\vee}(a) = 0$ *whenever* $\|\log a\| \geqslant N$.

See Theorem II.1.1 of [1].

Set $G_- = KA_0^+K$. It is an open dense subset of G. If

$$x = k_1 a k_2$$

is any point in G_-, define

$$f(x) = \tau(k_1)F^\vee(a)\tau(k_2).$$

The last lemma states that the function f has bounded support. It is our candidate for the inverse Fourier transform of F. It is not yet clear that f extends to a smooth function on G. However, we do know that $F^\vee(a)$ differs from (5.2) by a finite sum of residues. The main difficulty in the proof of Theorem 2 is to interpret these residues.

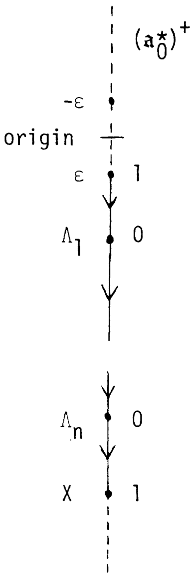
6. THE CASE OF REAL RANK 1

In order to get a feeling for what is required in general, we should recall Campoli's argument if G has real rank 1. This simply means that the integrals in (5.2) are over one dimensional spaces. The resulting residues will be evaluated at a finite number of points, $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$, in the closure of $-(\mathfrak{a}_0^*)^+$. Then $F^\vee(a)$ equals the sum of $F^\vee_{P_0}(a)$ and a function

$$F^\vee_{cusp}(a) = \sum_{j=0}^{n} \operatorname*{Res}_{\Lambda=\Lambda_j} \Phi(a,\mu(\Lambda)F(\Lambda),\Lambda), \tag{6.1}$$

on A_0^+. (Of course, it is understood that the residues are taken with respect to some isomorphism of $\mathfrak{a}_0^*$ with $\mathbb{R}$.)

Let us illustrate the process with a diagram, in which $\mathfrak{a}_0^*$ is represented by a broken vertical line.



Each large dot stands for an integral over an imaginary space, of dimension 0 or 1, which lies above the dot. With the arrows, we have shown how to move the contour over the point ε to the contour over X. The contour over $-\varepsilon$ is moved the same way, except that there might also be a contribution from a residue at the origin.

We would like to show that $\check{F}_{cusp}$ extends to a τ spherical function on G which is a sum of matrix coefficients of discrete series. Consider (6.1) as a function of F. For each j there is a finite dimensional subspace S_j of the symmetric algebra on $\mathfrak{a}^*_{0,\mathbb{C}}$ such that (6.1) depends only on the vector

$$\bigoplus_{j=0}^{n} d_{S_j} F(\Lambda_j).$$

It follows from Lemma 1 that there is a function $g \in C_c^\infty(G,\tau)$ such that (6.1) equals

$$\sum_{j=0}^{n} \operatorname*{Res}_{\Lambda=\Lambda_j} \Phi(a,\mu(\Lambda)\hat{g}(\Lambda),\Lambda).$$

Now apply what we have shown so far to the function

$$G(\Lambda) = \hat{g}(\Lambda)$$

in $PW(G,\tau)$. Then

$$\begin{aligned} G^{\vee}(a) &= G^{\vee}_{cusp}(a) + G^{\vee}_{P_0}(a) \\ &= F^{\vee}_{cusp}(a) + G^{\vee}_{P_0}(a). \end{aligned}$$

On the other hand, g is a Schwartz function, so that

$$g(x) = g_{cusp}(x) + G^{\vee}_{P_0}(x), \qquad x \in G,$$

for a uniquely determined function g_{cusp} which is a sum of matrix coefficients of discrete series. It follows that

$$F^{\vee}_{cusp}(a) - G^{\vee}(a) = g_{cusp}(a) - g(a)$$

for each $a \in A_0^+$. However, both $G^{\vee}(a)$ and $g(a)$ are of bounded support on A_0^+. This means that $F^{\vee}_{cusp}(a)$ equals $g_{cusp}(a)$ outside a bounded set. Since both functions are analytic, $F^{\vee}_{cusp}$ extends to a smooth, τ spherical function on G which is a sum of matrix coefficients of discrete series.

By its definition, $F^{\vee}_{P_0}(x)$ is a smooth, τ spherical function on G. Therefore the function

$$f(a) = F^{\vee}(a),$$

which we know equals

$$F^{\vee}_{cusp}(a) + F^{\vee}_{P_0}(a),$$

extends to a smooth, τ spherical function on G. Since it has bounded support on A_0^+, it belongs to $C_c^{\infty}(G,\tau)$. Moreover,

$$f(x) = F^{\vee}_{cusp}(x) + F^{\vee}_{P_0}(x)$$

must be the decomposition of f according to associativity classes of cuspidal parabolic subgroups. (If G is not cuspidal - that is, G has no discrete series - the function $F^{\vee}_{cusp}$ will of course be zero.) It follows without difficulty that

$$\hat{f}(\Lambda) = F(\Lambda).$$

This gives the proof of Theorem 2.

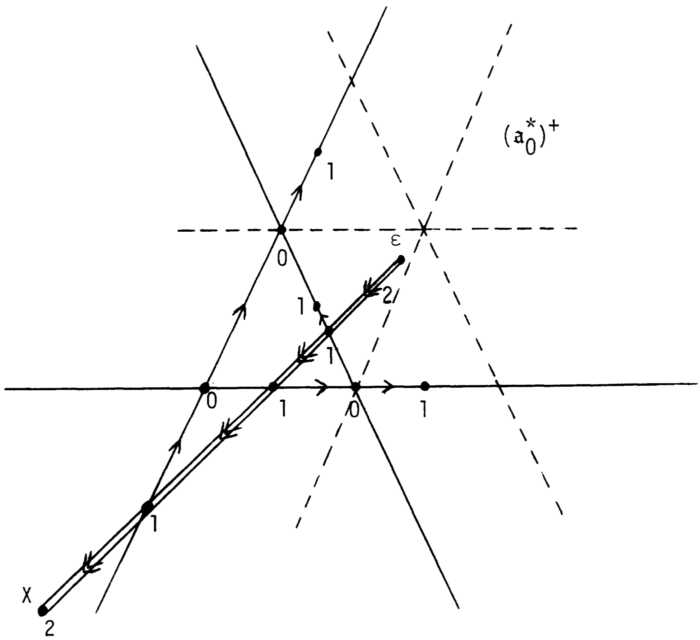
7. HIGHER RANK

If the real rank of G is greater than 1, it is considerably more difficult to interpret the residues. We will do nothing more than try to get a feeling for the main ideas by looking at the case of $SL(3, \mathbb{R})$.

For $G = SL(3, \mathbb{R})$, the space $\mathfrak{a}_0^*$ has dimension 2. For simplicity we will assume that for each positive root, the function

$$\Phi(a, \mu(\Lambda)F(\Lambda), \Lambda)$$

has exactly one associated singular hyperplane which meets the negative chamber $-(\mathfrak{a}_0^*)^+$. This leaves three singular hyperplanes to contend with, which we will represent in the diagram below by unbroken lines. The broken lines stand for the walls of the chambers in $\mathfrak{a}_0^*$. Each large dot stands for an integral over an imaginary space of dimension 0, 1 or 2, which lies above the dot.



The diagram again illustrates what happens when we move the contour of integration from $\varepsilon + i\mathfrak{a}_0^*$ to $X + i\mathfrak{a}_0^*$. As we cross each of the three singular hyperplanes, we pick up a residue consisting of an integral over a space of dimension 1. We would expect these terms to give the contribution from induced discrete series (induced, that is, from maximal parabolic subgroups). Such representations are of course tempered, and can correspond only to the points on the singular hyperplanes which are *closest* to the origin. We must therefore move the contours of the 1-dimensional integrals to 1-dimensional imaginary spaces over these points. In so doing, we pick up further residues, at points on the diagram labelled with 0. We would expect the sum of these to give the contribution from the discrete series. Since $G = SL(3, \mathbb{R})$ has no discrete series, the sum should vanish.

It is clear that there will be some bookkeeping problems for general groups. However, it is possible to handle them with an induction hypothesis. Consider the 1-dimensional residues on the *horizontal* singular hyperplanes (of which there is just one in our diagram). Such residues are eventually moved over to the vertical line which passes through the origin. This vertical line corresponds to the Levi component of a maximal parabolic subgroup. In fact, the *geometry* of the 1-dimensional residues on the horizontal singular hyperplanes will be identical to the geometry of the 0-dimensional residues for the Paley-Wiener theorem for the Levi subgroup.

We assume inductively that Theorem 2 is true for the Levi component M of any proper parabolic subgroup of G. To exploit this, however, we need something more. We require a natural procedure for lifting functions from $M\_$ to functions on $G\_$ which generalizes the Eisenstein integral. Such a procedure is provided by a theorem of a Casselman.

8. THE THEOREM OF CASSELMAN

Suppose that J is a meromorphic function from $\mathfrak{a}_{0,\mathbb{C}}^*$ to A_0 such that the function

$$\Phi(a, J(\Lambda), \Lambda)$$

is analytic at $\Lambda = \Lambda_0$. Let $D = D_\Lambda$ be any differential operator on

$\mathfrak{a}^*_{0,\mathbb{C}}$. Define

$$\Psi(x) = \tau(k_1)(D_{\Lambda_0}\Phi(a,J(\Lambda_0), \Lambda_0))\tau(k_2)$$

for any point

$$x = k_1 \; a \; k_2, \qquad a \in A_0^+, \qquad k_1, k_2 \in K,$$

in $G\_$. Then Ψ is a $\mathfrak{Z}$ finite, τ spherical function from $G\_$ to V_τ. Let $A(G\_,\tau)$ be the space spanned by all functions obtained in this way. We would expect $A(G\_,\tau)$ to be the space of *all* $\mathfrak{Z}$ finite, τ spherical functions from $G\_$ to V_τ. However, I have not thought about this question. Let $A(G,\tau)$ be the subspace of functions in $A(G\_,\tau)$ which extend to smooth functions on G. Again, we would expect $A(G,\tau)$ to be the space of all $\mathfrak{Z}$ finite, τ spherical functions from G to V_τ.

Suppose that

$$P = NAM$$

is a parabolic subgroup of G. The Levi component M is reductive, so we can define the space $A(M\_,\tau)$ as above. (It consists of functions from $M\_$ to V_τ which are spherical with respect to the restriction of τ to $K \cap M$.) If ϕ belongs to the subspace $A(M,\tau)$ of $A(M\_,\tau)$, and

$$x = nmak, \qquad n \in N, \qquad m \in M, \qquad a \in A, \qquad k \in K,$$

is any point in G, define

$$\phi_P(x) = \phi(m)\tau(k).$$

We also write, as usual,

$$H_P(x) = \log a,$$

an element in the Lie algebra $\mathfrak{a}$ of A. Then the Eisenstein integral

$$E_P(x,\phi,\lambda) = \int_K \tau(k)^{-1}\phi_P(kx)e^{(\lambda+\rho)(H_P)kx))}dk,$$

as a function of x, belongs to $A(G,\tau)$. It depends analytically on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

THEOREM 3. (Casselman) The Eisenstein integral can be extended in a natural way to a linear map from $A(M\_,\tau)$ to $A(G\_\ ,\tau)$, which depends *meromorphically* on a point $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

The theorem seems quite remarkable to me. The map certainly cannot be defined by an integral as above, for the integral in general will not converge. However, the map turns out to be just what is needed. It allows one to identify the sum of 1-dimensional residues in §7 with a wave packet of Eisenstein integrals associated to the maximal parabolic subgroups. One can then identify the 0-dimensional residues with the discrete series of G by following the argument of §6.

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