HARMONIC ANALYSIS OF THE SCHWARTZ SPACE ON A REDUCTIVE LIE GROUP I

James Arthur

INTRODUCTION

Let G be a reductive Lie group. The Plancherel theorem for G has recently been established by Harish-Chandra. According to this theorem, there is a linear isometry, $\underline{F}$, from $L^2(G)$ onto $L^2(\hat{G})$. $L^2(\hat{G})$ is a certain Hilbert space of Hilbert-Schmidt operator valued functions on $\hat{G}$.

The Schwartz space of G, $\underline{C}(G)$, is a Fréchet space of functions on G. It is a dense subspace of $L^2(G)$, and its injection into $L^2(G)$ is continuous. Motivated by the abelian case, we can ask whether it is possible to characterize the image of $\underline{C}(G)$ in $L^2(\hat{G})$ under $\underline{F}$. It turns out that there is a natural candidate, $\underline{C}(\hat{G})$, for this image space. $\underline{C}(\hat{G})$ is a Fréchet space which is defined by a family of seminorms on $L^2(\hat{G})$. It is the object of this and a subsequent paper to show that the restriction of $\underline{F}$ to $\underline{C}(G)$ defines a topological isomorphism from $\underline{C}(G)$ onto $\underline{C}(\hat{G})$.

We shall not actually define $L^2(\hat{G})$ and $\underline{C}(\hat{G})$ until the next paper. In order to make such a definition, of course, we shall have to have a collection of irreducible unitary representations of G whose complement in $\hat{G}$ has Plancherel measure zero. This collection is obtained by inducing certain representations from cuspidal parabolic subgroups of G. In §2 we shall gather various facts about the representations of G and its subgroups. In particular we will discuss the representations

$$\pi(\sigma,\lambda) \quad , \sigma \in \underline{E}_2(M), \lambda \in \underline{a}_c \ ,$$

which are induced from cuspidal parabolic subgroups

$$P = N A M$$

of G.

The most difficult part of the proof of our theorem is to show that $\underline{C}(G)$ is mapped <u>onto</u> $\underline{C}(\hat{G})$. It is to this end that most of our labour will be directed. Our first step is the weak estimate in Lemma 5.4. Our next ingredient is a careful study of the asymptotic estimates of [5(e)], §27. This is the content of §6. These estimates lead to the definition of certain meromorphic functions, the c functions, which play an important role in Harish-Chandra's proof of the Plancherel theorem. They are essential to our proof as well. We begin their study in §7.

With an eye towards future applications, we have extended some of the Kunze-Stein and Knapp-Stein theory of intertwining operators from minimal parabolic subgroups to more general cuspidal subgroups of G. Our starting point is a result of Langlands (Lemma 3.1). This result, together with Lemma 3.2, allows us to express each intertwining operator as a product of operators associated to cuspidal subgroups of rank one (Lemma 4.4). Product formulas of this nature are helpful in dealing with the rather delicate convergence questions raised in Lemma 4.1.

We conclude this paper by using the results of §4 to study the c functions. In fact, the c functions and the intertwining operators are related by a simple formula, which we establish in Lemma 8.1. Assuming the analytic continuation of the c functions for parabolic rank one, a result that must be proved on its own in any treatment, we then give simple proofs of several of the results in [5(f)]. These include the analytic continuation of the c functions, the functional equations for the c functions and Eisenstein integral, and the product formula for the scalar

$$d(s : \lambda) = c(s : -\bar{\lambda})^* \; c(s : \lambda).$$

The proof of our main theorem will be completed in the next paper. It is a generalization of the author's doctoral dissertation, submitted to Yale University in 1970, which dealt with the case of real rank one. The author is deeply indebted to Robert Langlands, who originally suggested the problem, and who has been generous with his advice and encouragement.

CONTENTS

§1. PROPERTIES OF G

We adopt the following notational conventions. If G is a Lie group, we will denote its Lie algebra by $\mathfrak{g}$, and the complexification of its Lie algebra by $\mathfrak{g}_{\mathbb{C}}$. We shall write G^{o} for the connected component of 1 in G. If $X(G)$ is the group of all continuous homomorphisms of G into $\mathbb{R}^{\times}$, we write

$$G^{1} = \bigcap_{\chi \in X(G)} \ker |\chi|$$

Finally, we write G_1 for the analytic subgroup of G whose Lie algebra is $\mathfrak{g}_1$, the derived subalgebra of $\mathfrak{g}$.

From now on, G will be a fixed Lie group such that $\mathfrak{g}$ is reductive. We make the following assumptions G:

(1.1) G^{o} is of finite index in G.

(1.2) G_1 has finite center.

(1.3) If $G_{\mathbb{C}}^{Ad}$ is the adjoint group of $\mathfrak{g}_{\mathbb{C}}$ then $Ad(G)$ is contained in $G_{\mathbb{C}}^{Ad}$.

Fix a mamimal compact subgroup K of G. Then

$$K^{o} = K \cap G^{o}$$

is both the connected component of 1 in K and a maximal compact subgroup of G^{o}. K is the normalizer of K^{o} in G. From this it follows easily that K meets every connected component of G; for if $g \in G^{+}$, an arbitrary connected component of G, $g^{-1} K^{o} g$ is another maximal compact subgroup of G^{o}, so that

$$(g^o)^{-1}\,(g^{-1}K^o g)(g^o) = (g\,g^o)^{-1}\cdot K^o\cdot(g\,g^o)$$

equals K^o for some $g^o \in G^o$. Then $g\,g^o$ belongs to $K \cap G^+$.

Let C be the center of G^o. Fix a maximal vector subgroup ${}^{(1)}A$ of C. Then $C = C_K \cdot {}^{(1)}A$, where $C_K = C \cap K$. As agreed above, $\underline{k}$, $\underline{c}_K$, and ${}^{(1)}\underline{a}$ are the Lie algebras of K, C_1 and ${}^{(1)}A$ respectively, and

$$\underline{g} = \underline{g}_1 \oplus \underline{c}_K \oplus {}^{(1)}\underline{a}\ .$$

Let Θ be the Cartan involution of $\underline{g}_1$ with respect to $\underline{k}_1 = \underline{k} \cap \underline{g}_1$. We extend Θ to an involution of $\underline{g}$ by defining

$$\Theta\,(X + Y) = X - Y\ ,\quad X \in \underline{c}_K\ ,\ Y \in {}^{(1)}\underline{a}\ .$$

Fix for once and for all a real symmetric bilinear form B on $\underline{g}$ such that

(1.4) $B(\Theta X, \Theta Y) = B(X, Y)$, $X, Y \in \underline{g}$.

(1.5) $B([X, Y], Z) + B(Y, [X, Z]) = 0$, $X, Y, Z \in \underline{g}$.

(1.6) The quadratic form

$$|X|^2 = -B(X, \Theta X)\ ,\quad X \in \underline{g}\ ,$$

is positive definite on $\underline{g}$.

It is clear that such a B exists. For example we could choose B such that $\underline{c}$ and $\underline{g}_1$ are orthogonal and such that the restriction of B to $\underline{g}_1$ is the Killing form. Our assumptions on (G, K, Θ, B) are precisely those of [5(f)], §2.

It is convenient to extend the form B to a symmetric bilinear form on $\underline{g}_c$ in the obvious way. Also, the quadratic form above extends uniquely to a Hermitian norm $|\ \ |$ on $\underline{g}_c$.

From (1.3) and (1.5) it follows that for $X , Y \in \underline{g}_{\mathbb{C}}$, and $x \in G$,

$$B(Ad(x)X , Ad(x)Y) = B(X , Y) .$$

If $k \in K$, $Ad(k)$ belongs to the analytic subgroup of $G_{\mathbb{C}}^{Ad}$ whose Lie algebra is $(\underline{k}_1)_{\mathbb{C}}$. It follows that

$$Ad(k)\,\Theta Y = \Theta\, Ad(h) Y , \; Y \in \underline{g} .$$

Therefore, we have

$$|Ad(k) Y| = |Y| , \; Y \in \underline{g} .$$

A subalgebra $\underline{p}$ of $\underline{g}$ is said to be <u>parabolic</u> if $\underline{p}_{\mathbb{C}}$ contains some maximal solvable subalgebra of $\underline{g}_{\mathbb{C}}$. A subgroup P of G is called a <u>parabolic</u> subgroup if it is the normalizer of some parabolic subalgebra $\underline{p}$ of $\underline{g}$. Then P is closed, and its Lie algebra is $\underline{p}$. It is well known that $G^{o} = (G^{o} \cap P) K^{o}$. Since K meets every connected component of G , we have $G = P \cdot K$.

Suppose that P is a parabolic subgroup of G , with Lie algebra $\underline{p}$. Let $\underline{n}$ be the nilpotent radical of $\underline{p}$, and let N be the analytic subgroup of P whose Lie algebra is $\underline{n}$. Define

$$L = P \cap \Theta(P)$$

and

$$M = L^{1} .$$

Let A be the maximal θ-stable vector subgroup of L.

Then

$$P = NAM.$$

$\underline{m}$, $\underline{n}$, and $\underline{a}$ are the Lie algebras of N, M and A respectively, and we have the direct sum decomposition

$$\underline{p} = \underline{n} + \underline{a} + \underline{m}.$$

The dimension, q, of $\underline{a}$, is called the <u>parabolic rank</u> of P. Notice that G itself is parabolic. Its parabolic rank, ${}^{(1)}q$, is the dimension of ${}^{(1)}\underline{a}$.

For any element α in the dual space of $\underline{a}$, let

$$\underline{n}(\alpha) = \{X \in \underline{n} : [H, X] = \alpha(H)\cdot X, \ H \in \underline{a}\}$$

The set, Σ, of elements α such that $\underline{n}(\alpha)$ is not empty is called the set of roots of (P, A). A root α is said to be <u>simple</u> if it cannot be written in the form $\alpha = \beta + \gamma$, for β and γ in Σ. Let Φ be the set of simple roots of (P, A). Then it is known that Φ forms a basis for the dual space of the B-orthogonal complement of ${}^{(1)}\underline{a}$ in $\underline{a}$. Furthermore, any element in Σ can be written uniquely as a nonnegative integral combination of roots in Φ. As usual we define

$$\underline{a}^+ = \{H \in \underline{a} : \alpha(H) > 0 \ \text{for} \ \alpha \in \Phi\},$$

and

$$A^+ = \exp \underline{a}^+.$$

We also define

$$\rho = \frac{1}{2} \sum_{\alpha\in\Sigma} \dim (\underline{n}(\alpha)) \cdot \alpha .$$

Sometimes it will be convenient to denote parabolic subgroups of G by P together with some left or right superscript. Then we shall index all the objects associated with P in the same way.

Let $K_M = K \cap M$, and let Θ_M and B_M be the restrictions of Θ_M and B to $\underline{m}$. Then (M, K_M, Θ_M, B_M) satisfies all the assumptions we made on (G, K, Θ, B).

The map

$$N \times A \times M \times K \longrightarrow G$$

given by

$$(n, a, m, k) \longrightarrow n\, a\, m\, k$$

is a surjective diffeomorphism. If $x \in G$, there are elements x_N, x_A, x_M and x_K in N, A, M and K, respectively, such that

$$x = x_N\, x_A\, x_M\, x_K .$$

x_N and x_A are uniquely determined, while x_M and x_K are uniquely determined modulo K_M. As usual, we define $H(x)$ to be the element in $\underline{a}$ such that

$$\exp H(x) = x_A .$$

At this point, we shall agree upon normalizations for the various Haar measures which arise.

Suppose that $P = N A M$, as above. Let $d X$ be the Euclidean measure on $\underline{n}$ associated with the restriction of the Euclidean norm $| \ |$ to $\underline{n}$. We normalize the Haar measure $d n$ on N by

$$\int_N \varphi(n)\, d n = \int_{\underline{n}} \varphi(\exp X)\, d X, \quad \varphi \in C_c^\infty(N).$$

More generally, for any subspace $\underline{q}$ of $\underline{n}$, we define a measure on $\exp \underline{q}$ in terms of the Euclidean measure on $\underline{q}$. Similarly, we choose the Haar measure $d a$ on A to be the measure which corresponds under the exponential mapping to the Euclidean measure on $\underline{a}$.

Let $d k$ be the Haar measure on K for which the volume of K is one. Suppose for the moment that P is a minimal parabolic subgroup of G. Then M is contained in K, so that

$$G = N A K.$$

Let $d x$ be the Haar measure on G such that for any $\varphi \in C_c^\infty(G)$,

$$\int_G \varphi(x)\, d x = \int_N \int_A \int_K \varphi(a n h)\, d k\, d a\, d n.$$

Since any two minimal parabolic subgroups of G are conjugate under K, $d x$ is independent of P.

Suppose once again that P is arbitrary. If, in the above discussion, we replace G by M, we obtain a normalized Haar measure $d m$ on M. It is evident that for any $\varphi \in C_c^\infty(G)$,

$$\int_G \varphi(x)\, d x = \int_N \int_A \int_M \int_K \varphi(a m n k)\, d k\, d m\, d a\, d n,$$

and

$$\int_G \varphi(x)\, d x = \int_N \int_A \int_M \int_K \varphi(n a m k)\, e^{-2\rho(H(a))}\, d k\, d m\, d a\, d n.$$

We shall normalize all our Haar measures according to these conventions without further comment.

Let us fix, for once and for all, a minimal parabolic subgroup

$$ {}^{(o)}P = {}^{(o)}N\ {}^{(o)}A\ {}^{(o)}M $$

with Lie algebra

$$ {}^{(o)}\underline{p} = {}^{(o)}\underline{a} + {}^{(o)}\underline{n} + {}^{(o)}\underline{m} . $$

Let ${}^{(o)}\widehat{M}$ be the normalizer of ${}^{(o)}A$ in K. ${}^{(o)}M$ is a normal subgroup of ${}^{(o)}\widehat{M}$. The quotient group,

$$ \Omega = {}^{(o)}\widehat{M}/{}^{(0)}M , $$

is called the restricted Weyl group of G. Ω operates on ${}^{(o)}\underline{a}_{c}$ in the obvious way. It preserves the bilinear form B.

Fix a Cartan subalgebra ${}^{(o)}\underline{b}$ of ${}^{(o)}\underline{m}$. Then

$$ \underline{h} = {}^{(o)}\underline{b} + {}^{(o)}\underline{a} $$

is a Cartan subalgebra of $\underline{g}$. Choose any ordering on the dual of ${}^{(o)}\underline{a}$ such that for each root of $({}^{(o)}P, {}^{(o)}A)$ is positive. Fix a compatible ordering on the dual space of $\underline{h}$. Denote the set of positive roots of $(\underline{g}, \underline{h})$, with respect to this ordering, by Δ. The restriction of the bilinear form B to $\underline{h}_{\mathbb{C}}$ is nondegenerate. For convenience we shall denote this restricted form by $\langle\ ,\ \rangle$. We use $\langle\ ,\ \rangle$ to identify $\underline{h}_{\mathbb{C}}$ with its dual space. In particular, we regard Δ as a subset of $\underline{h}_{\mathbb{C}}$.

Suppose that $\underline{a}$ is any subspace of ${}^{(o)}\underline{a}$. We extend any linear functional on $\underline{a}_{\mathbb{C}}$ to a linear functional on $\underline{h}_{\mathbb{C}}$ by making it vanish on the orthogonal complement of $\underline{a}_{\mathbb{C}}$ in $\underline{h}_{\mathbb{C}}$. We can then identify this functional with a vector in $\underline{h}_{\mathbb{C}}$. In particular, if $P = NAM$ is a parabolic subgroup of G such that $\underline{a}$ is contained in ${}^{(o)}\underline{a}$, we will regard Φ and Σ as subsets of $\underline{h}_{\mathbb{C}}$.

We introduce a convention of indexing the collection of subsets

$$\{{}^{(o)}_{(u)}\Phi : u \in \underline{J}\}$$

of ${}^{(o)}\Phi$ by a partially ordered set $\underline{J}$. We denote the greatest and least elements in $\underline{J}$ by 1 and 0 respectively. Then ${}^{(o)}_{(1)}\Phi = {}^{(o)}\Phi$, while ${}^{(o)}_{(o)}\Phi$ is the empty set. For any $u \in \underline{J}$, let ${}^{(u)}\underline{a}$ be the set of points $H \in {}^{(o)}\underline{a}$ such that $\langle\alpha, H\rangle = 0$ for each root α in ${}^{(o)}_{(u)}\Phi$. Define ${}^{(u)}A = \exp {}^{(u)}\underline{a}$. Let ${}^{(u)}L$ be the centralizer of ${}^{(u)}A$ in G, and write ${}^{(u)}M = ({}^{(u)}L)^1$. Denote the set of elements in ${}^{(o)}\Sigma$ which are orthogonal to ${}^{(u)}\underline{a}$ by ${}^{(o)}_{(u)}\Sigma$. Define

$${}^{(u)}\underline{n} = \bigoplus_{\alpha \in {}^{(o)}\Sigma - {}^{(o)}_{(u)}\Sigma} {}^{(o)}\underline{n}(\alpha)$$

and

$${}^{(u)}N = \exp {}^{(u)}\underline{n}.$$

Then

$$ {}^{(u)}P = {}^{(u)}N \cdot {}^{(u)}L $$

is called a standard parabolic subgroup of G. Let ${}^{(u)}\Phi$ and ${}^{(u)}\Sigma$ be the sets of distinct nonzero vectors in ${}^{(u)}\mathfrak{a}$ obtained by taking the orthogonal projections onto ${}^{(u)}\mathfrak{a}$ of the roots in ${}^{(o)}\Phi$ and ${}^{(o)}\Sigma$ respectively. Then ${}^{(u)}\Sigma$ is the set roots of $({}^{(u)}P, {}^{(u)}A)$, while ${}^{(u)}\Phi$ is the set of simple roots.

It is known that any parabolic subgroup of G is conjugate to one and only one standard parabolic subgroup. Suppose that ${}^{(u)}P$, $u \in \underline{J}$, is a standard parabolic subgroup of G. Then

$$ {}^{(o)}_{(u)}P = {}^{(u)}M \cap {}^{(o)}P $$

will be our fixed minimal parabolic subgroup of ${}^{(u)}M$. The elements $v \in \underline{J}$ such that $v \leq u$ index the standard parabolic subgroups

$$ {}^{(v)}_{(u)}P = {}^{(u)}M \cap {}^{(v)}P $$

of ${}^{(u)}M$. Any object associated to the group ${}^{(v)}_{(u)}P$ in the manner that objects have been associated to $P = {}^{(u)}P$ will be denoted by the left superscript (v) and the left subscript (u). For example ${}^{(v)}_{(u)}N$ is the nilpotent radical of ${}^{(v)}_{(u)}P$, while ${}^{(v)}_{(u)}H$ is a map from

$$ {}^{(v)}_{(u)}M = {}^{(u)}M $$

to ${}^{(v)}_{(u)}\mathfrak{a}$. Notice that

$$ {}^{(v)}N = {}^{(u)}N \cdot {}^{(v)}_{(u)}N , $$

and that ${}^{(v)}_{(u)}\mathfrak{a}$ is the orthogonal complement of ${}^{(u)}\mathfrak{a}$ in ${}^{(v)}\mathfrak{a}$.

We need to recall Harish-Chandra's results on the center of a universal enveloping algebra. Let $\underline{G}$ be the universal enveloping algebra of $\underline{g}_{\mathbb{C}}$, and let $\underline{Z}$ be the center of $\underline{G}$. We can identify the universal enveloping algebra of $\underline{h}_{\mathbb{C}}$ with

$$S = S(\underline{h}_{\mathbb{C}}) ,$$

the symmetric algebra on $\underline{h}_{\mathbb{C}}$. We therefore can identify S with a subalgebra of $\underline{G}$. Our bilinear form $\langle\ ,\ \rangle$ can be extended to a bilinear map from $S \times S$ to $\mathbb{C}$. For each $\alpha \in \Delta$, the set of positive roots of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$, fix root vectors X_α and $X_{-\alpha}$ for α and $-\alpha$ so that $B(X_\alpha, X_{-\alpha}) = 1$. Then to any $Z \in \underline{Z}$ there is associated a unique element $\gamma'(Z)$ in S such that $Z - \gamma'(Z)$ belongs to

$$\Sigma_{\alpha \in \Delta} \underline{G} X_\alpha .$$

If $\delta = \frac{1}{2} \Sigma_{\alpha\in\Delta} \alpha$, the map

$$H \longrightarrow H + \langle H, \delta\rangle I , \quad H \in \underline{h} ,$$

extends to an isomorphism β of S. Define

$$\gamma(Z) = \beta^{-1}(\gamma'(Z)) , \quad Z \in \underline{Z} .$$

Then the map

$$Z \longrightarrow \gamma(Z) , \quad Z \in \underline{Z} ,$$

is an isomorphism from $\underline{Z}$ onto J, the subalgebra of elements in S which are invariant under the Weyl group of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$. For a proof of these facts, see [5(b)], Lemmas 18 and 19.

Similarly, there is an isomorphism

$$\gamma_L : \underline{Z}_L \longrightarrow J_L$$

from the center of the universal enveloping algebra of $\underline{m}_{\mathbb{C}} + \underline{a}_{\mathbb{C}}$ onto the subalgebra of elements in S which are invariant under the Weyl group of $(\underline{m}_{\mathbb{C}} + \underline{a}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$. Since J is a subalgebra of J_L, we can define

$$\mu = \gamma_L^{-1} \circ \gamma ,$$

an injective homomorphism from $\underline{Z}$ into $\underline{Z}_L$. The map

$$Y_M + Y_A \longrightarrow Y_M + Y_A + \langle Y_A, \rho \rangle I , \quad Y_M \in \underline{m} , \ Y_A \in \underline{a} ,$$

extends to an isomorphism ϵ of the universal enveloping algebra of $\underline{m}_{\mathbb{C}} + \underline{a}_{\mathbb{C}}$, which preserves $\underline{Z}_L$. Given any $Z \in \underline{Z}$, $\epsilon(\mu(Z))$ may be characterized as the unique element in $\underline{Z}_L$ such that $Z - \epsilon(\mu(Z))$ belongs to $\underline{G}\ \underline{n}_{\mathbb{C}}$, ([5(d)], Lemma 13 and its corollary). In [5(e)], §45, it is shown that $Z - \epsilon(\mu(Z))$ actually belongs to $\Theta(\underline{n})_{\mathbb{C}} \cdot G \cdot \underline{n}_{\mathbb{C}}$.

It is known that the maps γ, γ_L and μ are independent of our choice of ordering on the dual spaces of $\underline{h}$ and $\underline{a}$. In particular we could replace P by its opposite parabolic subgroup, $\exp(\Theta(\underline{n}))\cdot A M$, so long as we altered the compatible ordering on the dual of $\underline{g}_{\mathbb{C}}$ accordingly. In the above discussion the roles of $\Theta(\underline{n})$ and $\underline{n}$ would be interchanged, and ϵ would have to be replaced by ϵ^{-1}. From this remark it follows that for any $Z \in \underline{Z}$ $\epsilon^{-1}(\mu(Z))$ is the unique element in $\underline{Z}_L$ such that

$$Z - \epsilon^{-1}(\mu(Z))$$

belongs to $\underline{n}_{\mathbb{C}} \cdot \underline{G} \cdot \Theta(\underline{n})_{\mathbb{C}}$.

It will be necessary to make use of one particular element in $\underline{Z}$. Let $\{H_1, \ldots, H_n\}$ be an orthonormal basis of $\underline{h}_c$. Define

$$Z_G = H_1^2 + \ldots + H_n^2 + \sum_{\alpha \in \Delta} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) + \langle \delta , \delta \rangle I .$$

It is well known that Z_G lies in the center of $\underline{G}$. We have normalized X_α and $X_{-\alpha}$ above in such a way that

$$[X_\alpha , X_{-\alpha}] = \alpha .$$

It follows that

$$Z_G = H_1^2 + \ldots + H_n^2 + 2\delta + 2\sum_{\alpha > 0} X_{-\alpha} X + \langle \delta , \delta \rangle I ,$$

so that

$$\gamma' (Z_G) = H_1^2 + \ldots + H_n^2 + 2\delta + \langle \delta , \delta \rangle I .$$

Therefore,

$$(1.7) \qquad \gamma (Z_G) = H_1^2 + \ldots + H_n^2$$

Let Θ_K and B_K be the restrictions of Θ and B to $\underline{k}$. Then (K, K, Θ_K , B_K) satisfies all the assumptions we made on (G, K, Θ, B). Fix, for once and for all, a Cartan subalgebra ${}^{(1)}\underline{b}$ of $\underline{k}$ which, we may assume, contains ${}^{(o)}\underline{b}$. In exactly the same way as above, we can define the element Z_K in the center of the universal enveloping algebra of $\underline{k}_c$.

We shall conclude this section by recalling Harish-Chandra's definition of the Schwartz space of G. $\underline{G}$ can be identified with the algebra of left invariant differential operator on G . There is a canonical anti-isomorphism

$$Y \longrightarrow Y^R \quad , \quad Y \in \underline{G} \; ,$$

from $\underline{G}$ onto the algebra of right invariant differential operators on G. For $Y_1, Y_2 \in \underline{G}$, the differential operators $Y_1{}^R$ and Y_2 commute. For $f \in C^\infty(G)$, the value of $Y_1{}^R \cdot Y_2 \cdot f$ at any $x \in G$ is denoted by

$$f(Y_1 ; x ; Y_2)$$

Define two positive functions Ξ and σ on G by

$$\Xi(x) = \int_K e^{\langle {}^{(o)}\rho , {}^{(o)}H(kx) \rangle} \, dk \, ,$$

and

$$\sigma(k_1 \cdot \exp H \cdot k_2) = |H| \, , \qquad k_1, k_2 \in K \, , \quad H \in {}^{(o)}\underline{a} \, .$$

σ is well defined on G since $G = K \cdot {}^{(o)}A \cdot K$, and

$$|Ad(k) \cdot H| = |H| \, , \quad k \in K \, , \; H \in {}^{(o)}\underline{a} \, .$$

For each pair of elements Y_1 and Y_2 in $\underline{G}$, and every real number m, define a semi-norm on $C^\infty(G)$ by

$$||f||_{Y_1, Y_2, m} = \sup_{x \in G} \{ |f(Y_1 ; x ; Y_2)| \, \Xi(x)^{-1} (1 + \sigma(x))^m \} \, ,$$

$f \in C^\infty(G)$.

Then $\underline{C}(G)$, the Schwartz space of G, is defined to be the set of all $f \in C^\infty(G)$ such that for every (Y_1, Y_2, m),

$$||f||_{Y_1, Y_2, m} < \infty \, .$$

$\underline{C}(G)$, together with this collection of semi-norms, is a topological vector space. It is known that $C_c^\infty(G)$ is dense in $\underline{C}(G)$, that $\underline{C}(G)$ is dense in $L^2(G)$, and that the inclusions

$$C_c^\infty(G) \subset \underline{C}(G) \subset L^2(G)$$

are continuous.

§2. INDUCED REPRESENTATIONS

If H is any reductive Lie group, we shall denote the collection of irreducible unitary representations of H by $E(H)$. We shall write $E_2(H)$ for the subcollection of square integrable representations. The relation of unitary equivalence partitions these collections into equivalence classes, which we denote by $\underline{E}(H)$ and $\underline{E}_2(H)$ respectively. If $\omega \in \underline{E}_2(H)$, and $\sigma \in E(H)$ is a representation in the class of ω, which acts on the Hilbert space V, let

$$d_\sigma = d_\omega$$

be the formal degree of σ. Recall that this is a positive number, depending on a choice of Haar measure on H, such that the Schur orthogonality relations hold. Namely, for vectors ϕ_1, ϕ_2, ψ_1 and ψ_2 in V.

$$\int (\sigma(x)\phi_1, \phi_2)\overline{(\sigma(x)\psi_1, \psi_2)}\, dx = d_\sigma^{-1}(\phi_1, \psi_1)\overline{(\phi_2, \psi_2)} .$$

A parabolic subgroup $P = NAM$ of G is said to be <u>cuspidal</u> if $\underline{E}_2(M)$ is not empty. From [5(e)], we know that this is the case if and only if $\underline{m}$ has a Cartan subalgebra which is contained in $\underline{k} \cap \underline{m}$. We shall begin this section with a brief discussion of $\underline{E}_2(M)$. For more details, we refer the reader to [12]. (See also [7(b)].)

Suppose that $P = NAM$ is a standard cuspidal subgroup of G. Suppose $\underline{b}_M$ is a Cartan subalgebra of $\underline{m}$ which is contained in $\underline{k}$. We may as well assume that $\underline{b}_M$ contains ${}^{(o)}\underline{b}$. Let d_M be one half the sum of the positive roots of $(m_{\mathbb{C}}, \underline{b}_{M,\mathbb{C}})$ with respect to some fixed ordering of the dual space of $\underline{b}_{M,\mathbb{C}}$. We define $\Lambda(\underline{b}_M, M)$ to be the set of real linear functionals ν on $i\,\underline{b}_M$ such that the function

$$H \longrightarrow e^{(\nu - d_M)(H)}, \quad H \in \underline{b}_M,$$

lifts to a function on $B_M^o = \exp \underline{b}_M$. For $s \in W(\underline{b}_M, \underline{m})$, the Weyl group of $(\underline{m}_{\mathbb{C}}, \underline{b}_{M,\mathbb{C}})$, it is known that $s\, d_M - d_M$ is an integral linear combination of the roots of $(\underline{m}_{\mathbb{C}}, \underline{b}_{M,\mathbb{C}})$. In particular, the function

$$H \longrightarrow e^{(s\, d_M - d_M)(H)}, \quad H \in \underline{b}_M,$$

lifts to a function on B_M^o. It follows that $\Lambda(\underline{b}_M, M)$ is invariant under $W(\underline{b}_M, \underline{m})$.

Let $\Lambda'(\underline{b}_M, M)$ be the set of regular elements in $\Lambda(\underline{b}_M, M)$. If M^+ is any subgroup of M which contains M^o, let $W(\underline{b}_M, M^+)$ be the normalizer of $\underline{b}_M$ in M^+, modulo the centralizer of $\underline{b}_M$ in M^+. Then $W(\underline{b}_M, M^+)$ is a subgroup of $W(\underline{b}_M, \underline{m})$. There is a one-to-one correspondence between $\underline{E}_2(M^o)$ and the set of orbits of $W(\underline{b}_M, M^o)$ in $\Lambda'(\underline{b}_M, M)$. Suppose that ν is an element in a given orbit and that σ is the corresponding representation. Then the value of the character Θ_σ of σ on B_M^o is given by

$$\Theta_\sigma(\exp H) = \gamma(M^o)\cdot \varepsilon(\nu) \cdot \prod_{\alpha \in P_M} (1-e^{-\alpha(H)})^{-1} \cdot \sum_{s \in W(\underline{b}_M, M^o)} \varepsilon(s)\cdot e^{(s\nu - d_M)(H)}$$

for $H \in \underline{b}_M$. Here P_M is the set of positive roots of $(\underline{m}_{\mathbb{C}}, \underline{b}_{M,\mathbb{C}})$, $\varepsilon(s)$ is the determinant of s, $\varepsilon(\lambda) = \pm 1$, and $\gamma(M^o)$ is a constant which depends only on M^o.

These facts follow from [5(e)], Theorem 16. Harish-Chandra's results actually apply only to a group which is both semisimple and acceptable. However, by considering finite covers of M^o one can successively remove these two restrictions.

Let $\bar{M}$ be the subgroup of elements m in M such that the map

$$x \longrightarrow m^{-1} x\, m, \quad x \in M^o ,$$

is an inner automorphism on M^o. Then $\bar{M}$ is a normal subgroup of M which contains M^o. Let $C(M^o)$ be the center of M^o, and let $C(\bar{M})$ be the centralizer of M^o in $\bar{M}$. Then $C(\bar{M})$ is compact and

$$C(M^o) \setminus C(\bar{M}) = M^o \setminus \bar{M} .$$

In addition we have

$$\bar{M} = C(\bar{M})\, M^o ,$$

and

$$C(\bar{M}) \cap M^o = C(M^o) .$$

Suppose $\chi \in E\,(C(\bar{M}))$ and $\sigma \in E_2(M^o)$. We shall say that χ and σ are <u>compatible</u> if the character

$$c \longrightarrow \chi(c) \cdot \sigma(c), \quad c \in C(M^o),$$

of the compact abelian group $C(M^o)$ is trivial. To every compatible pair

$$(\chi,\sigma), \quad \chi \in E(C(\bar{M})), \ \sigma \in E_2(M^o),$$

there is an obvious representation in $E_2(\bar{M})$. Every representation in $E_2(M)$ can be obtained this way. If σ is associated to $\nu \in \Lambda'(\underline{b}_M, M)$ as above, we shall denote the corresponding representation in $E_2(\bar{M})$ by $\bar{\sigma}_{\chi,\nu}$. Then for $c \in C(\bar{M})$ and $H \in \underline{b}_M$, the value of the character of $\bar{\sigma}_{\chi,\nu}$ at $c \cdot \exp H$ equals

$$(2.1)\quad \mathrm{Tr}\,(\chi(c)) \cdot \gamma(M^{o}) \cdot \varepsilon\,(\nu) \cdot \prod_{\alpha\in P_M} (1 - e^{-\alpha(H)})^{-1} \sum_{s\in W(\underline{b}_M, M^{o})} \varepsilon(s)\, e^{(s\nu - d_m)(H)} .$$

Suppose that $C(B_M^o)$ is the centralizer of B_M^o in M. Let M^{Ad} and $M_{\mathbb{C}}^{Ad}$ be the adjoint groups of $\underline{m}$ and $\underline{m}_{\mathbb{C}}$ respectively, and let B_M^{Ad} be the image of B_M^o in M^{Ad}. Let $B_{M,\mathbb{C}}^{Ad}$ be the analytic subgroup of $M_{\mathbb{C}}^{Ad}$ whose Lie algebra is the complexification of the Lie algebra of B_M^{Ad}. If c is any element in $C(B_M^o)$, $Ad\,(c)$ belongs to the centralizer of B_M^{Ad} in $M_{\mathbb{C}}^{Ad}$, which is just $B_{M,\mathbb{C}}^{Ad}$. However, there is an n such that c^n belongs to M^o. In particular, $Ad(c)^n$ belongs to B_M^{Ad}. This can only happen if $Ad(c)$ itself belongs to B_M^{Ad}. We have shown that $C(B_M^o)$ is contained in $\bar{M}$.

Let $N(B_M^o)$ and $\bar{N}(B_M^o)$ be the normalizers of B_M^o in M and $\bar{M}$ respectively. There is a natural map ι from $W(\underline{b}_M, M) = C(B_M^o)\backslash N(B_M^o)$ into $\bar{M}\backslash M$. Since any two compact Cartan subgroups of M^o are M^o- conjugate, $N(B_M^o)$ meets every connected component of M. It follows that ι is surjective. The kernal of ι is $C(B_M^o)\backslash \bar{N}(B_M^o)$, a group which is isomorphic to $W(\underline{b}_M, M^o)$.

Since M normalizes $\bar{M}$, there is an action of the group

$$W(\underline{b}_M, M^o) \backslash W(\underline{b}_M, M) = \bar{M} \backslash M$$

on $\underline{E}_2(\bar{M})$. From the formula (2.1), we see that this action is fixed point free. Let $\sigma_{\chi,\nu}$ be the representation of M obtained by inducing $\bar{\sigma}_{\chi,\nu}$ from $\bar{M}$ to M. Then by the general theory of induced representations, $\sigma_{\chi,\nu}$ is irreducible, and σ_{χ_1,ν_1} is equivalent to σ_{χ_2,ν_2} if and only if the equivalence classes of $\bar{\sigma}_{\chi_1,\nu_1}$ and $\bar{\sigma}_{\chi_2,\nu_2}$ belong to the same $W(\underline{b}_M, M^0)\backslash W(\underline{b}_M, M)$ orbit. $\sigma_{\chi,\nu}$ belongs to $E_2(M)$, and any representation in $E_2(M)$ is obtained this way.

Let $\underline{h}_M$ be the orthogonal complement of $\underline{a}$ in $\underline{h}$. Then $\underline{h}_M$ is a Cartan subalgebra of $\underline{m}$. It will be convenient to index our representations by elements in $\underline{h}_{M,\mathbb{C}}$ rather than by linear functionals on $i\,\underline{b}_M$. Accordingly, we fix for once and for all an element $y = y_M$ in the centralizer of $\underline{h}_M \cap \underline{b}_M$ in $M_{\mathbb{C}}^{Ad}$ such that

$$Ad(y) \cdot \underline{b}_{M,\mathbb{C}} = \underline{h}_{M,\mathbb{C}} .$$

If ν is in the dual space of $\underline{b}_{M,\mathbb{C}}$, we define an element ν^y in $\underline{h}_{\mathbb{C}} = \underline{b}_{M,\mathbb{C}} \oplus \underline{a}_{\mathbb{C}}$ by

$\langle\nu^y, H_1 + H_2\rangle = \nu(Ad(y^{-1}) H_1)$, $H_1 \in \underline{h}_{M,\mathbb{C}}$, $H_2 \in \underline{a}_{\mathbb{C}}$. Any element in $\Lambda(\underline{b}_M, M)$ can be regarded as a linear functional on $\underline{b}_{M,\mathbb{C}}$. We define $\Lambda(M)$ and $\Lambda'(M)$ to be the set of points ν in $\underline{h}_{\mathbb{C}}$ such that ν^y belongs to $\Lambda(\underline{b}_M, M)$ and $\Lambda'(\underline{b}_M, M)$ respectively.

<u>LEMMA 2.1</u>: If β is a root of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$, the set of numbers

$$\{\langle\mu, \beta\rangle : \mu \in \Lambda'(M)\}$$

generates a lattice in $\mathbb{R}$.

PROOF: Define the real vector spaces

$$\underline{u} = \mathrm{Ad}(y)\cdot i\,\underline{b}_M ,$$
$$\underline{v}_1 = (\underline{u} + \underline{a}) \cap \underline{g}_{1,\mathbb{C}} ,$$

and

$$\underline{u}_1 = \underline{u} \cap \underline{v}_1 = \underline{u} \cap \underline{g}_{1,\mathbb{C}} ,$$

where, as we recall, $\underline{g}_1$ is the derived subalgebra of $\underline{g}$. Then $\underline{v}_1$ is the real linear span in $\underline{h}_{\mathbb{C}}$ of the roots of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$. Let $L(M)$ be the set of points ν^y, where ν is a real linear functional on $i\,\underline{b}_M$ such that the function

$$H \longrightarrow e^{2\nu(H)} , \quad H \in \underline{b}_M ,$$

lifts to a character on B_M^o. Then $L(M)$ is a lattice in $\underline{u}$. It is evident that $(d_M)^y$ belongs to $L(M)$, so that $L(M)$ contains $\Lambda'(M)$. Define $L_1(M)$ to be the projection of $L(M)$ onto $\underline{u}_1$. It is a lattice in $\underline{u}_1$.

Let $L_1^-(G)$ be the lattice in $\underline{v}_1$ generated by the roots of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$ and define

$$L_1^-(M) = L_1^-(G) \cap \underline{u} = L_1^-(G) \cap \underline{u}_1 .$$

Suppose that γ is a root of $(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$. Let ν be the unique linear functional on $i\,\underline{b}_M$ such that ν^y is the orthogonal projection of γ onto $\underline{u}_1$. ν is the restriction to $i\,\underline{b}_M$ of a root of $(\underline{g}_{\mathbb{C}}, \underline{b}_{M,\mathbb{C}} + \underline{a}_{\mathbb{C}})$. It follows that the function

$$H \longrightarrow e^{2\nu(H)}, \quad H \in \underline{b}_M,$$

extends to a character on B_M^o. In particular, ν^y belongs to $L(M) \cap \underline{u}_1$, and hence to $L_1(M)$. We have shown that $L_1^-(M)$ is contained in $L_1(M)$.

Now $Ad(y) \circ \Theta \circ Ad(y^{-1})$ is an automorphism of $\underline{g}_{\mathbb{C}}$. It sends the above root γ onto the vector $2\nu^y - \gamma$ which, as a result, is also a root of $(\underline{g}_{\mathbb{C}}, \underline{h}_c)$. Consequently, $2\nu^y$ belongs to $L_1^-(M)$. From this fact it follows that

$$L_1^-(M) \otimes \mathbb{R} = \underline{u}_1 .$$

In other words, $L_1^-(M)$ is a lattice in $\underline{u}_1$. It is therefore of finite index in $L_1(M)$.

The bilinear form B is G-invariant, so its restriction to any simple factor of $\underline{g}_1$ must be a multiple of the Killing form. It follows that if β and γ are roots of $(\underline{g}_c, \underline{h}_c)$, $\frac{2\langle\beta,\gamma\rangle}{\langle\beta,\beta\rangle}$ is an integer. Since $L_1^-(M)$ is of finite index in $L_1(M)$, the set

$$\{\langle\mu_1, \beta\rangle : \mu_1 \in L_1(M)\}$$

is a lattice in $\mathbb{R}$. On the other hand, if $\mu \in L(M)$, and μ_1 is the projection of μ onto $\underline{u}_1$,

$$\langle\mu, \beta\rangle = \langle\mu_1, \beta\rangle.$$

The lemma follows. $\square$

We shall require another simple lemma of this type. The restriction of the form $<\ ,\ >$ to ${}^{(o)}\underline{a}$ is positive definite. Recall that ${}^{(o)}\underline{\Phi}$ is a basis of ${}^{(o)}_{(1)}\underline{a}$, the orthogonal complement of ${}^{(1)}\underline{a}$ in ${}^{(o)}\underline{a}$. Let

$$ {}^{(o)}\hat{\underline{\Phi}} = \{\hat{\alpha} : \alpha \in {}^{(o)}\underline{\Phi}\} $$

be the basis of ${}^{(o)}_{(1)}\underline{a}$ which is dual to ${}^{(o)}\underline{\Phi}$ with respect to $<\ ,\ >$.

Suppose that $u \in \underline{J}$. Then

$$ {}^{(u)}\hat{\underline{\Phi}} = \{\hat{\alpha} : \alpha \in {}^{(o)}\underline{\Phi} - {}^{(o)}_{(u)}\underline{\Phi}\} $$

is the basis for ${}^{(u)}_{(1)}\underline{a}$ dual to ${}^{(u)}\underline{\Phi}$. Define

$$ {}^{(u)}\underline{H}_o' = \sum_{\hat{\alpha} \in {}^{(u)}\hat{\underline{\Phi}}} \hat{\alpha} , $$

and

$$ H_o = {}^{(u)}\underline{H}_o = {}^{(u)}H_o' <{}^{(u)}\underline{H}_o' , {}^{(u)}\underline{H}_o' >^{-\frac{1}{2}} . $$

Then H_o is a unit vector in ${}^{(u)}\underline{a}^+$.

<u>LEMMA 2.2</u> : Let s belong to the Weyl group of $(\underline{g}_c , \underline{h}_c)$. Then the set of numbers

$$ \{<\mu , s H_o> ; \mu \in \Lambda'(M)\} $$

generates a lattice in $\mathbb{R}$.

PROOF: Define $L_1(M)$ and $L_1^-(M)$ as in the proof of the last lemma. If $\mu_1^- \in L_1^-(M)$, $s^{-1}\mu_1^-$ is an integral linear combination of roots of $(\underline{g}_c , \underline{h}_c)$. The projection of $s\mu_1^-$ onto ${}^{(o)}_{(1)}\underline{a}$ is an integral combination of elements in ${}^{(o)}\Phi$. It follows that

$$\langle \mu_1^- , s\, {}^{(u)}H_o' \rangle = \langle s^{-1} \mu_1^- , {}^{(u)}H_o' \rangle$$

is an integer. The assertion of the lemma becomes a consequence of the definition of $L_1(M)$ and the fact that $L_1^-(M)$ is of finite index in $L_1(M)$.

□

The Weyl group W_M of $(\underline{m}_{\mathbb{C}}, \underline{h}_{M,\mathbb{C}})$ acts on $\underline{h}_{\mathbb{C}}$, and preserves $\Lambda'(M)$.

If μ is in $\Lambda'(M)$, we shall write $\{\mu\}$ for the orbit of μ in $\Lambda'(M)$ under the action of the above group. Suppose that $\omega \in \underline{E}_2(M)$, that σ is a representation in the class ω, and that

$$\sigma = \sigma_{\chi,\nu}, \quad \nu \in \Lambda'(\underline{b}, M).$$

Then if μ is any element in the orbit $\{\nu^y\}$, we shall write

$$\mu = \mu(\sigma) = \mu(\omega).$$

We define the <u>absolute</u> <u>value</u>

$$|\omega| = |\sigma|$$

of this representation to be equal to $|\mu|$. This depends only on the orbit of μ in $\{\nu^y\}$.

Suppose that we are given the above representation σ together with an element $\lambda \in \underline{a}_{\mathbb{C}}$. Define a representation $\sigma \otimes \xi_\lambda$ of $L = MA$ by

$$(\sigma \otimes \xi_\lambda)(m a) = e^{\langle \lambda, H(a) \rangle} \sigma(m), \quad m \in M, \ a \in A.$$

Then it follows from [5(e)] and our discussion above that for any $\mu \in \Lambda'(M)$ such that $\mu = \mu(\sigma)$, and any $V \in \underline{Z}_L$,

(2.2) $(\sigma \otimes \xi_\lambda)(V) = \langle \gamma_L(V), \mu + \lambda \rangle I$.

For any $\omega \in \underline{E}_2(M)$ define $\underline{C}_\omega(M)$ to be the closed subspace of $\underline{C}(M)$, the Schwartz space on M , generated by functions of the form

$$m \longrightarrow (\sigma(m)\phi_1 , \phi_2) , m \in M ,$$

where σ is a representation in the class ω , and ϕ_1 and ϕ_2 are K_M - finite vectors in the space on which σ acts. For any orbit $\{\mu\}$ of W_M in $\Lambda'(M)$, let $\underline{C}_{\{\mu\}}(M)$ be the direct sum of all the spaces $\underline{C}_\omega(M)$, for which there is an element $\mu_1 = \mu_1(\omega)$ in $\{\mu\}$. Finally, let $\underline{C}_o(M)$ be the smallest closed subspace of $\underline{C}(M)$ which contains each of the spaces

$$\underline{C}_\omega(M) , \omega \in \underline{E}_2(M) .$$

$\underline{C}_o(M)$ is often called the space of <u>cusp forms</u> on M .

Before going on to induced representations, we shall set up some notation that we will use in connection with representation of K . In the above discussion we can replace (G , K , Θ , B) by (K , K , Θ , B). The only possibility for P is the group K itself. The Cartan subalgebra $^{(1)}\underline{b}$ of $\underline{k}$ takes the place of both $\underline{b}_M$ and $\underline{h}$. By applying the above definitions to K , we can define the absolute value $|\eta|$ of any irreducible representation η in $E(K)$. Recall the element Z_K defined in § 1.. It is a consequence of applying the formulas (1.7) and (2.2) to K , that

(2.3) $$\eta(Z_K) = |\eta|^2 , \eta \in E(G).$$

If π is any finite dimensional representation of K, and

$$\pi = \bigoplus_{i=1}^{n} \tau_i \quad , \quad \tau_i \in \mathcal{E}(K) ,$$

we define

$$|\pi| = \sum_{i=1}^{n} |\tau_i| .$$

From the Weyl dimension formula and the fact that K^0 has finite index in K, we observe that there is a polynomial p such that for any finite-dimensional representation τ of K,

(2.4) $\deg \tau \leq P(|\tau|)$.

If ν is any weight occurring in some irreducible component of the tensor product of two finite dimensional representations τ_1 and τ_2 of K,

$$|\nu| \leq |\tau_1| + |\tau_2| .$$

It follows that

$$|\tau_1 \otimes \tau_2| \leq \deg \tau_1 \cdot \deg \tau_2 \cdot (|\tau_1| + |\tau_2|) .$$

Therefore there is a polynomial P such that for any τ_1 and τ_2,

(2.5) $|\tau_1 \otimes \tau_2| \leq P(|\tau_1|) \cdot P(|\tau_2|)$.

Suppose that $P = NAM$ is a standard cuspidal subgroup of G. We define the absolute value of any finite dimensional representation of $K_M = K \cap M$ just as above. Then it is clear that there is a polynomial p such that if τ is a finite dimensional representation of K and γ_M is the restriction of γ to K_M,

$$|\gamma_M| \leq p\ (|\gamma|)\ .$$

A <u>double representation</u> τ of K consists of a vector space V together with a left and a right action of K on V which commute with each other. As usual we denote this action by

$$v \longrightarrow \tau(k_1)\ v\ \tau(k_2)\ ,\quad v \in V\ ,\quad k_1, k_2 \in K\ .$$

If γ_1 and γ_2 are finite dimensional representations of K on spaces V_1 and V_2 we define a double representation τ of K on $L\ (V_2\ , V_1)$, the space of linear maps from V_2 to V_1 equipped with the Hilbert-Schmidt norm, by

$$\tau(k_1)\quad S\quad \tau(k_2) = \gamma_1(k_1) \circ S \circ \gamma_2(k_2)\ ,$$

for $S \in L(V_2, V_1)$ and $k_1\ , k_2 \in K$. In this case we shall write $\tau = (\gamma_1\ , \gamma_2)$. If γ_1 and γ_2 are irreducible, we define

$$|\tau| = |\gamma_1| + |\gamma_2|\ .$$

Any finite dimensional double representation τ is a direct sum of representations of this form. We can define $|\tau|$ by insisting that

$$|\tau_1 \oplus \tau_2| = |\tau_1| + |\tau_2|\ .$$

Finally, if M is as above, we write τ_M for the restriction of τ to M. Then there is a polynomial p such that for any finite dimensional double representation τ of K,

(2.6) $$|\tau_M| \leq p\,(|\tau|).$$

For convenience, we shall denote the class of all finite dimensional unitary double representations of K by $F(K, K)$.

Suppose that $\omega \in \underline{E}_2(\hat{M})$ and that σ a representation in the class of ω which acts on the Hilbert space H_σ. Let $\underline{H}(\sigma)$ be the Hilbert space of measurable functions ϕ from $NA \backslash G$ to H_σ such that

(i) $\phi(m x) = \sigma(m) \phi(x)$, $m \in M$, $x \in G$,

and

(ii) $\|\phi\|_2^2 = \int_K |\phi(k)|^2 \, dk < \infty$.

For $\lambda \in \underline{a}_{\mathbb{C}}$ we have the induced representation $\pi(\sigma, \lambda)$ of G on $\underline{H}(\sigma)$ defined by

$$(\pi(\sigma, \lambda : y) \phi)(x) = \phi(x y) \cdot e^{\langle \lambda + \rho,\, H(xy) \rangle} e^{-\langle \lambda + \rho,\, H(x) \rangle},$$

for $\phi \in \underline{H}(\sigma)$ and $x, y \in G$. This representation is unitary if λ is imaginary.

By differentiation, we can lift $\pi(\sigma, \lambda)$ to a representation of $\underline{G}$ on the space of smooth vectors in $\underline{H}(\sigma)$, a dense subspace of $\underline{H}(\sigma)$. Suppose that ϕ is a smooth vector in $\underline{H}(\sigma)$. Then for $Z \in \underline{Z}$ and $x \in G$,

$$\begin{aligned}(\pi(\sigma, \lambda : Z) \phi)(x) &= (\pi(\sigma, \lambda : x)\, \pi(\sigma, \lambda : Z) \phi)(1) \\ &= (\pi(\sigma, \lambda : Ad(x) Z)\, \pi(\sigma, \lambda : x) \phi)(1) \\ &= (\pi(\sigma, \lambda : Z) \phi_x)(1),\end{aligned}$$

since

$$\phi_x = \pi(\sigma, \lambda : x)\phi$$

is a smooth vector. If $X \in \underline{n}_{\mathbb{C}}$, and $Y \in \underline{G}$, it is clear that

$$(\pi(\sigma, \lambda : X Y)\phi_x)(1) = (\pi(\sigma, \lambda : X)\pi(\sigma, \lambda : Y)\phi_x)(1).$$

But we observed in §1 that $Z - \epsilon^{-1}(\mu(Z))$ belongs to $\underline{n}_{\mathbb{C}}\underline{G}$.
It follows that

$$(\pi(\sigma,\lambda : Z)\phi)(x) = (\pi(\sigma, \lambda : \epsilon^{-1}(\mu(Z)))\phi_x)(1).$$

If $\mu = \mu(\sigma)$, it follows from (2.2) that this last expression equals

$$<\gamma_L(\epsilon^{-1}(\mu(Z))), \mu + \lambda + \rho > \phi_x(1).$$

By the definition of ϵ, this is the same as

$$<\gamma_L(\mu(Z)), \mu + \lambda> \phi_x(1)$$
$$= <\gamma(Z), \mu + \lambda> \phi(x).$$

We have shown that

$$(2.7) \qquad \pi(\sigma, \lambda : Z)\phi = <\gamma(Z), \mu + \lambda>\phi.$$

If Z_G is the element in $\underline{Z}$ defined in §1, we can combine this formula with (1.7) to obtain

$$(2.8) \qquad \pi(\sigma, \lambda : Z_G) = (|\sigma|^2 + <\lambda, \lambda>) I.$$

The restriction of $\pi(\sigma, \lambda)$ to K gives a unitary representation $\pi(\sigma)$ of K which is independent of λ. For $\eta \in \underline{E}(K)$ let $[\omega : \eta]$ denote the multiplicity of η in $\pi(\sigma)$. (This number depends only on the class ω of σ.) Let σ_{K_M} and η_M be the restrictions of σ and η to K_M. Then $\pi(\sigma)$ is equivalent to the representation obtained by inducing σ_{K_M} from K_M to K. It follows from Frobenius reciprocity that

$$[\omega : \eta] = \sum_{\delta \in \underline{E}(K_M)} [\sigma_{K_M} : \delta] [\eta_M : \delta] ;$$

where $[\sigma_{K_M} : \delta]$ and $[\eta_M : \delta]$ are the multiplicities of δ in σ_{K_M} and η_M respectively.

However, by [5(a)] and our previous discussion, there is a constant N, which depends only on M, such that

$$[\sigma_{K_M} : \delta] \leq N \deg(\delta) , \quad \delta \in \underline{E}(K_M) .$$

It follows that

$$(2.9) \quad [\omega : \eta] \leq N \deg(\eta) .$$

<u>LEMMA 2.3</u>: There is a constant C_M such that if ω and η are classes in $\underline{E}_2(M)$ and $\underline{E}(K)$ respectively, with

$$[\omega : \eta] > 0 ,$$

then

$$|\omega|^2 \leq |\eta|^2 + C_M .$$

PROOF: If $[\omega : \eta]$ is positive; there must be a δ in $\underline{E}(K_M)$ such that both $[\sigma_{K_M} : \delta]$ and $[\eta_M : \delta]$ are positive. By [5(e)], Lemma 71, there is a constant C_1 such that if $[\sigma_{K_M} : \delta]$ is positive, then

$$|\sigma|^2 \leq |\delta|^2 + C_1 .$$

However, it is easy to see that there is a C_2 such that if $[\eta_M : \delta]$ is positive, then

$$|\delta|^2 \leq |\eta|^2 + C_2 .$$

This fact follows by the method of [5(e)], Lemma 71, or alternatively, by a highest weight argument. These two inequalities establish the lemma. □

Suppose that for each $\omega \in \underline{E}_2(M)$, we select a representation $\sigma_\omega \in \omega$ on a Hilbert space H_ω. For each ω, choose an orthogonal basis $\{\phi_i : i \in I(\omega)\}$ of $\underline{H}(\sigma_\omega)$ such that for every i, ϕ_i transformsunder $\pi(\sigma_\omega)$ according to an irreducible reprepresentation $\mathfrak{z}(\phi_i)$ of K. We shall require the following

<u>LEMMA 2.4</u>: There is a number n such that

$$\sum_{\omega \in \underline{E}_2(M)} \sum_{i,j \in I(\omega)} (1 + |\omega|^2)^{-n} (1 + |\mathfrak{z}(\phi_i)|^2)^{-n} (1 + |\mathfrak{z}(\phi_j)|^2)^{-n}$$

is finite.

PROOF: The series in the lemma equals

$$\sum_{\omega \in \underline{E}_2(M)} (1 + |\omega|^2)^{-n} \cdot \Big(\sum_{i \in I(\omega)} (1 + |\mathfrak{z}(\phi_i)|^2)^{-n} \Big)^2$$

$$= \sum_{\omega \in \underline{E}_2(M)} (1+|\omega|^2)^{-n} \cdot \Big(\sum_{\mathfrak{z} \in \underline{E}(K)} \deg(\mathfrak{z}) \cdot [\omega : \mathfrak{z}] (1 + |\mathfrak{z}|^2)^{-n} \Big)^2 .$$

By (2.9) this expression is bounded by

$$\Big(\sum_{\omega \in \underline{E}_2(M)} (1 + |\omega|^2)^{-n} \Big) \cdot \Big(\sum_{\mathfrak{z} \in \underline{E}(K)} N \cdot \deg(\mathfrak{z})^2 \cdot (1 + |\mathfrak{z}|^2)^{-n} \Big)^2$$

It follows from our earlier discussion that

$$\sum_{\omega \in \underline{E}_2(M)} (1 + |\omega|^2)^{-n} \leq [M/M^\circ] \sum_{\mu \in \Lambda'(M)} (1 + |\mu|^2)^{-n} .$$

Since $\Lambda'(M)$ is contained in a lattice in a Euclidean space, the right hand series converges for large enough n. Similarly, for any fixed integer m, we may choose n so large that

$$\Sigma_{\gamma \in \underline{E}(K)} (1 + |\gamma|^2)^{-(n-m)} < \infty.$$

We fix m so that

$$\sup_{\gamma \in \underline{E}(K)} (\deg(\gamma)^2 (1 + |\gamma|^2)^{-m}) < \infty .$$

This can be done in view of (2.4). Our proof is complete.

□

If we combine this result with Lemma 2.3 we immediately obtain

<u>COROLLARY 2.5</u>: There is a number n such that

$$\Sigma_{\omega \in \underline{E}_2(M)} \quad \Sigma_{i,j \in I(\omega)} \; (1 + |\gamma(\phi_i)|^2)^{-n} \quad (1 + |\gamma(\phi_j)|^2)^{-n}$$

is finite. □

For each $\sigma \in E_2(M)$, we have a decomposition

$$\underline{H}(\sigma) = \bigoplus_{\gamma \in \underline{E}(K)} \underline{H}_{\{\gamma\}}(\sigma)$$

where the restriction of $\pi(\sigma)$ to $\underline{H}_{\{\gamma\}}(\sigma)$ is equivalent to a finite number of copies of γ. If F is a finite subset of $\underline{E}(K)$, define

$$\underline{H}_F(\sigma) = \bigoplus_{\gamma \in F} \underline{H}_{\{\gamma\}}(\sigma)$$

If F_M is a finite subset of $\underline{E}(K_M)$, we define the subspace H_{σ,F_M} of H_σ the same way. Suppose that F_M is the set of irreducible components in the restriction to K_M of each representation in F. Then we write $H_{\sigma,F}$ for H_{σ,F_M}. If $\phi \in \underline{H}_F(\sigma)$, it is clear that for each $k \in K$, $\phi(k)$ belongs to $H_{\sigma,F}$.

We shall sometimes write $\underline{H}^0(\sigma)$ and H^0_σ for the union, over all finite subsets F of $\underline{E}(K)$, of the spaces $\underline{H}_F(\sigma)$ and $H_{\sigma,F}$.

§3. THE MAPPINGS $\Omega(\underline{a}, \underline{a}')$

Suppose $u \in \underline{J}$. Let us say that a linear subspace of ${}^{(u)}\underline{a}$ is distinguished if it is of the form ${}^{(v)}\underline{a}$, for some index $v \in \underline{J}$ such that $u \leq v$. If $\underline{a}$ and $\underline{a}'$ are two distinguished subspaces of ${}^{(o)}\underline{a}$, we write $\Omega(\underline{a},\underline{a}')$ for the set of distinct mappings from $\underline{a}$ onto $\underline{a}'$ that can be obtained by restricting transformations in Ω to $\underline{a}$. The spaces $\underline{a}$ and $\underline{a}'$, as well as the corresponding standard parabolic subgroups P and P', are said to be associated if $\Omega(\underline{a},\underline{a}')$ is not empty. We shall write $\Omega(\underline{a})$ for the union, over all distinguished subspaces $\underline{a}'$ which are associated to $\underline{a}$, of the sets $\Omega(\underline{a},\underline{a}')$.

If $\underline{a}$ is a distinguished subspace of ${}^{(o)}\underline{a}$, we shall say that an element $\alpha \in \underline{a}$ is a root of $\underline{a}$ if either α or $-\alpha$ is a root of (P, A). We shall say that α is a simple root of $\underline{a}$ if it belongs to Φ. Let us say that a point H in $\underline{a}_c$ is P-regular if for every root α of $\underline{a}$,

$$\langle\alpha, H\rangle \neq 0 .$$

We shall denote the set of P-regular points in $\underline{a}$ by $\underline{a}_r$.

Suppose that P is standard parabolic subgroup of G. A subset B of the roots of $\underline{a}$ is called a fundamental system if any root of $\underline{a}$ can be written uniquely as a linear combination of elements in B with integral coefficients all of the same sign.

The convex set

$$c_B = \{H \in \underline{a} : \alpha(H) > 0 , \alpha \in B\}$$

is called the <u>Weyl</u> <u>chamber</u> of B. If $B = \Phi$, the set of simple roots of (P,A), c_B is denoted by $\underline{a}^+$. It is clear that Weyl chambers corresponding to different fundamental systems in $\underline{a}$ are disjoint.

If $\underline{a}'$ is another distinguished subspace of ${}^{(o)}\underline{a}$ and $s \in \Omega(\underline{a}, \underline{a}')$, $s^{-1}(\Phi')$ is a fundamental system for $\underline{a}$. All fundamental systems obtained this way are distinct. For if $t \in \Omega(\underline{a}, \underline{a}'')$ and

$$t^{-1}(\Phi'') = s^{-1}(\Phi') ,$$

$s\,t^{-1}(\Phi') = \Phi''$. st^{-1} is the restriction to a'' of some element

$$Ad(w) , w \in G .$$

We must have

$$w \cdot N' A' \cdot w^{-1} = N''A'' ,$$

and therefore

$$w P' w^{-1} = P'' .$$

Since no distinct standard parabolic subgroups are conjugate, P' equals P''. Therefore w belongs to P'. However, any element in P' which normalizes A' also centralizes A'. It follows that $s = t$.

It is known that Ω acts simply transitively on the Weyl chambers of ${}^{(o)}\underline{a}$. In particular, there is an element s in Ω which maps ${}^{(o)}\underline{a}^+$ onto $-{}^{(o)}\underline{a}^+$. This means that $s({}^{(o)}\Phi) = -{}^{(o)}\Phi$. It follows that if $\underline{a}$ is a one dimensional distinguished subspace of ${}^{(o)}\underline{a}$ there is another one dimensional distinguished subspace $\underline{a}'$, and an element $s \in \Omega(\underline{a}, \underline{a}')$ such that

$$s(\underline{a}^+) = -(\underline{a}')^+ .$$

By the remark above, $\underline{a}$ and $\underline{a}'$ are uniquely determined.

Now suppose that $\underline{a}$ is a distinguished subspace of ${}^{(o)}\underline{a}$ of arbitrary dimension. For each simple root α in Φ ,

$$\underline{a}^\alpha = \{H \in \underline{a} : \langle\alpha , H\rangle = 0\}$$

is a distinguished subspace of ${}^{(o)}\underline{a}$. Let P^α be the corresponding standard parabolic subgroup of G . Then

$$P_\alpha = P \cap M^\alpha$$

is a maximal standard parabolic subgroup of M^α .

In particular

$$\underline{a}_\alpha = \underline{a} \cap \underline{m}^\alpha$$

is a one dimensional distinguished subspace of ${}^{(o)}\underline{a} \cap \underline{m}^\alpha$. Applying the above remark we choose a second distinguished subspace $\underline{a}'_\alpha$ of ${}^{(o)}\underline{a} \cap \underline{m}^\alpha$, and an element $s_\alpha \in \Omega^\alpha(\underline{a}_\alpha , \underline{a}'_\alpha)$ such that

$$s_\alpha(\underline{a}^+_\alpha) = -(\underline{a}'_\alpha)^+ .$$

s_α can be regarded as a map from $\underline{a}$ to

$$\underline{a}' = \underline{a}^\alpha \oplus \underline{a}'_\alpha$$

which leaves $\underline{a}^\alpha$ pointwise fixed. With this interpretation s_α is called the <u>simple</u> <u>reflection</u> corresponding to α .

The following lemma is due to Langlands ([7(a)] , Lemma 2.13).

LEMMA 3.1: Let $\underline{a}^{(1)}, \ldots\ldots, \underline{a}^{(r)}$ be the distinct distinguished subspaces of ${}^{(o)}\underline{a}$ which are associated to $\underline{a}$. If E is the set of fundamental systems of roots of $\underline{a}$, then $\underline{a}$ is the closure of $\bigcup_{B \in E} c_B$. For any B there is a unique i and a unique $s \in \Omega(\underline{a}, \underline{a}^{(i)})$ such that

$$c_B = s^{-1}(\underline{a}^{(i)})^+ .$$

PROOF: We have already noted that if i and s exist they must be unique. Recall that an element $H_o \in \underline{a}$ is P-regular whenever it belongs to no hyperplane of the form

$$\{H \in \underline{a} : \langle\alpha , H\rangle = 0\} , \alpha \in \Sigma .$$

We shall call H_o P-semiregular if it belongs to precisely one such hyperplane. We shall say that a polygonal path joining two P-regular points in $\underline{a}$ is P-semiregular if it is a union of a collection of P-regular points together with a finite number of P-semiregular points.

Suppose that $s \in \Omega(\underline{a}, \underline{a}^{(i)})$ and that H_o is a point in

$$c_s = s^{-1}(\underline{a}^{(i)})^+ .$$

Since any root of $\underline{a}$ is of the form $s^{-1}(\beta)$, for β a root of $\underline{a}^{(i)}$, H_o must be P-regular. Suppose that H_1 is another P-regular point of $\underline{a}$ such that the line segment joining H_o to H_1 is the union of a collection of P-regular points with one P-semiregular point H_2. Then H_2 belongs to a unique hyperplane

$$\{H \in \underline{a} : \langle \alpha , s H \rangle = 0\}$$

for α a simple root of $\underline{a}^{(i)}$. If $s_\alpha \in \Omega(\underline{a}^{(i)}, \underline{a}^{(j)})$ is the corresponding simple reflection, it follows from the definition of s_α that

$$c_{s_\alpha s} = (s_\alpha s)^{-1} (\underline{a}^{(j)})^+$$

is the connected component of H_1 in $\underline{a}_r$. We note that, conversely, if s, H_o and α are given, we can find $H_1 \in c_{s_\alpha s}$ such that the line segment joining H_o and H_1 is as above. Our lemma follows from the fact that any two P-regular points can be joined by a P-semiregular polygonal path.

□

It follows from the proof of the lemma that any $s \in \Omega(\underline{a}, \underline{a}^{(i)})$ can be written as a composition of simple reflections. The minimum number of simple reflections in any such decomposition is called the <u>length</u> of s. In fact, we see easily from the proof of the lemma, that for every P-semiregular polygonal path between $\underline{a}^+$ and $s^{-1} (\underline{a}^{(i)})^+$ which contains n P-semiregular points there is a canonical decomposition of s as a product of n simple reflections. Any such decomposition of s arises this way. The path may be taken to be a line segment if and only if n is the length of s.

Any root α of (P , A) is said to be <u>reduced</u> if there is no t, $0 < t < 1$, such that $t\alpha$ is Σ. Let $\overline{\Sigma}$ be the set of reduced roots of (P , A).

Suppose $s \in \Omega(\underline{a}, \underline{a}')$. Choose associated spaces

$$\underline{a}_o = \underline{a}', \underline{a}_1, \ldots\ldots, \underline{a}_n = \underline{a},$$

and simple reflections

$$s_{\alpha_i} \in \Omega(\underline{a}_i, \underline{a}_{i-1}), \quad \alpha_i \in \Phi_i, \ 1 \leq i < n,$$

such that

$$s = s_{\alpha_1} \cdots s_{\alpha_n}$$

is a decompositinn of s of minimal length. The following lemma generalizes a result in [2].

LEMMA 3.2: The roots

$$\beta_i = s_{\alpha_n}^{-1} s_{\alpha_{n-1}}^{-1} \cdots s_{\alpha_{i+1}}^{-1}(\alpha_i), \ 1 \leq i \leq n,$$

are positive and distinct. They are precisely those reduced roots β in $\bar{\Sigma}$ such that $s(\beta) < 0$.

PROOF: It is clear that for any root α of $\underline{a}$, $s(\alpha)$ is reduced if and only if α is reduced.

First we shall prove the lemma for the case that $n = 1$. Then $s = s_\alpha$, for α a simple root of $\underline{a}$. Following our earlier notation, s_α leaves the hyperplane $\underline{a}^\alpha$ pointwise fixed, but maps $-\alpha$ onto a simple root α' of $\underline{a}'$. The other simple roots $\{\gamma_1, \ldots, \gamma_{r-1}\}$ and $\{\gamma_1', \ldots, \gamma_{r-1}'\}$, of $\underline{a}$ and $\underline{a}'$ respectively, can be indexed in such a way that for every i, γ_i and γ_i' have the same projections onto $\underline{a}^\alpha$. It follows that there is a c_i in $\mathbb{R}$ such that

$$s_\alpha(\gamma_i) = \gamma_i' + c_i \alpha', 1 \leq i \leq r-1.$$

Since $s_\alpha(\gamma_i)$ is a root of $\underline{a}'$, c_i must be a nonnegative integer. From these equations we see that the image under s_α of any positive reduced root of $\underline{a}$ other than α is a positive root of $\underline{a}'$.

Suppose now that n is arbitrary. Let $\ell(s)$ be the number of roots $\beta \in \overline{\Sigma}$ such that $s(\beta)$ is negative. We shall show that $\ell(s)$ equal the length of s. Let $t \in \Omega(\underline{a}, \underline{a}'')$ be the unique map such that

$$t^{-1}(\underline{a}'')^{+} = -\underline{a}^{+} .$$

It is a consequence of the remarks following Lemma 3.1 that there is a minimal decomposition of t into simple reflections of the form

$$t = s_{\alpha_{n+1}} \cdots\cdots s_{\alpha_{n+m}} \cdot s_{\alpha_1} \cdots\cdots s_{\alpha_n} .$$

Appealing again to our remarks following Lemma 3.1, we see that $n + m$, the length of t, equals the number of positive reduced roots of $\underline{a}$.

Therefore

$$m + n = \ell(t) .$$

From the fact that

$$\ell(\alpha_i) = 1 \ , \ 1 < i \leq m + n \ ,$$

we obtain

$$m + n \leq \ell(s_{\alpha_{n+1}}^{-1} t) + 1 \leq \ldots\ldots \leq \ell(s) + m \leq \ell(s_{\alpha_1}^{-1} s) + m + 1$$

$$\leq \ldots\ldots \leq \ell(s_{\alpha_n}) + m + n - 1 = m + n \ .$$

In particular, $\ell(s)$ equals n , the length of s .

Finally, suppose that β is a positive root of $\underline{a}$ such that $s(\beta)$ is negative. Our proof will be complete if we can show that β is one of the roots

$$\{\beta_i : 1 \leq i \leq n\} \ .$$

We can certainly choose an i such that $s_{\alpha_{i+1}} \cdots s_{\alpha_n}(\beta)$ is positive but such that $s_{\alpha_i} \ldots s_{\alpha_n}(\beta)$ is negative. This can only happen if

$$s_{\alpha_{i+1}} \ldots s_{\alpha_n}(\beta) = \alpha_i \ .$$

Therefore

$$\beta = s_{\alpha_n}^{-1} \ldots s_{\alpha_{i+1}}^{-1}(\alpha_i) = \beta_i \ ,$$

and we are done. □

We conclude this section with some notation that will be of use later. Suppose that P is a standard parabolic subgroup of G . For each root β of (P , A) , we have the root space $\underline{n}(\beta)$. Define

$$\underline{v}(\beta) = \{ X \in \underline{g} : [H , X] = - \langle\beta , H\rangle X , H \in \underline{a}\} .$$

If α belongs to $\overline{\Sigma}$, let $\Sigma(\alpha)$ be the set of roots in Σ of the form $t\alpha$, for $t \geq 1$. Define

$$\underline{n}_\alpha = \bigoplus_{\beta\in\Sigma(\alpha)} \underline{n}(\beta)$$

and

$$\underline{v}_\alpha = \bigoplus_{\beta\in\Sigma(\beta)} \underline{v}(\beta) .$$

Then

$$\underline{n} = \bigoplus_{\alpha\in\overline{\Sigma}} \underline{n}_\alpha .$$

Let V be the analytic subgroup of G whose Lie algebra is the nilpotent subalgebra

$$\underline{v} = \bigoplus_{\alpha\in\overline{\Sigma}} \underline{v}_\alpha$$

of $\underline{g}$. Notice that

$$\overline{P} = V A M = \Theta(N) A M$$

is a parabolic subgroup of G. In particular, if $\underline{q}$ is a linear subspace of $\underline{v}$, we have, by the conventions of $\S 1$, a measure dv on $\exp \underline{q}$.

Suppose that α belongs to $\overline{\Sigma}$. Let N_α and V_α be the analytic subgroups of G corresponding to the Lie algebras $\underline{n}_\alpha$ and $\underline{v}_\alpha$ respectively. Let $\underline{a}^\alpha$ be the hyperplane

$$\{H \in \underline{a} ; \alpha(H) = 0\}$$

in $\underline{a}$. Let L^α be the centralizer of $\underline{a}^\alpha$ in G. Define

$$M^\alpha = (L^\alpha)^1 , \text{ and } K^\alpha = K \cap M^\alpha .$$

If $\underline{m}^\alpha$ is Lie algebra of M^α, $\Theta(\underline{m}^\alpha) = \underline{m}^\alpha$. If we denote the restrictions of Θ and B to $\underline{m}^\alpha$ by Θ^α and B^α respectively, $(M^\alpha, K^\alpha, \Theta^\alpha, B^\alpha)$ satisfy all the conditions we asked of (G, K, Θ, B) at the beginning of this paper. We define

$$\underline{a}_\alpha = \underline{a} \cap \underline{m}^\alpha ,$$

$$A_\alpha = \exp \underline{a}_\alpha ,$$

and

$$P_\alpha = N_\alpha A_\alpha M .$$

Then P_α is a maximal cuspidal subgroup of M^α. P_α, N_α, and V_α are the intersections of M^α with P, N and V respectively.

Notice that if α is a simple root of (P, A), our notation coincides with that introduced at the beginning of this section. In this case we define the vector ρ^α in $\underline{a}_\alpha$ by

$$\langle\rho_\alpha , H\rangle = \frac{1}{2} \operatorname{Tr} (\operatorname{ad} H)_{\underline{n}_\alpha} , \quad H \in \underline{a}_\alpha .$$

<u>LEMMA 3.3</u>: Suppose that α is a simple root of (P, A). Then ρ_α is the projection of ρ onto $\underline{a}_\alpha$.

PROOF: We have

$$\underline{n} = \underline{n}^\alpha \oplus \underline{n}_\alpha ,$$

where $\underline{n}^\alpha$ comes from the standard parabolic subgroup, P^α, of G. For $H \in \underline{n}_\alpha$,

$$\langle \rho , H \rangle = \frac{1}{2} \operatorname{Tr} (\operatorname{ad} H)_{\underline{n}}$$
$$= \frac{1}{2} \operatorname{Tr} (\operatorname{ad} H)_{\underline{n}_\alpha} + \frac{1}{2} \operatorname{Tr} (\operatorname{ad} H)_{\underline{n}^\alpha}$$
$$= \langle \rho_\alpha , H \rangle + \frac{1}{2} \operatorname{Tr} (\operatorname{ad} H)_{\underline{n}^\alpha} .$$

The algebra $\underline{m}^\alpha$ normalizes $\underline{n}^\alpha$. The map

$$Y \longrightarrow \frac{1}{2} \operatorname{Tr} (\operatorname{ad} Y)_{\underline{n}^\alpha} , \quad Y \in \underline{m}^\alpha ,$$

is a Lie algebra homomorphism from $\underline{m}^\alpha$ to $\mathbb{R}$. Since the center of $\underline{m}^\alpha$ is contained in $\underline{m}^\alpha \cap \underline{k}$, this homomorphism must vanish on $\underline{a}_\alpha$. It follows that

$$\langle \rho - \rho_\alpha , H \rangle = 0 , \quad H \in \underline{a}_\alpha .$$

In other words, ρ_α is the projection of ρ onto $\underline{a}_\alpha$. □

Suppose that $s \in \Omega(\underline{a}, \underline{a}')$. Let $\bar{\Sigma}_s$ and Σ_s be the sets of roots α in $\bar{\Sigma}$ and Σ respectively such that $s(\alpha)$ is negative. Define

$$\underline{v}_s = \oplus_{\alpha \in \bar{\Sigma}_s} \underline{v}_\alpha .$$

Let w be any representative of s in ${}^{(o)}\tilde{M}$. Then

$$\underline{v}_s = \operatorname{Ad}(w^{-1}) \underline{n}' \cap \underline{v} .$$

The group

$$V_s = w^{-1} N' w \cap V$$

is the analytic subgroup of G with Lie algebra $\underline{v}_s$.

Let c be a Weyl chamber in $\underline{a}$. Choose the distinguished subspace $\underline{a}'$ of ${}^{(o)}\underline{a}$, and $s \in \Omega(\underline{a}, \underline{a}')$ such that $s^{-1}(\underline{a}')^+ = c$. We define a vector ρ_c in $\underline{a}$ by

$$<\rho_c , H> = \frac{1}{2} \operatorname{Tr} (\operatorname{ad} sH)_{\underline{n}'} , \quad H \in \underline{a} .$$

Then $\rho_c = s^{-1}(\rho')$. We define a second vector ρ_s in $\underline{a}$ by

$$<\rho_s , H> = -\frac{1}{2} \operatorname{Tr} (\operatorname{ad} H)_{\underline{v}_s} , \quad H \in \underline{a} .$$

It is clear that

(3.1) $s(\rho - 2\rho_s) = \rho'$.

In other words,

$$\rho - 2\rho_s = \rho_c .$$

§4. INTERTWINING OPERATORS

In this section we shall discuss the intertwining operators between the induced representations we have defined. The operators are defined by the sort of integrals which are familiar from the work of Kunze and Stein, and Knapp and Stein. Following the method of Gindikin and Karpelevic [4] and Schiffman [9], we shall prove a product formula for these integrals.

We will need a lemma which will insure the convergence of the intertwining integrals. Suppose that $\underline{a}$ and $\underline{a}'$ are distinguished subspaces of ${}^{(o)}\underline{a}$, and that $s \in \Omega(\underline{a}, \underline{a}')$. As usual we have the standard parabolic subgroup $P = NAM$ of G corresponding to $\underline{a}$. Let $\underline{a}^+(s)$ be the set of λ in $\underline{a}$ such that

$$\langle \lambda , \beta \rangle > 0$$

for each root β in $\overline{\Sigma}_s$. We shall write Ξ_M for the function on M used to define the Schwartz space on M.

LEMMA 4.1: Suppose that Γ is a compact subset of $\underline{a}^+(s)$ and that c is any real number. Then there are constants d and d_o such that for any $m \in M$ and any suitably large real number r, the integral over V_s of

$$(4.1) \qquad \Xi_M(m\, v_M) \cdot (1 + \sigma(m\, v_M))^{-r} \cdot (1 + \sigma(v_A))^{d} \cdot \{\sup_{\lambda \in \Gamma} e^{\langle \lambda + \rho, H(v) \rangle}\}$$

is bounded by

$$d_o \quad \Xi_M(m) \cdot (1 + \sigma(m))^{-(r/2 - d)} ,$$

We shall postpone the proof of this lemma until the end of this section.

Let $T(s)$ be the set of points λ in $\underline{a}_{c}$ whose real parts lie in $\underline{a}^{+}(s)$. Suppose that σ is a representation in $E_2(M)$ which acts on the Hilbert space H_σ.

<u>LEMMA 4.2</u>: For $\lambda \in T(s)$, $\xi \in H_\sigma^{o}$, $\phi \in \underline{H}^{o}(\sigma)$, and $x \in G$, the integral

$$(4.2) \quad \int_{V_s} (\phi(v\, w^{-1}x), \xi)\, e^{\langle \lambda+\rho,\, H(v\, w^{-1}x)\rangle}\, e^{-\langle s\lambda+\rho',\, H'(x)\rangle} dv$$

is absolutely convergent. It equals

$$(4.3) \quad \int_{V_s} (\sigma(w^{-1}x_M, w)\, \phi(v\, w^{-1}x_K), \xi)\, e^{\langle \lambda+\rho, H(v)\rangle}\, dv.$$

PROOF: First of all we shall prove that (4.3) is absolutely convergent. Then we shall show that with a suitable change of variables we obtain the integral (4.2). Since $\phi \in \underline{H}^{o}(\sigma)$, we may choose a finite set $\{\xi_1,\ldots\ldots,\xi_t\}$ of vectors in H_σ^{o} and functions $c_1,\ldots,c_t$ on K such that for $k \in K$,

$$\phi(k) = \Sigma_{i=1}^{t}\ c_i(k)\, \xi_i$$

Then

$$|(\sigma(w^{-1}x_M, w)\quad \phi(v\, w^{-1}x_K), \xi) \cdot e^{\langle \lambda+\rho, H(v)\rangle}|$$

is bounded by

$$\Sigma_{i=1}^{t}\ |\ c_i(v_K w^{-1}\ x_K)| \cdot |(\sigma(w^{-1}x_M, w\ v_M)\ \xi_i, \xi)| \cdot e^{\langle \lambda_{\mathbb{R}}+\rho, H(v)\rangle}.$$

Since ξ_i and ξ are K_M-finite vectors in H_σ, the function

$$m \longrightarrow (\sigma(m)\ \xi_i, \xi), \quad m \in M,$$

is in $\underline{C}(M)$, ([5 (e)], Lemma 65, Corollary 1). The integrability of (4.3) now follows from Lemma 4.1.

To see the equality of (4.2) and (4.3), decompose x as $x_{N'} \cdot x_{M'} \cdot x_{A'} \cdot x_K$. Replace the variable v in (4.3) by

$$(w^{-1} x_{M'} x_{A'} w)^{-1} \cdot v \cdot (w^{-1} x_{M'} x_{A'} w) .$$

If ρ_s is defined as in §3, we have

$$d((w^{-1} x_{M'} x_{A'} w)^{-1} \cdot v \cdot (w^{-1} x_{M'} x_{A'} w)) = e^{2\langle \rho_s, s^{-1}H'(x)\rangle} dv .$$

In view of (3.1), the integral (4.3) becomes

$$\int_{V_s} (\phi(v w^{-1} x_{M'} x_{A'} x_K), \xi) \cdot e^{\langle \lambda+\rho, H(v w^{-1} x_{M'} x_{A'} x_K)\rangle} \cdot e^{-\langle s\lambda+\rho', H'(x)\rangle} dv.$$

Finally, we note that this integral over V_s may be replaced by the integral over

$$w^{-1} N' w \cap N \backslash w^{-1} N' w .$$

In other words we can replace the variable v by $v \cdot w^{-1} x_{N'} w$. This gives the integral (4.2). $\square$

For any $k \in K$, let $(R(w : \lambda)\phi)(k)$ be the vector in H_σ^o such that for any $\xi \in H_\sigma^o$,

$$((R(w : \lambda)\phi)(k), \xi) = \int_{V_s} (\phi(v w^{-1} k), \xi) \; e^{\langle \lambda+\rho, H(v)\rangle} dv .$$

Then $(R(w : \lambda)\phi)(k)$ is a linear combination of the vectors $\xi_1, \ldots, \xi_t$ introduced in the proof of the lemma. For any $x \in G$, we define

$$(R(w : \lambda)\phi)(x) = \sigma(w^{-1} x_{M'} w) \cdot (R(w : \lambda)\phi)(x_K) .$$

Then $R(w : \lambda)\,\phi$ is a vector in $\underline{H}^o\,(w\,\sigma)$. For any $\xi \in H^o_\sigma$, $((R(w : \lambda)\,\phi)(x)\,,\,\xi)$ equals (3.2). The map

$$\lambda \longrightarrow R\,(w : \lambda)\,,\ \lambda \in T\,(s)\,,$$

is an analytic function with values in the space of linear maps from $\underline{H}^o\,(\sigma)$ to $\underline{H}^o\,(w\,\sigma)$.

Suppose that f is a left and right K-finite function in $C_c^\infty\,(G)$. Then $\pi\,(\sigma, \lambda : f)$ is a linear operator on $\underline{H}^o\,(\sigma)$. For s, ϕ, λ, and ξ as above,

$$((R(w : \lambda)\,\pi\,(\sigma\,,\,\lambda : f)\,\phi)\,(k),\,\xi)$$

$$= \int_{V_s}\int_G f\,(y)\,((\pi\,(\sigma\,,\,\lambda : y)\,\phi)(v\,w^{-1}k),\,\xi)\cdot e^{<\lambda+\rho,H(v)>}dy\,dv$$

$$= \int_{V_s}\int_G f\,(y)\,(\phi(v\,w^{-1}k\,y),\,\xi)\cdot e^{<\lambda+\rho,H(v\,w^{-1}k\,y)>}dy\,dv$$

$$= \int_G f(y)\int_{V_s}(\phi\,(v\,w^{-1}k\,y),\,\xi)\cdot e^{<\lambda+\rho,\,H(v\,w^{-1}k\,y)>}dv\,dy$$

$$= \int_G f(y)\,((R(w : \lambda)\,\phi)\,(k\,y),\,\xi)\cdot e^{<s\,\lambda+\rho',\,ky>}dy$$

$$= \int_G f(y)\,((\pi\,(w\,\sigma,\,s\,\lambda : y)\,R\,(w : \lambda)\,\phi)\,(k),\,\xi)\,dy$$

$$= ((\pi\,(w\,\sigma,\,s\lambda : f)\,R\,(w : \lambda)\,\phi)\,(k),\,\xi)\,.$$

We have obtained the intertwining property

$$(4.4)\qquad R\,(w : \lambda)\,\pi\,(\sigma,\,\lambda : f) = \pi\,(w\,\sigma,\,s\,\lambda : f)\,R\,(w : \lambda)\,.$$

If α is any root in $\bar{\Sigma}$, and λ belongs to $\underline{a}_{\mathbb{C}}$, we write

$$\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

for the projection of λ onto $(\underline{a}_\alpha)_{\mathbb{C}}$. Suppose now that α is a simple root. If $\lambda = \rho$, it follows from Lemma 3.3 that the definition of ρ_α given in §3 is the same as above. Let w_α be a representative of s_α in ${}^{(o)}\widetilde{M}$. s_α can be regarded as a simple reflection on $\underline{a}_\alpha$ as well as on $\underline{a}$. If $\lambda \in T(s_\alpha)$ we have the operator $R_\alpha(w_\alpha : \lambda_\alpha)$ associated to the parabolic subgroup P_α of M^α. Its domain, $\underline{H}^o_\alpha(\sigma)$, is a space of functions from M^α to H_σ. If $\phi \in \underline{H}^o(\sigma)$, and $k \in K$, the function

$$\phi_k : m \longrightarrow \phi(m k), \quad m \in M^\alpha,$$

belongs to $\underline{H}^o_\alpha(\sigma)$. For $x \in G$, we define

$$(R^G_\alpha(w_\alpha : \lambda_\alpha)\phi)(x) = (w_\alpha\sigma)(x_M)\ (R_\alpha(w_\alpha : \lambda_\alpha)\phi_{x_K})(1).$$

$R^G_\alpha(w_\alpha : \lambda_\alpha)$ is an operator whose domain is $\underline{H}^o(\sigma)$.

<u>LEMMA 4.3</u>: If α is a simple root of (P, A) and $\lambda \in T(s_\alpha)$,

$$R(w_\alpha : \lambda) = R^G_\alpha(w_\alpha : \lambda_\alpha).$$

PROOF: By Lemma 3.2, α is the only element in $\overline{\Sigma}$ which is transformed into a negative root by s_α. Therefore $V_{s_\alpha} = V_\alpha$. For $v \in V_{s_\alpha}$, we have $v_K = v_{K^\alpha}$, and $H(v) = H_\alpha(v)$.

For $\phi \in \underline{H}^o(\sigma)$, $k \in K$, and $\xi \in H^o_\sigma$,

$$((R(w_\alpha : \lambda)\phi)(k), \xi)$$

$$= \int_{V_{s_\alpha}} (\sigma(v_M)\cdot\phi(v_K \cdot w_\alpha^{-1} k), \xi)\ e^{\langle \lambda+\rho, H(v)\rangle}\, dv$$

$$= \int_{V_\alpha} (\sigma(v_M) \cdot \phi(v_{K^\alpha} \cdot w_\alpha k), \xi) \, e^{\langle \lambda_\alpha + \rho_\alpha, H_\alpha(v) \rangle} \, dv$$

by Lemma 3.3. But this last expression is just

$$((R^G_\alpha(w_\alpha : \lambda_\alpha) \phi)(k), \xi).$$

The lemma is proved. □

Suppose that $s \in \Omega(\underline{a}, \underline{a}')$. We choose associated spaces

$$\underline{a}_0 = \underline{a}', \underline{a}_1, \ldots\ldots, \underline{a}_n = \underline{a},$$

and simple reflections

$$s_{\alpha_i} \in \Omega(\underline{a}_i, \underline{a}_{i-1}), \quad 1 \leq i \leq n,$$

such that

$$s = s_{\alpha_1} \cdots\cdots s_{\alpha_n}$$

is a decomposition of s of minimal length. Define

$$s_i = s_{\alpha_{i+1}} \cdots\cdots s_{\alpha_n}, \quad 0 \leq i \leq n.$$

Then $s_0 = s$ and s_n is the identity. For each i, let w, w_{α_i}, and w_i be representatives in ${}^{(0)}\widehat{M}$ of s, s_{α_i} and s_i respectively. We may assume that they have been chosen so that

$$w_i = w_{\alpha_{i+1}} \cdots\cdots w_{\alpha_n}, \quad 0 \leq i \leq n-1.$$

<u>LEMMA 4.4</u>: For $\lambda \in T(s)$,

$$R(w : \lambda) = R(w_{\alpha_1} : s_1\lambda) \cdot R(w_{\alpha_2} : s_2\lambda) \cdots\cdots R(w_{\alpha_n} : s_n\lambda).$$

PROOF: We shall prove the lemma by induction on n. The length of s_1 is $n-1$, so we assume that the lemma is true with s replaced by s_1. By Lemma 3.2, $\bar{\Sigma}_s$ is the disjoint union of $\bar{\Sigma}_{s_1}$ and the reduced root

$$\beta = s_1^{-1}(\alpha_1)$$

of $\underline{a}$. Therefore $T(s)$ is precisely the union of $T(s_1)$ with the set of λ such that the number

$$\langle s_1 \lambda_{\mathbb{R}} , \alpha_1 \rangle = \langle \lambda_{\mathbb{R}} , \beta \rangle$$

is positive. It is clear that

$$\underline{v}_s = \underline{v}_\beta \oplus \underline{v}_{s_1} .$$

Furthermore,

$$V_s = V_{s_1} V_\beta ;$$

and the normalized Haar measure on V_s is the product of the normalized Haar measures on V_{s_1} and V_β.

For $\phi \in \underline{H}^o(\sigma)$, $k \in K$, $\lambda \in T(s)$, and $\xi \in H_\sigma^o$, $((R(w, \lambda)\phi)(k), \xi)$ equals

$$\int_{V_s} (\sigma(v_M)\phi(v_K w^{-1} k), \xi) \quad e^{\langle \lambda + \rho , H(v) \rangle} dv$$

$$\int_{V_\beta} \int_{V_{s_1}} (\sigma((v_1 x)_M \phi((v_1 x)_K w_1^{-1} \cdot w_{\alpha_1}^{-1} k), \xi) \quad e^{\langle \lambda + \rho, H(v_1 x) \rangle} dv_1 dx$$

We write $M(v_1 x)$ and $K(v_1 x)$ for $(v_1 x)_M$ and $(v_1 x)_K$ respectively. Decomposing x as $x_N x_A x_M x_K$, we obtain

$$\int_{V_\beta} \int_{V_{s_1}} (\sigma(M(x_N^{-1} v_1\, x_N\, x_A\, x_M)) \cdot \phi\, (K(x_N^{-1}\, v_1\, x_N\, x_A\, x_M\, x_K\, w_1^{-1}\; w_{\alpha_1}^{-1}\, k),\, \xi).$$
$$e^{\langle\lambda+\rho,\ H(x_N^{-1}\, v_1 x_N\, x_A)\rangle}\, dv_1\, dx\ .$$

By an argument similar to [4], we can assert that

$$x_N^{-1}\; v_1\; x_N = n \cdot v(x,\ v_1)\ ,$$

for uniquely determined elements $n \in N$ and $v(x,\ v_1) \in V_{s_1}$, and that for fixed x, the map

$$v_1 \longrightarrow v\ (x,\ v_1)$$

is a measure preserving diffeomorphism of V_{s_1}. It follows that our expression above equals

$$\int_{V_\beta} \int_{V_{s_1}} (\sigma(M\ (v\ x_A\ x_M) \cdot \phi\ (K(v\ x_A\ x_M\ x_K) \cdot w_1^{-1}\;\; w_{\alpha_1}^{-1}\ k),\ \xi)\ .$$
$$e^{\langle\lambda+\rho,\ \ H\ (v\ x_A)\rangle} dv\ dx\ .$$

In the integral over V_{s_1}, replace the variable x by $(x_A\ x_M) \cdot v \cdot (x_A x_M)^{-1}$. We have

$$d((x_A\ x_M) \cdot v \cdot (x_A\ x_M)^{-1}) = e^{-2\langle\rho_{s_1}\ ,\ H(x)\ \rangle} dv\ .$$

Therefore, $((R(w\ :\ \lambda)\ \phi)(k),\ \xi)$ equals

$$\int_{V_\beta\, V_{s_1}} (\sigma(M(x_A\ x_M\ v) \cdot \phi\ (K(x_A\ x_M\ v\ x_K)\ \ w_1^{-1}\ \ w_{\alpha_1}^{-1}\ k),\ \xi)$$
$$e^{\langle\lambda+\rho,\ H(x_A v)\rangle}\quad e^{-2\langle\rho_{s_1}\ ,\ H(x)\rangle}\ dv\ dx\ .$$

Recall from (3.1) that

$$s_1(\rho - 2\rho_{s_1}) = \rho_1 .$$

Therefore our expression equals

$$\int_{V_\beta}\int_{V_{s_1}} (\sigma(x_M)\cdot\sigma(v_M)\ \phi\ (v_K\ x_K\cdot w_1^{-1}\ w_{\alpha_1}^{-1}\ k),\ \xi)$$
$$e^{\langle\lambda+\rho,\ H(v)\rangle}\ e^{\langle\lambda+s_1^{-1}\ \rho_1, H(x)\rangle}\ dv\ dx$$

$$= \int_{V_\beta} (\sigma(x_M)\ (R(w_1 : \lambda)\ \phi)\ (w_1\ x_K\ w_1^{-1}\ w_{\alpha_1}^{-1}\ k),\ \xi)\cdot e^{\langle s_1\lambda+\rho_1,\ s_1H(x)\rangle}\ d\ x .$$

We change this expression to an integral over

$$(V_1)_{s_{\alpha_1}} = w_1\cdot V_\beta\cdot w_1^{-1} ,$$

replacing the variable x by $w_1\ x\ w_1^{-1}$. We have

$$(w_1\ x\ w_1^{-1})_{N_1} = w_1\ x_N\ w_1^{-1} .$$

Similar equations hold for the A_1, M_1, and K components of $w_1\ x\ w_1^{-1}$. Since

$$\sigma(x_M) = (w_1\sigma)\ (w_1\ x_M\ w_1^{-1}) ,$$

the integral becomes

$$\int_{(V_1)_{s_{\alpha_1}}} ((w_1\sigma)\ (y_{M_1})\cdot(R(w_1 : \lambda)\ \phi)\ (y_K\cdot w_{\alpha_1}^{-1}\ k),\ \xi) .$$
$$e^{\langle s_1\lambda+\rho_1, H_1(y)\rangle}\ dy$$

$$= ((R(w_{\alpha_1} : s_1\lambda)\ R\ (w_1 : \lambda)\ \phi)\ (k),\ \xi) .$$

The proof is complete upon applying the induction hypothesis to s_1.

□

COROLLARY 4.5: Let s' be an element in $\Omega(\underline{a}', \underline{a}'')$ with a representative w' in ${}^{(o)}\widetilde{M}$. Suppose that s and w are as in the lemma, and that the length of $s's$ is the sum of the lengths of s' and s. Then

$$R(w'w : \lambda) = R(w' : s\lambda) \; R(w : \lambda).$$

□

If $\phi \in \underline{H}(\sigma)$, we write

$$(w\phi)(x) = \phi(w^{-1} x_M, x_K) \quad , \quad x \in G .$$

Then w defines a map from $\underline{H}(\sigma)$ to $\underline{H}(w\sigma)$. Given $\lambda \in T(s)$,

$$r(s : \lambda) = w^{-1} \cdot R(w : \lambda)$$

is a linear operator on $\underline{H}^o(\sigma)$. For $\phi \in \underline{H}^o(\sigma)$, $\xi \in H^o_\sigma$ and $k \in K$,

$$((r(s : \lambda)\phi)(k), \xi) = \int_{V_s} (\sigma(v_M)\phi(v_K k), \xi) \; e^{\langle \lambda+\rho, H(v)\rangle} dv ,$$

so $r(s : \lambda)$ in fact does not depend on the representative w of s in ${}^{(o)}\widetilde{M}$.

It is convenient to restate Lemma 4.4 in terms of $r(s : \lambda)$. Given $\beta \in \overline{\Sigma}$, we define the vector $\rho(P_\beta)$ in $\underline{a}_\beta$ by

$$<\rho(P_\beta),\ H> = \frac{1}{2}\ \mathrm{Tr}\ (\mathrm{ad}\ H)_{\underline{m}_\beta}\ ,\ H \in \underline{a}_\beta\ .$$

For $\phi_\beta \in \underline{H}^o_\beta\ (\sigma)$, $\xi \in H^o_\sigma$, and $k \in K^\beta$, we write

$$((r_\beta(\lambda_\beta)\ \phi_\beta)\ (k),\ \xi) = \int_{V_\beta} (\phi_\beta(v\ k),\ \xi)\ \ e^{<\lambda_\beta + \rho(P_\beta),H_\beta(v)>}\ dv,$$

where, as we recall, λ_β is the projection of λ onto $(\underline{a}_\beta)_{\mathfrak{c}}$.

Then $r_\beta(\lambda_\beta)$ is an operator on $\underline{H}^o_\beta\ (\sigma)$. If β is a simple root, it follows from Lemma 4.3 that

$$r_\beta(\lambda_\beta) = r(s_\beta : \lambda)\ .$$

We define the operator $r^G_\beta\ (\lambda)$ on $\underline{H}^o\ (\sigma)$ by setting

$$(r^G_\beta\ (\lambda)\ \phi)\ (x) = \sigma(x_M) \cdot (r_\beta(\lambda_\beta)\ \phi_{x_K})\ (1)\ ,$$

where $\phi \in \underline{H}^o\ (\sigma)$, $x \in G$, and

$$\phi_{x_K}(m) = \phi\ (m\ x_K)\ ,\quad m \in M^\beta\ .$$

<u>LEMMA 4.6</u>: Let $\lambda \in T\ (s)$, and define

$$\beta_i = s_i^{-1}\ (\alpha_i),\ 1 \leq i \leq n\ .$$

Then

$$r\ (s : \lambda) = r^G_{\beta_1}\ (\lambda)\ \ldots\ldots\ r^G_{\beta_n}\ (\lambda)\ .$$

PROOF: From Lemma 4.4,

$$r\ (s : \lambda) = w^{-1}\ R\ (w : \lambda)$$

$$= w^{-1} \cdot R \cdot (w_{\alpha_1} : s_1\lambda) \cdot w_1 \cdot r\ (s_1 : \lambda)\ .$$

Now for $\phi \in \underline{H}^{o}(\sigma)$, $\xi \in H^{o}_{\sigma}$, and $k \in K$,

$$((w^{-1} R (w_{\alpha_1} : s_1\lambda) w_1 \phi) (k), \xi)$$

$$= \int_{(V_1)_{s_{\alpha_1}}} (\phi (w_1^{-1} v w_{\alpha_1}^{-1} w k), \xi) \; e^{\langle s \lambda + \rho_1, H_1(v)\rangle} dv$$

$$= \int_{(V_1)_{\alpha_1}} (\phi (w_1^{-1} v w_1 k), \xi) \cdot e^{\langle s \lambda + \rho_{\alpha_1}, H_1(v)\rangle} dv ,$$

by Lemma 3.3. In this integral we change the variable v to $w^{-1} v w$. The integral is then taken over V_{β}. It equals

$$\int_{V_{\beta_1}} (\phi (v k), \xi) \; e^{\langle \lambda_{\beta_1} + \rho(P_{\beta_1}), H_{\beta_1}(v)\rangle} dv$$

$$= ((r^{G}_{\beta_1}(\lambda) \phi) (k), \xi) .$$

Therefore,

$$r (s : \lambda) = r^{G}_{\beta_1}(\lambda) \cdot r (s_1 : \lambda) ,$$

and the lemma follows by induction on the length of s.

□

Suppose that F is a finite subset of $\underline{E}(K)$. If T is an operator on $\underline{H}(\sigma)$ for which $\underline{H}_F(\sigma)$ is an invariant subspace, we shall write $\det_F T$ for the determinant of the restriction of T to $\underline{H}_F(\sigma)$.

<u>COROLLARY 4.7</u>: The holomorphic function

$$\lambda \longrightarrow \det_F (r(s:\lambda)), \quad \lambda \in T(s),$$

is not identically zero.

PROOF: In view of the lemma, we have only to show that for any $\beta \in \overline{\Sigma}$, the holomorphic function

$$\nu \longrightarrow \det_F (r_\beta^G(\nu)),$$

whose domain is

$$\{\nu \in (\underline{a}_\beta)_{\mathbb{C}} : \langle \nu, \beta \rangle_{\mathbb{R}} > 0\},$$

does not vanish identically. This fact is essentially the rank one version of a result in [5(f)] (Lemma 9 and its corollary). One verifies the result easily by checking that for $\phi \in \underline{H}_F(\sigma)$,

$$\lim_{\langle \lambda, \beta \rangle_R \longrightarrow \infty} r_\beta^G(\lambda)\phi = \phi .$$

□

It is known that

$$\gamma(P) = \int_V e^{\langle 2\rho, H(v) \rangle} dv$$

is finite. This fact insures better convergence properties for the integral defining $R(w:\lambda)$ whenever λ belongs to $\rho + T(s)$. It is easy to show, using a variation on the proof of Lemma 4.4, that the integral

$$\int_{V_s} e^{\langle \lambda_{\mathbb{R}} + \rho, H(v) \rangle} dv$$

equals

$$\prod_{\beta\in\bar{\Sigma}_s} \int_{V_\beta} e^{\langle\lambda_{\mathbb{R}}+\rho(P_\beta),\ H_\beta(v)\rangle}\, dv .$$

If λ is in the closure of $\rho + T(s)$, this last expression, by the first statement of ([5 (e)], Lemma 65), is bounded by

$$\prod_{\beta\in\bar{\Sigma}_s} \gamma_\beta (P_\beta) .$$

In particular, setting $\lambda = \rho$ and $s = s_\ell$, the element of greatest length in Ω (a), we find that

$$(4.5) \qquad \gamma(P) = \prod_{\beta\in\bar{\Sigma}} \gamma_\beta (P_\beta) .$$

Suppose that $s \in \Omega(\underline{a}, \underline{a}')$ and $w \in {}^{(o)}\tilde{M}$ are as above, and that λ is an arbitrary point in the closure of $\rho + T(s)$. The domain of $R(w : \lambda)$ can be extended to the set of all bounded functions

$$\phi : G \longrightarrow H_\sigma$$

in $\underline{H}(\sigma)$. For if ϕ is any such function, and $k \in K$,

$$\int_{V_s} \|\phi(v w^{-1}k) \cdot e^{\langle\lambda+\rho,\ H(v)\rangle}\|\, dv$$

is bounded by

$$\gamma(P) \cdot \sup_{x\in G} \|\phi(x)\| .$$

We can define

$$(R(w:\lambda)\phi)(k) = \int_{V_s} \phi(v w^{-1}k) \cdot e^{\langle \lambda+\rho, H(v)\rangle} dv .$$

It will be necessary to have a formula for the adjoint of $R(w:\lambda)$. Since $P'\backslash G$ is compact, we can find a function β' in $C_c^{\infty}(G)$ such that for any $x \in G$,

$$\int_{N'} \int_{A'} \int_{M'} \beta'(n m a x)\, dn\, dm\, da = 1 .$$

Then if h is any continuous function on $P'\backslash G$,

$$\int_K h(k)\, dk = \int_G \beta'(x)\, h(x) \cdot e^{\langle 2\rho', H'(x)\rangle} dx .$$

Suppose that λ belongs to $\rho + T(s)$, and that ϕ and ϕ' are bounded functions in $\underline{H}(\sigma)$ and $\underline{H}(w\sigma)$ respectively. Then $(\phi', R(w:\lambda)\phi)$ equals

$$\int_G \int_{V_s} (\phi'(k), \phi(v w^{-1}x)) \cdot e^{\langle\bar{\lambda}+\rho, H(v w^{-1}x)\rangle} e^{-\langle s\bar{\lambda}+\rho', H'(x)\rangle} dv \cdot e^{\langle 2\rho', H'(x)\rangle} \beta'(x)\, dx .$$

By our remarks above, this double integral is absolutely convergent. It equals

$$\int_{V_s} \int_G (\phi'(w v x), \phi(x)) \cdot e^{\langle\bar{\lambda}+\rho, H(x)\rangle} e^{\langle -s\bar{\lambda}+\rho', H'(w v x)\rangle} \beta'(w v x)\, dx\, dv$$

$$= \int_G (\phi'(w\,x), \phi(x)) \cdot e^{<\bar{\lambda}+\rho,\ H(x)>} \cdot e^{<-s\bar{\lambda}+\rho',\ H'(w\,x)>} \{ \int_{wV_s w^{-1}} \beta'(n'wx) dn' \} dx .$$

Define

$$N_s = w^{-1} N' w \cap N$$

and

$$\tilde{N}_s = w^{-1} V' w \cap N .$$

It is a simple matter to check that for any integrable function h on the group $N_s A M$,

$$\int_M \int_A \int_{N_s} h(m\,a\,n)\, e^{<\rho,\ H(a)>}\, e^{<\rho',\ s\,H(a)>}\, d\,n \cdot d\,a \cdot d\,m$$

$$= \int_M \int_A \int_{N_s} h(m\,a\,n)\, d\,n\, d\,a\, d\,m.$$

Therefore, rewriting our above integral over G as a multiple integral over $M \times A \times N_s \times \tilde{N}_s \times K$, we obtain

$$\int_K \int_{\tilde{N}_s} (\phi'(w\,\tilde{n}\,k), \phi(k)) \cdot e^{<-s\bar{\lambda} + \rho',\ H'(w\,\tilde{n})>}$$

$$\{ \int_{w V_s w^{-1}} \int_M \int_{N_s} \int_A \beta'(n' w\,n\,a\,m\,\tilde{n}\,k)\, d\,n' \cdot d\,n \cdot d\,a \cdot d\,m \}\, d\,\tilde{n}\, d\,k$$

$$= \int_K \int_{w\,\hat{N}_s \cdot w^{-1}} (\phi'(v'\,w\,k), \phi(k)) \cdot e^{<-s\bar{\lambda}+\rho',\ H'(v')>}\, d\,v' \cdot d\,k$$

$$= (R(w^{-1} : -s\bar{\lambda})\,\phi', \phi) .$$

We have shown that

$$(4.6) \qquad R(w:\lambda)^* = R(w^{-1} : -s\bar{\lambda})$$

for any $\lambda \in \rho + T(s)$. By analytic continuation, this formula is true for any $\lambda \in T(s)$.

Our final task of this section is to prove Lemma 4.1. The proof is straightforward but somewhat lengthy. It entails combining the inductive arguments on the length of s with some inequalities of Harish-Chandra.

The group

$$^{(o)}P_M = M \cap {}^{(o)}P$$

is a minimal parabolic subgroup of M. We write $^{(o)}\underline{a}_M$, $^{(o)}\rho_M$, $^{(o)}\underline{a}_M^+$ and so on, for the various objects associated to the pair $(M, {}^{(o)}P_M)$. If m is any element in M, there are elements $h \in \exp(-({}^{(o)}\underline{a}_M^+))$ and $k, k' \in K_M$, such that

$$m = k' h k.$$

Notice that

$$\Xi_M(m) = \Xi_M(h)$$

and

$$\sigma(m) = \sigma(h).$$

In addition, for $v \in V_s$,

$$\Xi_M(m\ v_M) \cdot (1 + \sigma\ (m\ v_A))^{-r}$$

$$= \Xi_M(h \cdot k\ v_M k^{-1}) \cdot (1 + \ (h \cdot k\ v_M\ k^{-1}))^{-r}$$

$$= \Xi_M(h \cdot (k\ v\ k^{-1})_M) \cdot (1 + \sigma\,(h \cdot (k\ v\ k^{-1})_M))^{-r}$$

Since

$$(k\ v\ k^{-1})_A = v_A\ ,$$

we can alter the expression (4.1) by replacing m by h and v by $k\ v\ k^{-1}$. The map

$$v \longrightarrow k\ v\ k^{-1}\ , \quad v \in V_s\ ,$$

is a measure preserving transformation of V_s. Therefore, to prove Lemma 4.1, we may assume that $m = h$, where $-\log\,(h)$ belongs to ${}^{(o)}\mathfrak{a}_M^+$.

If we combine the second statement of [5 (e)], Lemma 90, with [5 (c)], Lemma 43, we find that there are constants c_o and c_1 such that for any $x \in M$,

$$\Xi_M\ (x) \leq c_o \cdot e^{<{}^{(o)}\rho_M,\,{}^{(o)}H_M(x)>} \cdot (1 + \sigma\,(x))^{c_1}\ .$$

In particular, for h and v as above,

$$\Xi_M\ (h\ v_M) \leq c_o \cdot e^{<{}^{(o)}\rho_M,\,{}^{(o)}H_M(h)>} \cdot e^{<{}^{(o)}\rho_M,\,{}^{(o)}H_M(v_M)>} \cdot (1 + \sigma(h\ v_M))^{c_1}$$

Take r to be any fixed positive number, and let $r_1 = r - c_1$. Then the expression (4.1), with m replaced by h, is bounded by

$$c_o \cdot e^{\langle {}^{(o)}\rho_M,\ {}^{(o)}H_M(h)\rangle} \cdot e^{\langle {}^{(o)}\rho,\ {}^{(o)}H(v)\rangle} \cdot (1 + \sigma(h\, v_M))^{-r_1} \ (1 + \sigma(v_A))^d$$

$$\sup_{\lambda \in \Gamma} \{e^{\langle \lambda,\ H(v)\rangle}\} .$$

Let ${}^{+}\underline{a}(s)$ be the closed subset of $\underline{a}$ consisting of nonnegative linear combinations of the roots in $\bar{\Sigma}_s$. We write ${}^{+}\underline{a}$ for the set of nonnegative linear combinations of the roots in $\bar{\Sigma}$. It is clear that there is a constant c_2 such that for any $\lambda \in \Gamma$, and $X \in {}^{+}\underline{a}(s)$,

$$(1 + \langle \lambda, X\rangle)^{-(r_1+d)} \leq c_2(1 + \sigma(\exp X))^{-(r_1+d)}$$

We shall show that $H(v)$ belongs to $-({}^{+}\underline{a}(s))$ for each $v \in V_s$. We assume inductively that this fact is true if s is replaced by any element in $\Omega(\underline{a})$ of length less than that of s. Let $s = s_{\alpha_1} \cdot s_1$, where s_{α_1} is a simple reflection and s_1 is an element in $\Omega(\underline{a})$ whose length is less than the length of s. By Lemma 3.2, $\bar{\Sigma}_s$ is the disjoint union of $\bar{\Sigma}_{s_1}$ and $\beta = s_1^{-1}(\alpha_1)$. If v is an element in V_s, we write

$$v = v_1\, x \ , \quad v_1 \in V_{s_1} \ , \ x \in V_\beta .$$

As we saw in the proof of Lemma 4.4, there are elements $n \in N$, $\tilde{v}_1 \in V_{s_1}$

and $k \in K$, such that

$$v_1 x = n\, x_M\, x_A\, \hat{v}_1\, k .$$

In particular,

$$H(v) = H(x) + H(\hat{v}_1) .$$

Now

$$H(x) = c(x)\ \beta ,$$

where $c(x)$ is a real number which, by [5 (e)], Lemma 85, is less than or equal to zero. Our assertion now follows from tbe induction hypothesis.

It follows that for $\lambda \in \underline{a}^+(s)$,

$$\langle \lambda , H(v) \rangle \leq 0 .$$

In particular, there is a constant c_3 such that for any $\lambda \in \underline{a}^+(s)$,

$$e^{\langle \lambda, H(v) \rangle} \leq c_3\ (1 - \langle \lambda, H(v) \rangle)^{-(r_1+d)} .$$

If $\lambda \in \Gamma$,

$$(1 + \sigma(v_A))^d \cdot e^{\langle \lambda, H(v) \rangle}$$

is bounded by

$$c_2\, c_3\, (1 + \sigma(v_a))^{-r_1}$$

Now

$$(1 + \sigma(h\, v_M))^{-r_1}\ (1 + \sigma(v_A))^{-r_1}$$

$$\leq \qquad (1 + \sigma(h\, v_M) + \sigma(v_A))^{-r_1} .$$

By [5 (e)], Lemma 10, this last expression is bounded by

$$(1 + \sigma(h\, v_M\, v_A))^{-r_1}.$$

We write

$$h\, v_M\, v_A = n \cdot h \cdot {}^{(o)}A(v) \cdot k, \qquad {}^{(o)}A(v) \in {}^{(o)}A,\ n \in {}^{(o)}N,\ k \in K.$$

It follows from [5 (e)], Lemma 90, that there is a constant c_4 such that

$$(1 + \sigma(h\, v_M\, v_A))^{-r_1} \leq c_4 \cdot (1 + \sigma(h \cdot {}^{(o)}A(v)))^{-r_1}.$$

Since

$$-{}^{(o)}H(v) = -\log({}^{(o)}A(v))$$

belongs to ${}^{+}({}^{(o)}\underline{a})$, the number

$$-\langle {}^{(o)}\rho, {}^{(o)}H(h \cdot {}^{(o)}A(v))\rangle = -\langle {}^{(o)}\rho_M, {}^{(o)}H_M(h)\rangle - \langle {}^{(o)}\rho, {}^{(o)}H(v)\rangle$$

is positive. It certainly can be bounded by a constant multiple of

$$|{}^{(o)}H(h \cdot {}^{(o)}A(v))| = \sigma(h \cdot {}^{(o)}A(v)).$$

Therefore, there exists a constant c_5 such that

$$(1 + \sigma(h \cdot {}^{(o)}A(v)))^{-r_1}$$

$$\leq c_5 (1 - \langle {}^{(o)}\rho_M, {}^{(o)}H_M(h)\rangle - \langle {}^{(o)}\rho, {}^{(o)}H(v)\rangle)^{-r_1}$$

$$\leq c_5 (1 - \langle {}^{(o)}\rho_M, {}^{(o)}H_M(h)\rangle)^{-r_1/2} (1 - \langle {}^{(o)}\rho, {}^{(o)}H(v)\rangle)^{-r_1/2}.$$

By [5 (c)], Lemma 36, there is a constant c_6 such that for any H in $-\underline{a}_M^{+}$,

$$e^{\langle {}^{(o)}\rho_M,\ H\rangle} \leq c_6 \quad \Xi_M\ (\exp H)\ .$$

We can also find another constant c_7 such that for any such H

$$(1 - \langle {}^{(o)}\rho_M,\ H\rangle)^{-r_1/2} \leq c_7 \quad (1 + \sigma\ (\exp H))^{-r_1/2}\ .$$

In other words, for h as above,

$$e^{\langle {}^{(o)}\rho_M,\ {}^{(o)}H_M\ (h)\rangle} \quad (1 - \langle {}^{(o)}\rho_M,\ {}^{(o)}H_M(h)\rangle)^{-r_1/2}$$

$$\leq \ c_6 \cdot c_7 \cdot \Xi_M(h) \cdot (1 + \sigma\ (h))^{-r_1/2}\ .$$

We have shown that the expression

(4.1), with m replaced by h, is bounded by a constant multiple of the product of

$$(4.7) \qquad e^{\langle {}^{(o)}\rho,\ {}^{(o)}H\ (v)\rangle} \quad (1 - \langle {}^{(o)}\rho,\ {}^{(o)}H\ (v)\rangle)^{-r_1/2}$$

and

$$\Xi_M\ (h) \cdot (1 + \sigma(h))^{-r_1/2}$$

To complete the proof of the lemma, we need only show that for suitably large r_1, the function of v defined by (4.7) is integrable over V_s .

Let w be a representative of s in ${}^{(o)}\tilde{M}$. The restriction of $Ad\ (w)$ to ${}^{(o)}\underline{a}_M$ maps ${}^{(o)}\underline{a}_M^+$ onto a Weyl chamber c' in ${}^{(o)}\underline{a}_{M'}$. By Lemma 3.1, we can choose an element w' in M_K' such that $Ad\ (w')$ maps c' onto ${}^{(o)}\underline{a}_{M'}^+$. Then

$$(w'w)^{-1} \cdot {}^{(o)}N_{M'} \cdot (w'\ w) = {}^{(o)}N_M\ ,$$

where

$$^{(o)}N_{M'} = {}^{(o)}N \cap M' .$$

It follows that

$$(w'w)^{-1} \cdot {}^{(o)}N \cdot (w' w) = (w'w)^{-1} \cdot N' \cdot {}^{(o)}N_{M'} \cdot (w'w) = w^{-1} N' w \cdot {}^{(o)}N_M .$$

Let ${}^{(o)}s$ be the restriction of $A d (w'w)$ to ${}^{(o)}\underline{a}$. Then the group

$$^{(o)}V_{({}^{(o)}s)} = (w'w)^{-1} \cdot {}^{(o)}N \cdot (w' w) \cap {}^{(o)}V$$

equals V_s. Thus, in order to prove the integrability of the function (4.7) we may assume that P is the minimal standard parabolic subgroup of $\hat{G}$.

We need to show that for $P = {}^{(o)}P$ and $s \in \Omega$,

$$\int_{V_s} e^{<\rho, H (v)>} (1 - <\rho, H(v)>)^{-m} dv$$

is finite whenever n is a suitably large positive number. Assume that this fact is true whenever s is replaced by any element in Ω whose length is less than that of s. In the integral we change the variables of integration the way we did in the proof of Lemma 4.4. Using the notation of Lemma 4.4, we obtain

$$\int_{V_s} e^{<\rho, H (v)>} (1- <\rho, H (v)>)^{-n} d v$$

$$= \int_{V_\beta} \int_{V_{s_1}} e^{<\rho, H (v_1 x)>} (1 - <\rho, H (v_1 x)>)^{-1} d v_1 d x$$

$$\int_{V_\beta}\int_{V_{s_1}} e^{<\rho,\, H(v)>}\, e^{<s_1^{-1}\rho,\, H(x)>}\,(1 - <\rho,\, H(v)> - <\rho,\, H(x)>)^{-n}\, dv\, dx\,.$$

This expression is bounded by a constant multiple of the product of

$$\int_{V_{s_1}} e^{<\rho,\, H(v)>}\,(1 - <\rho,\, H(v)>)^{-n/2}\, dv$$

and

$$\int_{V_\beta} e^{<s_1^{-1}\rho,\, H(x)>}\,(1 - <\rho,\, H(x)>)^{-n/2}\, dx\,.$$

For large n, the first integral is finite by our induction hypothesis. The second integral equals

$$(4.8)\qquad \int_{V_{s_{\alpha_1}}} e^{<\rho,\, H(x)>}\,(1 - <s_1\rho,\, H(x)>)^{-n/2}\, dx\,,$$

where, as we recall, s_{α_1} is the reflection corresponding to the simple root $\alpha_1 = s_1(\beta)$ of (P, A). Note that

$$<s_1\rho,\, H(x)> = c<\alpha_1,\, H(x)>,$$

where

$$c = <s_1\rho,\, \alpha_1>\cdot<\alpha_1,\alpha_1>^{-1} = <\rho,\, \beta>\cdot<\alpha_1,\alpha_1>^{-1}$$

is a positive number. If we employ the argument used to prove Lemma 4.3, we can interpret (4.8) as an integral associated with M^{α_1}, a group of real rank one. The convergence, for large n, of the resultant integral is a special case of [5(e)], Lemma 89.

We have just shown that for large r_1 the expression (4.7) is an integrable function of v. This completes the proof of Lemma 4.1. ☐

§5. THE EISENSTEIN INTEGRAL

Suppose P is a standard cuspidal subgroup of G and that τ is a double representation of K on a finite dimensional Hilbert space V. Then τ_M is the double representation of $K_M = K \cap M$ on V obtained by restricting τ to K_M. Define $\underline{C}(M, \tau_M)$ to be the set of functions ψ in $\underline{C}(M) \otimes V$ which are τ_M-spherical; that is, such that

$$\psi(k_1 m k_2) = \tau(k_1)\psi(m)\tau(k_2), \quad m \in M,\ k_1, k_2 \in K_M .$$

We also define

$$\underline{C}_o(M, \tau_M) = (\underline{C}_o(M) \otimes V) \cap \underline{C}(M, \tau_M),$$

$$\underline{C}_{\{\mu\}}(M, \tau_M) = (\underline{C}_{\{\mu\}}(M) \otimes V) \cap \underline{C}(M, \tau_M), \quad \mu \in \Lambda'(M),$$

and

$$\underline{C}_\omega(M, \tau_M) = (\underline{C}_\omega(M) \otimes V) \cap \underline{C}(M, \tau_M), \omega \in \underline{E}_2(M) .$$

These last three spaces are finite dimensional Hilbert spaces under the norm

$$||\psi||^2 = \int_M |\psi(m)|^2\, dm .$$

Fix $\psi \in \underline{C}_o(M, \tau_M)$. Extend the domain of ψ to G by

$$\psi(n\,a\,m\,k) = \psi(m)\,\tau(k), \quad n \in N,\ a \in A,\ n \in N,\ k \in K.$$

For $x \in G$, and $\lambda \in \underline{a}_c$, the <u>Eisenstein integral</u> is defined by

$$E(\psi : \lambda : x) = \int_K \tau(k^{-1})\psi(kx)\, e^{\langle \lambda+\rho,\, H(kx)\rangle}\, dk.$$

The importance of the Eisenstein integral is that it is closely related to the matrix coefficients of the representations defined in §2.

First of all we define an infinite dimensional double representation ρ of K. ρ acts on the Hilbert space $L^2(K\times K)$ by

$$(\rho(k_1')\, h\, \rho(k_2'))(k_1,k_2) = h(k_1k_1',\ k_2'\, k_2),$$

for $h \in L^2(K\times K)$, and $k_1, k_2, k_1', k_2' \in K$. Suppose that F is a finite subset of $\hat{\underline{E}}(K)$. Let V_F be the subspace of $L^2(K\times K)$ on which both the left and right actions of ρ decompose as a sum of classes in F. Let ρ_F be the restriction of ρ to V_F. ρ_F is a finite dimensional double representation of K.

Suppose that for $i = 1, 2$, τ_i is an irreducible representation of K, and h_i is a matrix coefficient of τ_i. Then if

$$h(k_1, k_2) = h_1(k_1)\cdot h_2(k_2), \quad k_1, k_2 \in K,$$

the Schur orthogonality relations gives the inequality

$$|h(k_1, k_2)|^2 \le \deg \tau_1 \cdot \deg \tau_2\ \ |h|^2, \quad k_1, k_2 \in K.$$

The right hand side is certainly bounded by the product of $|h|^2$ and a fixed polynomial in $|\tau_1| + |\tau_2|$. From this fact it follows easily that there is a polynomial P such that for any finite subset F of $\hat{\underline{E}}(K)$, and any $h \in V_F$, (5.1). $|h(k_1, k_2)| \leq P(|\rho_F|) \cdot |h|$, $k_1, k_2 \in K$.

Suppose that $\omega \in \underline{E}_2(M)$, and that σ is a representation in the class ω which acts on the Hilbert space H_σ . Let F be a finite subset of $\underline{E}(K)$ and let T be an operator in $\mathrm{End}(\underline{H}_F(\sigma))$. We regard T as an endomorphism on all of $\underline{H}(\sigma)$, and we define a function

$$K_T(k_1, k_2) , k_1, k_2 \in K,$$

with values in $\mathrm{End}(H_{\sigma,F})$ such that

(i), for $m_1, m_2 \in K \cap M$,

$$K_T(k_1 m_1, m_2 k_2) = \sigma(m_1)^{-1} K_T(k_1, k_2) \sigma(m_2)^{-1} ,$$

and

(ii), for any $\phi \in \underline{H}(\sigma)$,

$$(T\phi)(k_1) = \int_K K_T(k_1^{-1}, k_2) \phi(k_2) \, dk_2 .$$

These conditions specify K_T uniquely. In fact if $\{\phi_1, \ldots, \phi_n\}$ is an orthonormal basis of $\underline{H}_F(\sigma)$, and $\xi \in H_{\sigma;F}$,

$$K_T(k_1^{-1}, k_2)\xi = \Sigma_i (\xi, \phi_i(k_2)) \cdot (T\phi_i)(k_1) .$$

If T is any operator on a Hilbert space, we shall always write $||T||_2$ for the Hilbert-Schmidt norm of T. Now if $T \in \mathrm{End}(\underline{H}_F(\sigma))$, this norm makes $\mathrm{End}(\underline{H}_F(\sigma))$ into a finite dimensional Hilbert space

For $m \in M$, the operator $\sigma(m)\, K_T\, (k_2^{-1}, k_1^{-1})$ is of finite rank on H_σ. We define a function ψ_T from M to V_F by

$$(5.2)\quad \psi_T\,(k_1 : m : k_2) = \psi_T\,(m)(k_1, k_2)$$

$$= \mathrm{Tr}\,\{\sigma(m)\, K_T\, (k_2^{-1}\,, k_1^{-1})\}\,.$$

Let $\rho_{F,M}$ be the restriction of ρ_F to M. Recall that d_σ is the formal degree of σ.

LEMMA 5.1: The map

$$T \longrightarrow d_\sigma^{\frac{1}{2}}\ \psi_T\ ,\ T \in \mathrm{End}\ (\underline{H}_F(\sigma)),$$

is a linear isometry from $\mathrm{End}\ (\underline{H}_F(\sigma))$ onto $\underline{C}_\omega(M, \rho_{F,M})$.

PROOF: We must first of all check that ψ_T is a $\rho_{F,M}$ - spherical function. For $m_1, m_2 \in K_M$,

$$(\rho_F(m_1)\ \psi_T(m)\ \rho_F(m_2))(k_1,k_2)$$

$$= \mathrm{Tr}\ \{\sigma(m)\ K_T\ (k_2^{-1}\, m_2^{-1},\, m_1^{-1}\ k_1^{-1})\}$$

$$= \mathrm{Tr}\ \{\sigma(m)\ \sigma(m_2)\ K_T\ (k_2^{-1}\ , k_1^{-1})\ \sigma(m_1)\}$$

$$= \mathrm{Tr}\ \{\sigma(m_1\, m\, m_2) \cdot K_T(k_2^{-1}, k_1^{-1})\}$$

$$= \psi_T\ (k_1 : m_1\, m\, m_2 : k_2)\ .$$

Secondly, we observe that the kernal K_T, and hence the operator T, may be recovered from ψ_T. In fact

$$K_T(k_2^{-1}, k_1^{-1})$$
$$= d_\sigma \cdot \int_M \psi_T(k_1 : m : k_2) \cdot \sigma(m^{-1})\, d\, m .$$

Therefore our map is surjective. Finally, $d_\sigma ||\psi_T||_2^2$ equals

$$\int_{K \times K} \int_M d_\sigma \,|\, \mathrm{Tr} \,\{ \sigma(m)\, K_T(k_2^{-1}, k_1^{-1})\}|^2 \, d\, m \;\; d\, k_1 d\, k_2 .$$

From the Schur orthogonality relations, this is just

$$\int_{K \times K} ||K_T\, (k_2^{-1}, k_1^{-1})||_2^2 \, d\, k_1 \, d\, k_2$$
$$= ||T||_2^2 .$$

Therefore our map is an isometry. □

LEMMA 5.2: $E\,(\psi_T : \lambda : x)_{(1,1)} = \mathrm{Tr}\,\{\pi(\sigma, \lambda : x)\, T\}$.

PROOF: First of all, notice that if $\{\xi_\alpha\}$ is an orthonormal basis for H_σ, and $\{\phi_1, \ldots, \phi_n\}$ is an orthonormal basis of $\underline{H}_F(\sigma)$,

$$\psi_T(k_1 : m : k_2)$$
$$= \Sigma_{i,\alpha}\, (\sigma(m)\, (T\, \phi_i)(k_2), \xi_\alpha) \cdot (\xi_\alpha, \phi_i\, (k_1^{-1}))$$
$$= \Sigma_i\, (\sigma(m)\, (T\, \phi_i)\, (k_2), \phi_i\, (k_1^{-1})) .$$

We have

$$E(\psi_T : \lambda : x)_{(1,1)}$$
$$= \int_K (\tau(k^{-1})\, \psi_T(k\,x))_{(1,1)} \cdot e^{\langle\lambda+\rho,\, H(kx)\rangle}\, dk$$
$$= \int_K \psi_T(k^{-1} : (kx)_M : (kx)_K)\; e^{\langle\lambda+\rho,\, H(kx)\rangle}\, dk$$
$$= \int_K \Sigma_i\, (\sigma((kx)_M)\, (T\, \phi_i)((kx)_K),\, \phi_i(k)) \cdot e^{\langle\lambda+\rho,\, H(kx)\rangle}\, dk$$
$$= \int_K \Sigma_i((\pi(\sigma, \lambda : x)\, T\, \phi_i)(k),\, \phi_i(k))$$
$$= \mathrm{Tr}\,\{\pi(\sigma, \lambda : x)\, T\}\ . \qquad \square$$

Let $\mathrm{End}^{\circ}(\underline{H}(\sigma))$ be the union, over all finite subsets F of $\underline{E}(K)$, of the spaces $\mathrm{End}\ \underline{H}_F(\sigma)$. The map

$$T \longrightarrow \psi_T,\ T \in \mathrm{End}^{\circ}(\underline{H}(\sigma)),$$

has been used by Harish-Chandra to prove some of the results of [5(f)]. As we shall see later, it also provides a convenient framework for proving the theorem alluded to in the introduction.

Suppose that T is an arbitrary representation in $F(K,K)$, and that ψ is a function in $\underline{C}_o(M, \tau_M)$. It is easy to convince oneself that there is a finite subset F of $\underline{E}(K)$, a subrepresentation τ' of τ with $\psi \in \underline{C}_o(M, \tau_M')$, and a unitary intertwining operator between τ' and a subrepresentation of ρ_F. In other words, in studying the Eisenstein integral, we can take τ to equal ρ_F if we wish.

<u>LEMMA 5.3</u>: Suppose that τ is a double representation in $F(K,K)$, acting on V, and that Y_1 and Y_2 are elements in $\underline{G}$. Then there is a double representation $\tau' \in F(K,K)$, acting on the vector space

V', and a polynomial p, depending only on Y_1 and Y_2, so that

$$|\tau'| \leq p(|\tau|),$$

and such that the following property is satisfied: given $\omega \in \underline{E}_2(M)$, $\psi \in \underline{C}_\omega(M, \tau_M)$, $\lambda \in \underline{a}_c$, and $\xi \in V^*$, the dual space of V, there exist $\psi' \in \underline{C}_\omega(M, \tau'_M)$ and $\xi' \in (V')^*$ such that

(i) $||\psi'|| \leq p(|\omega| + |\lambda| + |\tau| \cdot ||\psi||$,

(ii) $|\xi'| \leq p(|\tau|) \cdot |\xi|$,

and

(iii) $\langle\xi, E(\psi : \lambda : Y_1 ; x; Y_2)\rangle = \langle\xi', E(\psi' : \lambda : x)\rangle$.

PROOF: We may assume that $\tau = \rho_F$, for some finite subset F of $\underline{E}(K)$, and that

$$\langle\xi, \phi\rangle = \phi(k_1, k_2), \quad \phi \in V_F,$$

for some pair of points k_1 and k_2 in K. Then $\psi = \psi_T$ for some $T \in \text{End}(\underline{H}_F(\sigma))$, and σ a fixed representation in the class ω. We have

$$\begin{aligned}
&\langle\xi, E(\psi_T : \lambda : Y_1 ; x; Y_2)\rangle \\
= \quad & E(\psi_T : \lambda : Y_1 ; x; Y_2)(k_1, k_2) \\
= \quad & \text{Tr}\,\{\pi(\sigma, \lambda : k_1) \cdot \pi(\sigma, \lambda : Y_1)\, \pi(\sigma, \lambda : x)\, \pi(\sigma, \lambda : Y_2) \\
& \qquad \pi(\sigma, \lambda : k_2) T\} \\
= \quad & \text{Tr}\,\{\pi(\sigma, \lambda : x)\, T'\} \\
= \quad & E(\psi_{T'} : \lambda : x)(1,1),
\end{aligned}$$

where

$T' = \pi(\sigma, \lambda : Y_2) \quad \pi(\sigma, \lambda : k_2) \cdot T \cdot \pi(\sigma, \lambda : k_1) \quad \pi(\sigma, \lambda : Y_1)$.

Let $\underline{G}_n$ be the space of elements in $\underline{G}$ whose degree is no greater than n. We assume that n is large enough that Y_1 and Y_2 both belong to $\underline{G}_n$. Suppose η_n is the adjoint representation of K on $\underline{G}_n$. If η is some representation in F, let $F'(\eta)$ be the set of elements in $\underline{E}(K)$ which occur in the decomposition of $\eta \otimes \eta_n$. Let F' be the union of the $F'(\eta)$ over all $\eta \in F$. We let τ' be the double representation $\rho_{F'}$. It follows from the remarks and definitions of §2, (in particular, the formulas (2.4) and (2.5)), that there is a polynomial p such that

$$|\rho_{F'}| \leq p(|\rho_F|) .$$

The operator T' belongs to End $(\underline{H}_{F'}(\sigma))$, so that $\psi_{T'}$ lies in $\underline{C}_\omega(M, \rho_{F',M})$. We let $\psi' = \psi_{T'}$. Define $\xi' \in (V_{F'})^*$ by

$$\langle\xi', \phi\rangle = \phi(1, 1) , \quad \phi \in V_{F'} .$$

If η' is the representation in F' of largest degree, it follows from the Schur orthogonality relations that

$$|\xi'| = \deg \eta' .$$

This is certainly bounded by a polynomial in $|\eta'|$, and hence by a polynomial, p, in $|\rho_F|$. Since $|\xi| \geq 1$, again by the Schur orthogonality relations, we have

$$|\xi'| \leq |\xi| \cdot p(\rho_F|) .$$

Our final task is to estimate $||\psi_{T'}||$, which by Lemma 5.1 equals $d_\sigma^{-\frac{1}{2}} \cdot ||T'||_2$. Let P' be the orthogonal projection of $\underline{H}(\sigma)$ onto $\underline{H}_{F'}(\sigma)$. Define

$$B_i = P' \cdot \pi(\sigma, \lambda : Y_i) \cdot P' \ , \ i = 1, 2,$$

and

$$\tilde{T} = \pi(\sigma, \lambda : k_2) \cdot T \cdot \pi(\sigma , \lambda : k_1) .$$

Then

$$T' = B_2 \ \tilde{T} \cdot B_1,$$

and

$$||T'||_2 \leq ||B_2|| \cdot ||\tilde{T}||_2 \cdot ||B_1|| = ||B_2|| \cdot ||T||_2 \cdot ||B_1||.$$

We have only to estimate the uniform operator norms

$$||B_i|| \ , \ i = 1, 2.$$

At this stage we must make use of a result of E. Nelson ([8], Lemma 6.3). Let $\{Y_1, \ldots, Y_d\}$ be an orthonormal basis of $\underline{g}$ with respect to the bilinear form

$$(X,Y) \longrightarrow - B(X, \Theta Y) \ , \ X, Y \in \underline{g} .$$

We assume that a subset of this basis forms a basis of $\underline{k}$. Define an element

$$V = I - (Y_1{}^2 + \ldots + Y_d{}^2)$$

in $\underline{G}$. It is easy to see that if Z_G and Z_K are the elements defined in §1,

$$V = 2 Z_K - Z_G + \gamma I ,$$

for some real number γ. When applied to our situation, Nelson's lemma affirms the existence of a constant C, depending only on Y_1 and Y_2, such that for any differentiable vector $\phi \in \underline{H} (\sigma)$,

$$|| \pi(\sigma, \lambda : Y_i) \phi || \leq C ||\pi (\sigma, \lambda : V^n) \phi|| \ , \ i = 1, 2.$$

If η is a class in $\underline{E}(K)$, and ϕ is a unit vector in $\underline{H}_{\{\eta\}}(\sigma)$, ϕ is differentiable. Furthermore, by (2.6),

$$||\pi(\sigma, \lambda : V^n)\,\phi|| = (2|\eta|^2 - |\sigma|^2 - \langle\lambda, \lambda\rangle + \gamma)^n .$$

From these properties, and the fact that for $\eta \in F^{\gamma}$, $|\eta|$ is bounded by a polynomial in $|\rho_F|$, we see that there is a polynomial p such that

$$||B_i|| \leq p(|\sigma| + |\lambda| + |\rho_F|) .$$

Therefore, by Lemma 5.1,

$$||\psi^{\gamma}|| = ||\psi_{T\gamma}|| \leq p(|\sigma| + |\lambda| + |\rho_F|) \cdot ||\psi_T|| .$$

With this inequality our proof is complete.

$\square$

<u>LEMMA 5.4:</u> Suppose that Y_1 and Y_2 are elements of $\underline{G}$. Then there is a polynomial p and a constant c_o, such that for $\tau \in F(K, K)$, $\omega \in \underline{E}_2(M)$, $\psi \in \underline{C}(M, \tau_M)$, $\lambda \in \underline{a}_c$, and $x \in G$, $|E(\psi : \lambda : Y_1 ; x; Y_2)|$

is bounded by

$$||\psi|| \cdot p(|\omega| + |\lambda| + |\tau|) \cdot \Xi(x) \cdot e^{c_o |\lambda_R| \cdot (1 + \sigma(x))} .$$

PROOF: First of all we shall estimate $\psi(m)$. Suppose that τ acts on the vector space V, and that $\xi \in V^*$. Then the function

$$m \longrightarrow \langle \xi, \psi(m)\rangle, \; m \in M,$$

lies in $\underline{C}_{\omega}(M)$. We apply the Sobolev lemma([10], Theorem 3.3) of Trombi and Varadarajan. There are elements $\{X_1^i, X_2^i \mid 1 \leq i \leq r\}$ in the universal enveloping algebra of $\underline{m}_a$ such that

$$\sup_{m \in M} \{ \Xi_M(m)^{-1} \quad |\langle \xi, \psi(m) \rangle| \}$$

is bounded by the sum over i of

(5.3) $$\left(\int_M |\langle \xi, \psi(X_1^i ; m ; X_2^i) \rangle|^2 \, d\, m \right)^{\frac{1}{2}} .$$

In the previous lemma replace G by M and consider the cuspidal subgroup M of M. We see that (5.3) is bounded by

$$\left(\int_M |\langle \xi' , \psi'(m) \rangle|^2 \, d\, m \right)^{\frac{1}{2}}$$
$$\leq ||\xi'|| \cdot ||\psi'||$$
$$\leq ||\xi|| \cdot ||\psi|| \cdot p(|\tau_M|) \cdot p(|\omega| + (|\tau_M|) .$$

Recalling the formula (2.6), we conclude that there is a polynomial p_1 such that

$$\sup_{m \in M} \{ \Xi_M(m)^{-1} \quad |\psi(m)| \} \leq p_1(|\omega| + |\tau|) \cdot ||\psi|| .$$

To go on with the proof of the lemma we note that Lemma 5.3 allows us to assume that $Y_1 = Y_2 = 1$. We have

$$|E\,(\psi : \lambda : x)|$$
$$= | \int_K \tau(k^{-1}) \quad \psi(k\ x) \cdot e^{\langle \lambda + \rho, H(k\ x) \rangle} \, d\ k|$$
$$\leq \int_K |\psi(k\ x)| \cdot e^{\langle \lambda_R + \rho,\ H(kx) \rangle} \quad d\ k .$$

If we define

$$\Xi_M(x) = \Xi_M(x_M), \ x \in G,$$

it is easily verified, ([5(e)], Corollary, Lemma 84), that

$$\Xi\,(x) = \int_K \Xi_M(k\ x) \cdot e^{\langle H(k\ x), \rho \rangle} \, d\ k.$$

It follows that $|E(\psi : \lambda : x)|$ is bounded by

$$||\psi|| \cdot p_1(|\omega| + |\tau|) \cdot \Xi(x) \left(\sup_{k \in K} e^{\langle \lambda_{\mathbb{R}}, H(k\,x)\rangle}\right).$$

Now it is an immediate consequence of [5(e)], Lemma 90, that

$$|H(x)| \leq c_o (1 + \sigma(x)),\ x \in G,$$

for some constant c_o. Therefore

$$\sup_{k \in K} e^{\langle \lambda_{\mathbb{R}}, H(k\,x)\rangle}$$

$$\leq \sup_{k \in K} e^{c_o |\lambda_{\mathbb{R}}| \cdot (1 + \sigma(k\,x))} = e^{c_o |\lambda_{\mathbb{R}}| \cdot (1 + \sigma(x))}.$$

This finishes our proof. □

We shall also need a variant of this lemma which we state as a corollary.

<u>COROLLARY 5.5</u>: Suppose that $D = D_\lambda$ is a differential operator with constant coefficients on the real vector space $i\,\underline{a}$. Then there is a polynomial p and a constant m such that for $\lambda \in i\,\underline{a}$, $x \in G$, and ψ as in the lemma, $|D_\lambda\, E(\psi : \lambda : x)|$ is bounded by

$$||\psi|| \cdot p(|\omega| + |\lambda| + |\tau|) \cdot (1 + \sigma(x))^m \cdot \Xi(x).$$

PROOF: Let q be the polynomial function on $\underline{a}$ such that

$$|D_\lambda\, E(\psi : \lambda : x)|$$

$$= \left|\int_K q(H(k\,x)) \cdot \tau(k^{-1})\psi(k\,x) \cdot e^{\langle \lambda + \rho, H(k\,x)\rangle}\, dk\right|.$$

This expression is bounded by

$$\int_K |\varphi(H(k\,x))| \cdot |\chi(k\,x)| \cdot e^{\langle \rho, H(k\,x)\rangle}\, d\,k.$$

Choose constants C and m such that for all k and x,

$$|\varphi(H(k\,x))| \leq C(1+\sigma(k\,x))^m = C(1+\sigma(x))^m .$$

Our corollary can be now proved by following the last part of the proof of the lemma. □

§6. THE MAIN ESTIMATES

The basis for our paper, and indeed for much of the harmonic analysis on G, is Harish-Chandra's asymptotic estimates for tempered τ-spherical eigenfunctions of $\underline{Z}$, ([5(e)], Part II). We want to study the dependence of these estimates on τ as well as on the $\underline{Z}$-eigenvalues. This has been done in [10], but only for the case that the eigenfunctions are square integrable. Accordingly, we review Harish-Chandra's work.

First of all, we summarize the results of Chevalley [3], and Harish-Chandra [5(c)], on finite groups generated by reflections. Suppose that ${}^{(u)}P$, $u \in \underline{J}$, is a standard parabolic subgroup of G. The Weyl groups, W and ${}^{(u)}W$, of $(\underline{g}_{\mathbb{C}}, {}^{(o)}\underline{h}_{\mathbb{C}})$ and $({}^{(u)}\underline{m}_{\mathbb{C}} + {}^{(u)}\underline{a}_{\mathbb{C}}, {}^{(o)}\underline{h}_{\mathbb{C}})$ act on ${}^{(o)}\underline{h}_{\mathbb{C}}$, and are both generated by reflections. They preserve the symmetric, nondegenerate bilinear form $\langle\ ,\ \rangle$. It is just these assumptions on ${}^{(o)}\underline{h}_{\mathbb{C}}$, W, ${}^{(u)}W$ and $\langle\ ,\ \rangle$ for which Harish-Chandra's results are valid, although they stated in [5(c)] in a slightly less general setting.

The groups W and ${}^{(u)}W$ act on S, the symmetric algebra on ${}^{(o)}\underline{h}_{\mathbb{C}}$. Let J and ${}^{(u)}J$ be the subalgebras of S which are invariant under W and ${}^{(u)}W$ respectively. If ${}^{(u)}W$ is of index r in W, there are homogenous elements

$$v_1 = 1,\ v_2,\ \ldots\ldots,\ v_r$$

which form a basis of ${}^{(u)}J$ as a free J-module, ([5(c)], Lemma 8). It is clear that for any $v \in {}^{(u)}J$, we can choose elements

$$z_{v,ij} \quad , \; i \leq 1, \; j \leq r \; ,$$

in J such that

$$v \, v_j = \Sigma_i \, z_{v,ij} v_i \quad , \quad j = 1, \ldots, r.$$

Suppose that Λ is any regular element in ${}^{(o)}\underline{h}_{\mathbb{C}}$. This means that Λ is not a fixed point of any of the mappings in W. Let J_Λ be the set of elements $z \in J$ such that $\langle z , \Lambda \rangle = 0$. ${}^{(u)}J \, J_\Lambda$ is an ideal in ${}^{(u)}J$. For any $v \in {}^{(u)}J$, let v^* be the projection of v onto ${}^{(u)}J \, / {}^{(u)}J \, J_\Lambda$. Then by [5(c)], Lemma 13, $v_1^*, \ldots, v_r^*$ is a basis of the $\mathbb{C}$-vector space ${}^{(u)}J \, / {}^{(u)}J \, J_\Lambda$. It is convenient to fix a Hilbert space E, independent of Λ, with orthonormal basis $\{e_1, \ldots, e_r\}$. Then for each regular Λ we identify E with the dual space of ${}^{(u)}J \, / {}^{(u)}J \, J_\Lambda$ by demanding that $\{e_1, \ldots, e_r\}$ be the dual basis to $\{v_1^*, \ldots, v_r^*\}$. By taking the transpose of the natural action of ${}^{(u)}J$ on ${}^{(u)}J \, / \, {}^{(u)}J \, J_\Lambda$ we obtain a representation

$$v \longrightarrow \gamma(\Lambda : v) \; , \; v \in {}^{(u)}J,$$

of ${}^{(u)}J$ on E. For $v \in {}^{(u)}J$,

$$\Sigma_i \; z_{v,ij} \; v_i - \Sigma_i \langle z_{v,ij} \; , \Lambda \rangle \, v_i$$

is in ${}^{(u)}J \, J_\Lambda$. It follows that the matrix of $\gamma(\Lambda : v)$ with respect to the above basis is the transpose of

$$(\langle z_{v,ij}, \Lambda \rangle)_{1 \leq i, j \leq r} .$$

Let ${}^{(u)}\Delta_+$ be the set of positive roots of $(\underline{g}_{\mathbb{C}}, {}^{(o)}\underline{h}_{\mathbb{C}})$ which do not vanish on ${}^{(u)}\underline{a}$. Define an element, ϖ, in S by

$${}^{(u)}\varpi = \prod_{\alpha \in {}^{(u)}\Delta_+} \alpha .$$

Let $\{s_1 = 1, s_2, \ldots, s_r\}$ be a fixed set of representatives of $W/{}^{(u)}W$ in W. Then it follows from [5(c)] (Lemma 15 and the discussion preceeding it), that if

$$f_i(\Lambda) = \Sigma_k \; v_k(s_i\Lambda) \; {}^{(u)}\varpi(s_i\Lambda)^{-1} \cdot e_k, \quad 1 \leq i \leq r,$$

then

$$\gamma(\Lambda : v) \; f_i(\Lambda) = \langle v, s_i\Lambda \rangle f_i(\Lambda) \quad , v \in {}^{(u)}J, 1 \leq i < r$$

In addition, there are elements

$$\tau^j \quad , 1 \leq j \leq r ,$$

in S such that

$$e_j = \Sigma_i \; \tau^j(s_i\Lambda) \; f_i(\Lambda), \quad 1 \leq j \leq r .$$

In particular, $\{f_i(\Lambda) : 1 \leq i \leq r\}$ is a basis of E, and the operator $\gamma(\Lambda : v)$ is semisimple.

Fix a standard cuspidal subgroup P of G. For $\delta > 0$, let $C(\underline{a}_{\mathbb{C}}, \delta)$ be the set of vectors $\lambda \in \underline{a}_{\mathbb{C}}$ such that the length of the real part of λ is less than δ. Let $C_r(\underline{a}_{\mathbb{C}}, \delta)$ denote the set of P-regular elements in $C(\underline{a}_{\mathbb{C}}, \delta)$. It is a consequence of Lemma 2.1 that there is a $\delta_o > 0$ such that for any $\lambda \in C_r(\underline{a}_{\mathbb{C}}, \delta)$, and any $\mu \in \Lambda'(M)$,

$$\Lambda = \mu + \lambda$$

is a regular point in ${}^{(o)}\underline{h}_{\mathbb{C}}$. We shall apply the estimates to the functions

$$\phi(x) = E(\psi : \lambda : x) ,$$

where λ ranges over $C_r(\underline{a}_{\mathbb{C}}, \delta_o)$ and ψ belongs to $\underline{C}_{\{\mu\}}(M, \tau_M)$. We will let μ vary over all elements in $\Lambda'(M)$, and τ vary over all double representations in $F(K,K)$. As above, we write

$$\Lambda = \mu + \lambda .$$

For ${}^{(u)}P$ as above, and $\tau \in F(K, K)$,

$${}^{(u)}\tau = \tau_{{}^{(u)}M}$$

is a finite dimensional unitary double representation of the maximal compact subgroup

$${}^{(u)}K = K_{{}^{(u)}M} = K \cap {}^{(u)}M$$

of ${}^{(u)}M$. Let ${}^{(u)}\underline{Z}$ and $\underline{Z}$ be the centers of the universal enveloping algebras of ${}^{(u)}\underline{m}_{\mathbb{C}} + {}^{(u)}\underline{a}_{\mathbb{C}}$ and $\underline{g}_{\mathbb{C}}$ respectively. As we saw in §1, there are isomorphisms

$$\gamma : \underline{Z} \longrightarrow J ,$$

$${}^{(u)}\gamma : {}^{(u)}\underline{Z} \longrightarrow {}^{(u)}J ,$$

and

$$^{(u)}\mu = {}^{(u)}\gamma^{-1} \circ \gamma .$$

For Λ as above, we define $\underline{Z}_\Lambda$ to be the set of Z in $\underline{Z}$ such that $\langle \gamma(Z), \Lambda \rangle = 0$. Then ${}^{(u)}\underline{Z} \cdot \underline{Z}_\Lambda$ is an ideal in ${}^{(u)}\underline{Z}$ and ${}^{(u)}\underline{Z}/{}^{(u)}\underline{Z}\,\underline{Z}_\Lambda$ is a complex vector space of dimension r. Define

$$V_i = {}^{(u)}\gamma^{-1}(v_i) \ , 1 \leq i \leq r .$$

If $V \in {}^{(u)}\underline{Z}$, define

$$z_{V,i\,j} = \gamma^{-1}(z_{{}^{(u)}\gamma(V),i\,j}) .$$

Then

$$V\,V_j = \Sigma_i\, {}^{(u)}\mu(z_{V,i\,j})\, V_i \ , 1 \leq j \leq r .$$

Define the operator $\Gamma(\Lambda : V)$ on E by

$$\Gamma(\Lambda : v) = \gamma(\Lambda : {}^{(u)}\gamma(V)).$$

Then the matrix of $\Gamma(\Lambda : V)$ is the transpose of

$$(\langle \gamma(z_{V,i\,j}), \Lambda \rangle)_{1 \leq i,\, j \leq r}$$

Its eigenvalues are

$$\langle {}^{(u)}\gamma(V), s_i \Lambda \rangle \ , 1 \leq i \leq r .$$

Suppose that a representation τ in $F(K,K)$ acts on the finite dimensional Hilbert space V_τ. Let $\underline{E}$ be the Hilbert space $V_\tau \otimes E$. The double representation τ of K, and the representation $\Gamma(\Lambda, \cdot)$ of ${}^{(u)}\underline{Z}$ both extend trivially to $\underline{E}$. For any $V \in {}^{(u)}\underline{Z}$ define

$$U_j(\Lambda : V) = V\,V_j - \Sigma_i \langle \gamma(z_{V,i\,j}), \Lambda \rangle V_i .$$

This is an element in ${}^{(u)}\underline{Z} \cdot {}^{(u)}\mu(\underline{Z}_\Lambda)$. For $m \in {}^{(u)}L$, let

$$^{(u)}d(m) = |\det (\mathrm{Ad}(m)|_{{}^{(u)}\underline{n}})|^{\frac{1}{2}} .$$

Finally, for $m \in {}^{(u)}L$, define

$$ {}^{(u)}\Phi(m) = \Sigma_j V_j ({}^{(u)}d(m) \phi(m)) \otimes e_j , $$

and

$$ {}^{(u)}\Psi_V(m) = \Sigma_j U_j(\Lambda : V) ({}^{(u)}d(m) \phi(m)) \otimes e_j . $$

Then ${}^{(u)}\Phi$ and ${}^{(u)}\Psi_V$ are ${}^{(u)}\tau$-spherical functions from ${}^{(u)}L$ to $\underline{E}$.

If V_j' is the differential operator ${}^{(u)}d(m)^{-1} V_j \circ {}^{(u)}d(m)$, we have

$$ {}^{(u)}\Phi(m) = \Sigma_j {}^{(u)}d(m) \cdot \phi(m ; V_j') \otimes e_j . $$

Recall from §1 the definition of the function Ξ on G. We define the function ${}^{(u)}\Xi$ on ${}^{(u)}L$ the same way. It is known ([5(e)], Lemma 47) that there are numbers c and d such that for $m \in {}^{(u)}L$,

$$ {}^{(u)}d(m) \; \Xi(m) \leq c \; {}^{(u)}\Xi(m) \; (1 + \sigma(m))^d $$

It follows from Lemma 5.4 that we may choose a polynomial p such that for $\lambda \in C_r(\underline{a}_{\mathbb{C}}, \delta)$,

$$ (6.1) \qquad |{}^{(u)}\Phi(m)| \leq ||\psi|| \cdot p(|\Lambda| + |\tau|) \cdot {}^{(u)}\Xi(m) \cdot (1 + \sigma(m))^d \cdot e^{c_0 |\lambda_{\mathbb{R}}| \cdot \sigma(m)} $$

From [5(e)], Corollary 1, Lemma 76, or as can be seen directly from the above definitions, we have the following differential equation

$$ {}^{(u)}\Phi(m;V) = \Gamma(\Lambda : V) {}^{(u)}\Phi(m) + {}^{(u)}\Psi_V(m) . $$

In particular, if $H \in {}^{(u)}\underline{a}$, H also lies in ${}^{(u)}\underline{Z}$, and the differential equation can be written

$$\frac{d}{dt}\left(e^{-t\Gamma(\Lambda:H)} \cdot {}^{(u)}\Phi(m \cdot \exp tH)\right) = e^{-t\Gamma(\Lambda:H)} \cdot {}^{(u)}\Psi_H(m \cdot \exp tH),$$

$t \in R$, $m \in {}^{(u)}L$

Transforming this into an integral equation, we obtain

$$(6.2) \qquad {}^{(u)}\Phi(m \exp TH) = e^{T\Gamma(\Lambda:H)}\, {}^{(u)}\Phi(m) + \int_0^T e^{(T-t)\Gamma(\Lambda:H)}\, {}^{(u)}\Psi_H(m \cdot \exp tH)\, dt, \quad T \in \mathbb{R},\ m \in {}^{(u)}L .$$

Let $H_0 = {}^{(u)}H_0$ be the unit vector in ${}^{(u)}\underline{a}^+$ defined in §2. Let β_0 be the smallest value of $\alpha(H_0)$ as α ranges over the simple roots of $({}^{(u)}P, {}^{(u)}A)$. We shall study the consequences of (6.2) for $H = H_0$. However, we must first estimate the function Ψ_{H_0}.

Let ${}^{(u)}L^+ = {}^{(u)}K \cdot {}^{(o)}A^+ \cdot {}^{(u)}K$.

<u>LEMMA 6.1</u>: There is a positive number δ, a polynomial p and a real number d such that for $\lambda \in C_r(\underline{a}_{\mathbb{C}}, \delta)$, $m \in {}^{(u)}L^+$ and $t \geq 0$,

$$|{}^{(u)}\Psi_{H_0}(m \exp tH_0)| \leq ||\Psi|| \cdot p(|\Lambda| + |\tau|) \cdot e^{-(\beta_0 - c_0|\lambda_{\mathbb{R}}|)t}\, {}^{(u)}\Xi(m)\,(1+\sigma(m))^d \cdot e^{c_0|\lambda_{\mathbb{R}}|\sigma(m)} .$$

PROOF: Since Ψ_{H_0} is ${}^{(u)}\tau$-spherical we may assume that m belongs to ${}^{(o)}A^+$. For any $V \in {}^{(u)}\underline{Z}$ let V' be the differential operator ${}^{(u)}d^{-1}\, V \circ {}^{(u)}d$. Then for $m \in {}^{(u)}L$,

$${}^{(u)}\Psi_{H_0}(m) = \Sigma_j\, {}^{(u)}d(m) \cdot \phi(m;\ U_j(\Lambda : H_0)') \otimes e_j .$$

We have

$$U_j(\Lambda : H_o)' = (\Sigma_i {}^{(u)}\mu(Z_{H_o,ij}) V_i)' - (\langle \gamma(Z_{H_o,ij}), \Lambda \rangle V_i)'$$

$$= \Sigma_i {}^{(u)}\mu(Z_{H_o,ij} - \langle \gamma(Z_{H_o,ij}), \Lambda \rangle I)' \cdot V_i'$$

where I is the identity element in $\underline{G}$. Define an element ζ_{ij} in ${}^{(u)}\underline{Z}$ to equal

$$(Z_{H_o,ij} - \langle \gamma(Z_{H_o,ij}), \Lambda \rangle I) - ({}^{(u)}\mu(Z_{H_o,ij} - \langle \gamma(Z_{H_o,ij}), \Lambda \rangle I)'.$$

Since the differential operator

$$Z_{H_o,ij} - \langle \gamma(Z_{H_o,ij}), \Lambda \rangle I$$

is in the center of $\underline{G}$, and annihilates ϕ, we have

$${}^{(u)}\Phi_{H_o}(m) = \Sigma_{ij} {}^{(u)}d(m)\, \phi(m ; \zeta_{ij} V_i') \otimes e_j .$$

On the other hand

$$\zeta_{ij} = Z_{H_o,ij} - {}^{(u)}\mu(Z_{H_o,ij})'$$

and is independent of Λ. We noted in §1 that for any $Z \in \underline{Z}$,

$$Z - {}^{(u)}\mu(Z)' \in \theta({}^{(u)}n) \cdot \underline{G} = {}^{(u)}\underline{V}\ \underline{G} .$$

Therefore for each i and j there are elements

$$X^k_{ij} \in {}^{(u)}\underline{V}$$

and

$$Y^k_{ij} \in \underline{G}, \qquad k = 1, \ldots, N(i,j),$$

such that

$${}^{(u)}\Phi_{H_o}(m) = \Sigma_{i,j,k} {}^{(u)}d(m) \cdot \phi(m; X^k_{ij} \cdot Y^k_{ij}) \otimes e_i .$$

If $H \in {}^{(o)}\underline{a}$, $Y \in \underline{G}$, and $X \in {}^{(u)}\underline{v}(\alpha)$, for α a root of $({}^{(u)}P, {}^{(u)}A)$, then

$$\phi(\exp H; X Y) = e^{-\alpha(H)} \phi(X ; \exp H ; Y) .$$

It follows that for $a \in {}^{(o)}A^{+}$ and $t \geq 0$

$|{}^{(u)}I_{H_o}(a \exp t H_o)|$ is bounded by

$$e^{-r_o t} \Sigma_{ijk}\, {}^{(u)}d\,(a \cdot \exp t H_o)\ |\phi\,(\Theta(X_{ij}^{k}); a \exp t H_o ; Y_{ij}^{k})| .$$

Now by [5(e)], Lemma 47, there are numbers c_1 and d_1 such that

$$ {}^{(u)}d(a \exp t H_o) \cdot \Xi(a \exp t H_o) \leq c_1 {}^{(u)}\Xi\,(a \exp t H_o) \cdot (1 + \sigma(a))^{d_1} (1+t)^{d_1} .$$

Our lemma now follows from (6.1). $\square$

If

$$\Xi = \xi_1 \otimes e_1 + \ldots + \xi_r \otimes e_r,\ \xi_1, \ldots, \xi_r \in V_\tau,$$

is an arbitrary vector in $\underline{E}$, define

$$t_i(\Xi) = \xi_i \quad , \quad 1 \leq i \leq r .$$

For Λ regular, we have

$$\underline{E} = \bigoplus_{i=1}^{r} (V_\tau \otimes f_i(\Lambda)) .$$

Let $F_\Lambda^{(1)}, \ldots, F_\Lambda^{(r)}$ be the projections on $\underline{E}$ relative to this decomposition. For $\Xi \in \underline{E}$, we have

$$\begin{aligned} F_\Lambda^{(i)} \cdot \Xi &= \Sigma_j\ \tau^j(s_i)\, t_j(\Xi) \otimes f_i(\Lambda) \\ &= \Sigma_{j,\ell}\ \tau^j(s_i\Lambda) \cdot v_\ell(s_i\Lambda) \cdot {}^{(u)}\varpi(s_i\Lambda)^{-1} \cdot t_j(\Xi) \otimes e_\ell . \end{aligned}$$

It follows that for $\leq i , k \leq r$,

(6.3) $t_k(F_\wedge^{(i)}\Xi) = \Sigma_j \tau^j(s_i \wedge) v_k(s_i \wedge) \cdot {}^{(u)}\varpi(s_i \wedge)^{-1} \cdot t_j(\Xi)$.

Since ${}^{(u)}\gamma(H_o) = H_o$, the set of eigenvalues of $\Gamma(\Lambda : H_o)$ is

$$Q = \{r_i = \langle H_o , s_i\Lambda \rangle ; 1 \leq i \leq r\} .$$

Let Q^o, Q^+, and Q^- be the sets of eigenvalues r_i such that $\langle H_o , s_i \Lambda\rangle$ is respectively equal to 0, greater than 0, or less than 0. Define

$$F_\wedge^o = \Sigma_{r_i \in Q^o} F_\wedge^{(i)} ,$$

$$F_\wedge^+ = \Sigma_{r_i \in Q^+} F_\wedge^{(i)}$$

and

$$F_\wedge^- = \Sigma_{r_i \in Q^-} F_\wedge^{(i)} .$$

We also define

$$\underline{E}_\wedge^o = F_\wedge^o (\underline{E}) , \underline{E}_\wedge^+ = F_\wedge^+ (\underline{E}) , \underline{E}_\wedge^- = F_\wedge^- (\underline{E}) .$$

Then

$$\underline{E} = \underline{E}_\wedge^o \oplus \underline{E}_\wedge^+ \oplus \underline{E}_\wedge^- .$$

We write ${}^{(u)}\Phi^o$, ${}^{(u)}\Phi^+$, and ${}^{(u)}\Phi^-$ for $F_\wedge^o \cdot {}^{(u)}\Phi$, $F_\wedge^+ \cdot {}^{(u)}\Phi$, and $F_\wedge^- \cdot {}^{(u)}\Phi$, respectively.

By Lemma 2.1, there is a positive number γ_o which is less than the absolute value of any nonzero number in the set

$$\{\langle H_o, s_i\mu\rangle : 1 \leq i \leq r , \mu \in \Lambda (M)\} .$$

LEMMA 6.2: For any γ, $0 < \gamma \leq \frac{\gamma_0}{2}$, we can choose a positive number δ and a polynomial p such that for $\lambda \in C_r(\underline{a}_e, \delta)$

(i) $\| e^{-t\Gamma(\Lambda : H_0)} \cdot F_\Lambda^+ \| \leq p(|\Lambda|)\, e^{-(\gamma_0 - \gamma)t}$, $t > 0$,

(ii) $\| e^{t\Gamma(\Lambda : H_0)} \cdot F_\Lambda^- \| \leq p(|\Lambda|)\, e^{-(\gamma_0 - \gamma)t}$; $t > 0$,

and

(iii) $\| e^{t\Gamma(\Lambda ; H_0)} \cdot F_\Lambda^0 \| \leq p(|\Lambda|)\, e^{\gamma|t|}$, $t \in R$.

PROOF: If we extend any operator T on E to the space $\underline{E} = V_\tau \otimes E$ by letting it act trivially on V_τ, its norm remains the same. Therefore, for the proof of this lemma, we can regard the projections F_Λ^+, F_Λ^-, and F_Λ^0 and the operator $\Gamma(\Lambda : H_0)$ as acting on E, a space whose dimension, r, is independent of τ and Λ.

We choose δ to be any positive number smaller than both $\frac{\gamma}{2}$ and δ_0. Given

$$\Lambda = \mu + \lambda, \ \lambda \in C_r(\underline{a}_e, \delta),$$

and an eigenvalue r_i of $\Gamma(\Lambda : H_0)$, we have

$$(r_i)_R = (\langle H_0, s_i\Lambda\rangle)_R = \langle H_0, s_i\mu\rangle + \langle H_0, s_i \lambda_R\rangle.$$

Since

$$|\langle H_0, s_i \lambda_R\rangle| < \frac{\gamma}{2}$$

we have

$$|(r_i)_R| > \gamma_0 - \frac{\gamma}{2}, \ \text{for } r_i \in Q^+ \cup Q^-,$$

and

$$|(r_i)_R| < \frac{\gamma}{2}, \ \text{for } r_i \in Q^0.$$

Let Γ^+, Γ^- and Γ^0 be closed positive rectangular curves in the complex plane, whose interiors contain the points in Q^+, Q^-, and Q^0 respectively, but no other eigenvalues of $\Gamma(\Lambda : H_0)$. We may choose the curves so that

(i), the arc length of each of the curves is bounded by a polynomial in $|\Lambda|$,

(ii), if z is any point on one of the curves then the distance from z to any point in Q is greater than $\frac{\gamma}{2}$, and

(iii), $z_{/R} \geq \gamma_0 - \gamma$, for $z \in \Gamma^+$,

$z_{/R} \leq -(\gamma_0 - \gamma)$, for $z \in \Gamma^-$,

$|z_{/R}| \leq \gamma$, for $z \in \Gamma^0$.

From spectral theory we know that

$$e^{-t\Gamma(\Lambda : H_0)} \cdot F_\Lambda^+ = \frac{1}{2\pi i} \cdot \int_{\Gamma^+} (z - \Gamma(\Lambda : H_0))^{-1} \cdot e^{-tz}\, dz$$

For $t \geq 0$, the norm of this operator is bounded by the product of a polynomial in $|\Lambda|$, and

$$e^{-(\gamma_0 - \gamma)t} \cdot \sup_{z \in \Gamma^+} \|(z - \Gamma(\Lambda : H_0))^{-1}\| .$$

The absolute value of any point z in Γ^+ is bounded by a polynomial in $|\Lambda|$. Therefore, for any such z the matrix of

$$z I - \Gamma(\Lambda : H_0)$$

with respect to the orthonormal basis $\{e_\cdot, \ldots, e_r\}$ has entries whose absolute values are bounded by a polynomial in $|\Lambda|$. On the other hand, the absolute value of the determinant of $z I - \Gamma(\Lambda : H_0)$ is bounded below by $(\frac{\gamma}{2})^r$. It follows that

$$\sup_{z \in \Gamma^+} \| (zI - \Gamma(\Lambda : H_o))^{-1} \|$$

is bounded by a polynomial in $|\Lambda|$.

We have shown that there is a polynimial p such that for $t \geq 0$,

$$\| e^{-\Gamma(\Lambda : H_o)} F_\Lambda^+ \| \leq p(|\Lambda|) \cdot e^{-(\gamma_o - \gamma)t} ,$$

as required. The other two estimates follow exactly the same way.

□

<u>LEMMA 6.3</u>: We can choose positive numbers β and δ, a polynomial p and a real number d, such that for $\lambda \in C_r(\underline{a}_c, \delta)$, $m \in {}^{(u)}L^+$, and $T \geq 0$,

$$| {}^{(u)}\Phi^+ (m \cdot \exp T H_o) | + | {}^{(u)}\Phi^- (m \cdot \exp T H_o) |$$

is bounded by

$$(6.4) \ldots \quad \| \psi \| \cdot p(|\Lambda| + |\tau|) \cdot e^{-\beta T} \cdot {}^{(u)}\Xi(m) \cdot (1 + \sigma(m))^d \cdot e^{c_o |\lambda_R| \cdot \sigma(m)} .$$

PROOF: We shall be using Lemma 6.1, and Lemma 6.2 with $\gamma = \frac{\gamma_o}{2}$. Choose δ such that for $\lambda \in C_r(\underline{a}_c, \delta)$ the inequalities of these lemmas are valid, and such that

$$c_o \delta \leq \inf \{ \frac{\beta_o}{2} , \frac{\gamma_o}{4} \} .$$

Choose β to be the minimum of $\frac{\beta_o}{4}$ and $\frac{\gamma_o}{8}$.

First we deal with ${}^{(u)}\Phi^{-}$. Since $F_{\bar{\Lambda}}$ commutes with $\Gamma(\Lambda : H_o)$, we see from (6.2) that

$$
{}^{(u)}\Phi^{-}(m \exp T H_o) = e^{T\Gamma(\Lambda : H_o)} \cdot {}^{(u)}\Phi^{-}(m \exp T H_o)
$$
$$
+ \int_o^T e^{(T-t)\Gamma(\Lambda : H_o)} \cdot {}^{(u)}\Phi_{H_o}(m \exp t H_o)\, dt
$$

It follows from Lemma 6.2 that there is a polynomial p_o such that $|{}^{(u)}\Phi^{-}(m \exp t H_o)|$ is bounded by the sum of

$$
(6.5) \qquad p_o(|\Lambda|) \cdot e^{-\frac{\gamma_o t}{2}} \cdot |{}^{(u)}\Phi(m \exp T H_o)|
$$

and

$$
(6.6) \qquad p_o(|\Lambda|) \cdot \int_o^T e^{-\frac{\gamma_o t}{2}(t-t)} |{}^{(u)}\Phi_{H_o}(m \exp t H_o)|\, dt .
$$

By (6.1) we can choose a polynomial p_1 such that (6.5) is bounded by

$$
||\psi|| \cdot p_1(|\Lambda|+|\tau|) \cdot e^{-\frac{\gamma_o t}{2}} \cdot {}^{(u)}\Xi(m) \cdot (1 + \sigma(m \exp T H_o))^d \cdot e^{c_o \cdot |\lambda_R| \cdot \sigma(m)} .
$$

Since

$$
\sigma(m \exp T H_o) \leq \sigma(m) + T ,
$$

we have

$$
e^{-\frac{\gamma_o T}{2}} \cdot (1 + \sigma(m \exp T H_o))^d \cdot e^{c_o \cdot |\lambda_R| \cdot \sigma(m \exp T H_o)}
$$
$$
\leq e^{-(\frac{\gamma_o}{2} - c_o\delta)T} \cdot (1 + \sigma(m))^d \cdot (1 + T)^d \cdot e^{c_o|\lambda_R| \cdot \sigma(m)}
$$
$$
\leq c_2 \cdot e^{-\beta T} \cdot (1 + \sigma(m))^d \cdot e^{c_o \cdot |\lambda_R| \cdot \sigma(m)} ,
$$

for some constant c_2. Therefore there is a polynomial p such that (6.5) is bounded by (6.4). The integral in (6.6) is dominated by

$$e^{-\frac{\gamma_o T}{4}} \int_o^{\frac{T}{2}} |{}^{(u)}\Phi_{H_o}(m \exp t H_o)| \, dt + \int_{\frac{T}{2}}^{T} |{}^{(u)}\Phi_{H_o}(m \exp t H_o)| \, dt$$

$$\leq e^{-\frac{\gamma_o T}{4}} \cdot \int_o^{\infty} |{}^{(u)}\Phi_{H_o}(m \exp t H_o)| \, dt + \int_{\frac{T}{2}}^{\infty} |{}^{(u)}\Phi_{H_o}(m \exp t H_o)| \, dt .$$

It follows from Lemma 6.1 that (6.6) is also bounded by (6.4) for some polynomial p .

Now we consider ${}^{(u)}\Phi^+$. Using a change of variables, we rewrite the integral equation (6.2) as

$${}^{(u)}\Phi^+(m) = e^{-t_1 \Gamma(\Lambda : H_o)} \, {}^{(u)}\Phi^+(m \exp t_1 H_o) - \int_o^{t_1} e^{-t\Gamma(\Lambda : H_o)} . E_\Lambda^+ \Phi_{H_o}(m \exp t H_o) \, dt ,$$

for any $t_1 \geq 0$ and $m \in {}^{(u)}L$. Since $\frac{\gamma_o}{2} - \delta$ is positive, we observe from (6.1) and Lemma 6.2 that

$$\lim_{t_1 \to \infty} | e^{-t_1 \Gamma(\Lambda : H_o)} . {}^{(u)}\Phi^+(m \exp t_1 H_o)| = 0 .$$

Therefore

$${}^{(u)}\Phi^+(m) = - \int_o^{\infty} e^{-t \Gamma(\Lambda : H_o)} \cdot E_\Lambda^+ \, \Phi_{H_o}(m \exp t H_o) \, d t .$$

Replace m by $m \exp T H_o$. We obtain the equation

$$^{(u)}\Phi^{+}(m \exp T H_o) = -\int_T^{\infty} e^{-(t-T)\,\Gamma(\Lambda : H_o)} \cdot E_{\Lambda}^{+} \Phi_{H_o}(m \exp t H_o)\, d t .$$

Using Lemmas 6.1 and 6.2 once again, we conclude that there is a polynomial p such that $|^{(u)}\Phi^{+}(m \exp T H_o)|$ is bounded by (6.4). This completes the proof of the lemma. $\square$

Next we turn our attention to $^{(u)}\Phi^{o}$. Before doing this, however, we shall prove a simple lemma. For any $\varepsilon > 0$, let $^{(u)}S(\varepsilon)$ be the set of elements H of norm 1 in $^{(u)}\underline{a}$ such that $\alpha(H) \geq \varepsilon$ for every root α of $(^{(u)}P, {}^{(u)}A)$.

<u>LEMMA 6.4</u>: Given $\varepsilon > 0$ we can find a real number N such that for $m \in {}^{(u)}L$, $H \in {}^{(u)}S(\varepsilon)$, and $t \geq N \cdot \sigma(m)$, $m \exp t H$ is in $^{(u)}L^{+}$.

PROOF: Choose N_o such that

$$|\alpha(Y)| \leq N_o \,||Y||, \; Y \in {}^{(o)}\underline{a} ,$$

for each root α of $(^{(u)}P, {}^{(u)}A)$. Any $m \in {}^{(u)}L$ can be written

$$m = k_1 \cdot \exp Y \cdot k_2, \quad k_1, k_2 \in {}^{(u)}K, \; Y \in {}^{(u)}\underline{a} .$$

Then $\sigma(m) = ||Y||$. The lemma follows with $N = N_o \, \varepsilon^{-1}$. $\square$

For $m \in {}^{(u)}L$, define

$$^{(u)}\Theta(m) = {}^{(u)}\Phi^{o}(m) + \int_0^{\infty} e^{-t\,\Gamma(\Lambda : H_o)} E_{\Lambda}^{o} \Phi_{H_o}(m \exp t H_o)\, d t .$$

In view of Lemmas 6.1, 6.2 and 6.4 there is a $\delta > 0$ such that this integral converges absolutely uniformly for m and λ belonging to compact subsets of ${}^{(u)}M \times C_r(\underline{a}_{c}, \delta)$. In particular, ${}^{(u)}\text{Ⓗ}(m)$ depends analytically on $\lambda \in C_r(\underline{a}_{c}, \delta)$. By [5(e)], Lemmas 56 and 58, ${}^{(u)}\text{Ⓗ}$ has the following properties:

(6.7) $\quad {}^{(u)}\text{Ⓗ}(m \exp H) = e^{\Gamma(\Lambda : H)}\ {}^{(u)}\text{Ⓗ}(m), \quad m \in {}^{(u)}L,\ H \in {}^{(u)}\underline{a},$

(6.8) $\quad {}^{(u)}\text{Ⓗ}(k_1 m\, k_2) = \tau(k_1)\ {}^{(u)}\text{Ⓗ}(m) \cdot \tau(k_2), \quad m \in {}^{(u)}L,\ k_1, k_2 \in {}^{(u)}K,$

(6.9) $\quad {}^{(u)}\text{Ⓗ}(m;\ V) = \Gamma(\Lambda :\ V)\ {}^{(u)}\text{Ⓗ}(m), \quad m \in {}^{(u)}M,\ V \in {}^{(u)}\underline{Z}$

<u>LEMMA 6.5</u>: We can choose positive numbers β and δ, a polynomial p and a real number d such that for $\lambda \in C_r(\underline{a}_{c}, \delta)$, $m \in {}^{(u)}L^+$, and $T \geq 0$,

$$\left| {}^{(u)}\Phi(m \exp T H_o) - {}^{(u)}\text{Ⓗ}(m \exp T H_o) \right|$$

is bounded by the expression (6.4).

PROOF: We shall be using Lemma 6.1, and Lemma 6.2 with $\gamma = \inf \{\frac{\gamma_o}{2}, \frac{\beta_o}{4}\}$. Choose δ such that for $\lambda \in C_r(\underline{a}_c, \delta)$ the inequalities of these lemmas are valid, and such that

$$c_o \delta \leq \frac{\beta_o}{4}.$$

We take β to be $\frac{\beta_o}{4}$.

Since $F^o_\Lambda\ {}^{(u)}\text{Ⓗ}(m) = {}^{(u)}\text{Ⓗ}(m)$, we observe from (6.7) that

$$\left| {}^{(u)}\Phi^o(m \exp T H_o) - {}^{(u)}\text{Ⓗ}(m \exp T H_o) \right|$$

is bounded by the product of

$$|e^{T\Gamma(\Lambda : H_o)} F^o_\Lambda|$$

and

$$|e^{-T\Gamma(\Lambda : H_o)} \, {}^{(u)}\Phi^o(m \exp T H_o) - {}^{(u)}\textcircled{H}(m)| .$$

By Lemma 6.2 there is a polynomial p_o such that the first term is bounded by

$$p_o(|\Lambda|) \cdot e^{\gamma T}.$$

On the other hand by (6.2) and the definition of ${}^{(u)}\textcircled{H}(m)$, the second term equals

$$|\int_T^\infty e^{-t\Gamma(\Lambda : H_o)} \cdot E^o_\Lambda \; {}^{(u)}\Phi_{H_o}(m \exp t H_o) \, d t|,$$

an expression which is bounded by

$$p_o(|\Lambda|) \cdot e^{\gamma T} \cdot \int_T^\infty |{}^{(u)}\Phi_{H_o}(m \exp t H_o)| \, d t .$$

Our result now follows from Lemma 6.1 . □

We define

$${}^{(u)}\theta_i(m) = t_i({}^{(u)}\textcircled{H}(m)) \quad , 1 \le i \le r , \; m \in {}^{(u)}L .$$

Let ${}^{(u)}\theta(m) = {}^{(u)}\theta_1(m)$. Then ${}^{(u)}\theta$ is a τ-spherical function from ${}^{(u)}L$ to V_τ .

COROLLARY 6.6: We can choose positive numbers β and δ, a polynomial p, and a real number d such that for $\lambda \in C_r(\underline{a}_c, \delta)$, $m \in {}^{(u)}L^+$, and $T \ge 0$,

$$\left| {}^{(u)}d\,(m \exp T H_o)\,\phi\,(m \exp T H_o) - {}^{(u)}\Theta\,(m \exp T H_o) \right|$$

is bounded by

$$\|\psi\| \cdot p(|\Lambda| + |\tau|) \cdot e^{-\beta T} \cdot {}^{(u)}\Xi(m) \cdot (1 + \sigma(m))^d \cdot e^{c_o \cdot |\lambda_{\mathbb{R}}| \cdot \sigma(m)} .$$

PROOF: For any $m \in {}^{(u)}L$,

$$ {}^{(u)}d\,(m) \cdot \phi\,(m) = t_1({}^{(u)}\Phi\,(m)) .$$

But

$$|t_1({}^{(u)}\Phi\,(m)) - t_1({}^{(u)}\textcircled{H}\,(m))| \leq |{}^{(u)}\Phi\,(m) - {}^{(u)}\textcircled{H}\,(m)| .$$

The corollary follows from Lemmas 6.3 and 6.5. □

LEMMA 6.7: There is a $\delta > 0$, a polynomial p, and constants N and d, such that for $\lambda \in \mathfrak{C}_r(\underline{a}_\epsilon, \delta)$ and $m \in {}^{(u)}M$,

$$|{}^{(u)}\textcircled{H}\,(m)| \leq p(|\Lambda| + |\tau|) \cdot {}^{(u)}\Xi(m) \cdot (1 + \sigma(m))^d \cdot e^{N \cdot |\lambda_{\mathbb{R}}| \cdot \sigma(m)} \cdot \|\psi\| .$$

PROOF: Let δ be the positive number given by the last lemma. For any $m_+ \in {}^{(u)}L^+$, $|{}^{(u)}\textcircled{H}\,(m_+)|$ is bounded by the sum of $|{}^{(u)}\Phi^o\,(m_+) - {}^{(u)}\textcircled{H}\,(m_+)|$ and $|{}^{(u)}\Phi^o\,(m_+)|$. The first term, by the last lemma, can be bounded by an expression

$$\|\psi\| \cdot p_1(|\Lambda| + |\tau|) \cdot {}^{(u)}\Xi\,(m_+) \cdot (1 + \sigma(m_+))^d \cdot e^{c_o \cdot |\lambda_{\mathbb{R}}| \cdot \sigma(m_+)} .$$

By Lemma 6.2 and (6.1), the second term is also bounced by an expression of this form.

By Lemma 6.4, there is a constant N_o, such that for any $m \in {}^{(u)}M$ and for $t \geq N_o \cdot \sigma\,(m)$, $m \cdot \exp t H_o$ belongs to ${}^{(u)}L^+$.

For any $m \in {}^{(u)}M$, let $t_o = N_o \cdot \sigma(m)$, and $m_+ = m \cdot \exp t_o H_o$. Then

$${}^{(u)}\textcircled{H}(m) = {}^{(u)}\textcircled{H}(m_+ \cdot \exp(-t_o H_o)) = e^{-t_o \Gamma(\Lambda : H_o)} \, {}^{(u)}\textcircled{H}(m_+)$$

$$= - e^{-t_o \Gamma(\Lambda : H_o)} \cdot F^o_\Lambda \cdot {}^{(u)}\textcircled{H}(m_+).$$

By Lemma 6.2, there is a polynomial p_o such that

$$| e^{-t_o \Gamma(\Lambda : H_o)} F^o_\Lambda |$$

$$\leq p_o(|\lambda|) \cdot e^{2 t_o \cdot |\lambda_R|} = p_o(|\lambda|) \cdot e^{2 N_o \cdot \sigma(m) \cdot |\lambda_R|}$$

We have

$$\sigma(m_+) \leq \sigma(m) + \sigma(\exp t_o H_o)$$

$$= \sigma(m) + N_o \sigma(m)$$

$$= (N_o + 1) \sigma(m) .$$

Since

$${}^{(u)}\Xi(m_+) = {}^{(u)}\Xi(m) ,$$

our lemma follows with $N = (N_o + 1)c_o + 2 N_o$. $\square$

It is easy to prove a version of Corollary 6.6 which allows us to replace H_o by a vector which varies over ${}^{(u)}S(\varepsilon)$, for any $\varepsilon > 0$, and to let m range over all of ${}^{(u)}L$. However, we will not need to use such a result. We prove only the following weaker statement.

LEMMA 6.8: Let ε and δ be suitably small positive numbers. Given $\lambda \in \mathcal{C}_r(\underline{a}_{\mathbb{C}}, \delta)$ and $m \in {}^{(u)}L$, we can find positive numbers D and d such that for all $H \in {}^{(u)}S(\varepsilon)$ and $T \geq 0$,

$$|{}^{(u)}d\ (m \exp T H)\ \phi(m \exp T H) - {}^{(u)}\Theta(m \exp T H)| \leq D\, e^{-dt}.$$

PROOF: Fix a positive number γ which is smaller than $\frac{1}{2}\varepsilon\, |\alpha|^{-1}$ for each simple root α of $({}^{(u)}P, {}^{(u)}A)$. Then for $H \in {}^{(u)}S\ (\varepsilon)$, H_o as above, and α a simple root of $({}^{(u)}P, {}^{(u)}A)$,

$$\alpha(H - \gamma H_o) \geq \frac{\varepsilon}{2}$$

It follows from Lemma 6.4 that for T large enough, $m \cdot \exp T(H- \gamma H_o)$ belongs to ${}^{(u)}L^+$. To prove our lemma we write $m \exp T H$ as the product of $m \cdot \exp T(H- \gamma H_o)$ and $\exp T \gamma H_o$. The proof follows from Corollary 6.6. □

§7. THE c FUNCTIONS

Fix a standard cuspidal subgroup P of G. Suppose we are given a Weyl chamber $c = c_B$ in $\underline{a}$ associated to a fundamental systems B of roots in $\underline{a}$. Suppose also that f is a function on A and that ν is a complex number. Following Harish-Chandra, we shall write

$$\lim_{(a \xrightarrow{c} \infty)} f(a) = \nu$$

if for every pair of positive nu bers η and ε, there is a number N such that

$$|f(a) - \nu| \leq \varepsilon$$

whenever the conditions

$$\sigma(a) \geq N$$

and

$$\beta(\log a) \geq \eta \cdot \sigma(a), \quad \beta \in B,$$

on $a \in A$ are satisfied.

Suppose that τ is a double representation in F (K,K) which acts on V_τ. Harish-Chandra defines a space $\underline{A}(G,\tau)$ to be the set of $\underline{Z}$ finite functions ϕ in $C^\infty(G,\tau)$ for which there exist constants C and d such that

$$|\phi(x)| \leq c \cdot \Xi(x) \cdot (1 + \sigma(x))^d, \quad x \in G.$$

If λ belongs to $i\,\underline{a}_r$, the set of purely imaginary, P-regular points in $\underline{a}_\varepsilon$, Lemma 5.4 insures that the function

$$\phi(x) = E(\psi : \lambda : x)$$

is in $\underline{A}(G, \tau)$. Given ${}^{(u)}P$, $u \in \underline{J}$, the function ${}^{(u)}\theta$ defined in the previous section, is called the <u>constant term</u> of ϕ along ${}^{(u)}P$. It is characterized as the unique function in $\underline{A}\,({}^{(u)}L, {}^{(u)}\tau)$ such that for every $m \in {}^{(u)}L$,

$$(7.1) \quad \lim_{\substack{{}^{(u)}\underline{a}^+ \\ (a \longrightarrow \infty)}} \{ e^{<{}^{(u)}\rho,\ \log a>}\ \phi\,(m\,a) - {}^{(u)}\Theta\,(m\,a)\} = 0,$$

([5(f)], Theorem 2). That ϕ and ${}^{(u)}\Theta$ are so related, even if we take λ to lie in $C_r(\underline{a}_c, \delta)$, for some $\delta > 0$, follows directly from Lemma 6.8 and the formula

$$ {}^{(u)}d\,(m\,a) = \prod_{\alpha \in {}^{(u)}\Delta_+} (e^{\frac{1}{2}<\alpha,\ \log a>} - e^{-\frac{1}{2}<\alpha,\ \log a>}),$$

$$m \in {}^{(u)}M, \quad a \in \exp\,({}^{(u)}\underline{a}^+)\ .$$

Harish-Chandra developed the theory of the last section to define the constant term of <u>any</u> function in $\underline{A}\,(G, \tau)$. In particular, if G is replaced by ${}^{(u)}M$, and ϕ is replaced by ${}^{(u)}\Theta$, one can define the constant term ${}^{(v)}_{(u)}({}^{(u)}\Theta)$ of ${}^{(u)}\Theta$ along ${}^{(v)}_{(u)}P$, whenever $v \leq u$. It is easy to show that for $v \leq u$,

$$(7.2) \qquad {}^{(v)}\Theta = {}^{(v)}_{(u)}\,({}^{(u)}\Theta).$$

We remark that by ([5(e)], Lemma 43), a function $\phi \in \underline{A}\,(G, \tau)$ is in $\underline{C}_o(G, \tau)$ if and only if ${}^{(u)}\Theta = 0$ for all standard parabolic subgroups ${}^{(u)}P \neq G$.

Suppose again that $\lambda \in i\,\underline{a}_r$, and that

$$\phi(x) = E\,(\psi : \lambda : x)\ .$$

Harish-Chandra has shown ([5(f)], Lemma 8), that if ${}^{(u)}P$ is not associated to P, then for each $a \in {}^{(u)}A$ the function

$$(7.3) \qquad m \longrightarrow {}^{(u)}\Theta\,(m\,a),\ m \in {}^{(u)}M,$$

is orthogonal to $\underline{\mathcal{C}}_o\,({}^{(u)}M,\ {}^{(u)}\tau)$. By analytic continuation, we can fix a small positive number δ such that this fact remains true whenever λ belongs to $\mathcal{C}_r(\underline{a}_{\mathfrak{c}},\ \delta)$.

Suppose now that ${}^{(u)}P$ is associated to P. It follows easily from the above remarks that if $\lambda \in \mathcal{C}_r(\underline{a}_{\mathfrak{c}},\ \delta)$, and ϕ, ${}^{(u)}\Phi$, and ${}^{(u)}Ⓗ$ are as in the last section, the function

$$m \longrightarrow {}^{(u)}Ⓗ\ (m)\ ,\ m \in {}^{(u)}M\ ,$$

is in $\underline{\mathcal{C}}_o\,({}^{(u)}M,\ {}^{(u)}\tau)$. We write

$${}^{(u)}Ⓗ\ (m)\ =\ \Sigma_{i=1}^{r}\ \ {}^{(u)}Ⓗ^{(i)}(m),$$

where ${}^{(u)}Ⓗ^{(i)} = F_{\wedge}^{(i)} \cdot {}^{(u)}Ⓗ$. Each of the functions ${}^{(u)}Ⓗ^{(i)}$ is square integrable on ${}^{(u)}M$. On the other hand, if $V \in {}^{(u)}\underline{Z}$, we have

$${}^{(u)}Ⓗ^{(i)}(m;V) = \Gamma(\Lambda:V)\ \ {}^{(u)}Ⓗ^{(i)}(m) = \langle{}^{(u)}\gamma(V), s_i\Lambda\rangle \cdot {}^{(u)}Ⓗ^{(i)}(m),$$

by (6.9) and the definition of $F_{\wedge}^{(i)}$. Thus, ${}^{(u)}Ⓗ^{(i)}$ is an eigenfunction of ${}^{(u)}\underline{Z}$, which we have noted is square integrable. Suppose that ${}^{(u)}Ⓗ^{(i)}$ is not identically zero in λ. From the results of [5(e)], we know that for any element V in the center of the universal enveloping algebra of ${}^{(u)}\underline{m}_{\mathfrak{c}}$, the set of numbers

$$\langle{}^{(u)}\gamma(V),\ s_i\Lambda\rangle = \langle{}^{(u)}\gamma(V),\ s_i\mu\rangle + \langle{}^{(u)}\gamma(V),\ s_i\lambda\rangle,\ \lambda \in \mathcal{C}_r(\underline{a}_{\mathfrak{c}},\ \delta),$$

must be discrete. It follows that $\langle{}^{(u)}\gamma(V),\ s_i\lambda\rangle$ equals zero for all such V and λ. This can only happen if s_i maps $\underline{a}$ into ${}^{(u)}\underline{a}$.

For any $s \in \Omega(\underline{a}, {}^{(u)}\underline{a})$ there is a unique element s_i whose restriction to $\underline{a}$ equals s. The function

$$t_1({}^{(u)}\textcircled{H}^{(i)}(m)) \ , \ m \in {}^{(u)}M,$$

is of course defined in terms of

$$\phi(x) = E(\psi : \lambda : x), \quad \psi \in \underline{C}_{\{\mu\}}(M, \tau_M), \quad \lambda \in C_r(\underline{a}_c, \delta), \ x \in G .$$

It depends linearly on ψ and holomorphically on λ. The orbit of $s_i\mu$ under ${}^{(u)}W$, which depends only on s and μ, is denoted $\{s\,\mu\}$. We have shown that tbere is an analytic function

$$c(s : \lambda), \quad \lambda \in C_r(\underline{a}_c, \delta),$$

with values in the space of linear maps from $\underline{C}_{\{\mu\}}(M, \tau_M)$ to $\underline{C}_{\{s\,\mu\}}({}^{(u)}M, {}^{(u)}\tau)$ such that for $m \in {}^{(u)}M$

$$t_1({}^{(u)}\textcircled{H}^{(i)}(m)) = (c(s : \lambda)\ \psi)\ (m).$$

It follows from (6.7) that for $H \in {}^{(u)}\underline{a}$ and $m \in {}^{(u)}M$,

$$(7.4) \quad {}^{(u)}\theta\ (m \cdot \exp H) = \sum_{s \in \Omega(\underline{a}, {}^{(u)}\underline{a})} (c(s : \lambda)\psi)(m) \cdot e^{\langle s\lambda\ ,\ H\rangle} .$$

Set $\underline{a}' = {}^{(u)}\underline{a}$, and choose $s \in \Omega(\underline{a}, \underline{a}')$. Define a map

$$s : \underline{C}(M, \tau_M) \longrightarrow \underline{C}(M', \tau_{M'})$$

by

$$(s\,\psi)(m') = \tau(w)\ \psi(w^{-1}\, m'\, w)\ \tau(w^{-1}), \quad \psi \in \underline{C}(M, \tau_M),\ m' \in M',$$

where w is any representative of s in ${}^{(o)}\widetilde{M}$. It is clear that if $\mu \in \Lambda'(M)$ and $\psi \in \underline{C}_{\{\mu\}}(M, \tau_M)$, then $s\,\psi \in \underline{C}_{\{s\mu\}}(M', \tau_{M'})$.

LEMMA 7.1: Suppose that $\underline{a}''$ is another distinguished subspace of ${}^{(o)}\underline{a}$ which is associated to $\underline{a}$, and that c is a Weyl chamber in $\underline{a}''$. Choose $t \in \Omega(\underline{a}'', \underline{a}')$ such that $t(c) = (\underline{a}')^+$. Then for $\psi \in \underline{C}_{\{\mu\}}(M, \tau_M)$, $\lambda \in C_r(\underline{a}_{\mathbb{C}}, \delta)$ and $m \in M''$,

$$\lim_{(a \xrightarrow{c} \infty)} \{e^{\langle \rho_c'', \log a\rangle} E(\psi : \lambda : m a) - \sum_{s \in \Omega(\underline{a}, \underline{a}'')} (t^{-1} c(ts:\lambda)\psi)(m) \cdot e^{\langle s\lambda, \log a\rangle}\}$$

equals zero.

PROOF: Let w be a representative of t in ${}^{(o)}\widetilde{M}$. Notice that $m' = w m w^{-1}$ belongs to M', and that $a' = w a w^{-1}$ belongs to $\exp(\underline{a}')^+$ if a is in $\exp(c)$. Furthermore $t\rho_c'' = \rho'$. It follows that

$$e^{\langle \rho_c'', \log a\rangle} E(\psi : \lambda : m a) - \sum_{s \in \Omega(\underline{a}, \underline{a}'')} (t^{-1} c(ts: \lambda)\psi)(m) \cdot e^{\langle s\lambda, \log a\rangle}$$

equals

$$e^{\langle \rho', \log a'\rangle} \tau(w)^{-1} E(\psi : \lambda : m'a') \tau(w)$$

$$- \sum_{s \in \Omega(\underline{a}, \underline{a}'')} \tau(w)^{-1} (c(ts : \lambda)\psi)(m') \cdot \tau(w) \cdot e^{\langle ts\lambda, \log a'\rangle}$$

The lemma follows from (7.1) and (7.4).

$\square$

If c is a Weyl chamber in $\underline{a}$ and δ is a positive number, we denote the set of points in $C(\underline{a}_{\mathbb{C}}, \delta)$ whose real parts lie in c by $C(\underline{a}_{\mathbb{C}}, \delta, c)$. For $s \in \Omega(\underline{a}, \underline{a}')$, we shall sometimes write

$C(\underline{a}_{\mathfrak{c}}, \delta, s)$ for $C(\underline{a}_{\mathfrak{c}}, \delta, s^{-1}((\underline{a}')^{+}))$. ...

LEMMA 7.2: Suppose that $s \in \Omega(\underline{a}, \underline{a}')$ and that c is the chamber $s^{-1}((\underline{a}')^{+})$. Then for any λ in $C(\underline{a}_{\mathfrak{c}}, \delta, c)$, and $m \in M$,

$$\lim_{(a \xrightarrow{c} \infty)} e^{\langle -\lambda + \rho_c, \log a\rangle} \cdot E(\psi : \lambda : m\,a) = (s^{-1}c\,(s : \lambda)\psi)(m).$$

PROOF: From the last lemma we know that

$$\lim_{(a \xrightarrow{c} \infty)} \{e^{\langle -\lambda + \rho_c, \log a\rangle} \cdot E(\psi : \lambda : m\,a) - (s^{-1}c(s : \lambda)\psi)(m)\}$$

equals

$$\Sigma_{\substack{r \in \Omega(\underline{a},\underline{a}) \\ r \neq 1}} (s^{-1}c(s\,r : \lambda)\psi)(m) \cdot \lim_{(a \xrightarrow{c} \infty)} e^{\langle \lambda - r\lambda, \log a\rangle}.$$

Fix a point H in c and a nontrivial element r in $\Omega(\underline{a},\underline{a})$. Notice that the points $\nu' = s\,\lambda_{R}$ and $H' = s\,H$ both lie in $(\underline{a}')^{+}$. In addition,

$$r' = s\,r\,s^{-1}$$

is a nontrivial element in $\Omega(\underline{a}', \underline{a}')$. It is enough to verify the inequality

$$\langle \nu' - r'\nu', H'\rangle > 0.$$

The points ν' and H' both belong to the closure of ${}^{(o)}\underline{a}^{+}$. Apply Lemma 3.1 to the pair $(M', M' \cap {}^{(o)}P)$. If ${}^{(o)}\underline{a}^{+}_{M'}$ is the positive Weyl chamber in $\underline{m}' \cap {}^{(o)}\underline{a}$, there is a unique element ${}^{(o)}r$ in Ω such that

(i), the restriction of ${}^{(o)}r$ to $\underline{a}'$ equals r',

and (ii), ${}^{(o)}r$ maps ${}^{(o)}\underline{a}^{+}_{M'}$ onto itself.

Let ν_o be a fixed point of ${}^{(o)}r$ in ${}^{(o)}\underline{a}^+_{M'}$, of sufficiently small norm. Then both $\nu' + \nu_o$ and $H' + \nu_o$ belong to ${}^{(o)}\underline{a}^+$. It follows from [5(c)] (Corollary 1 of Lemma 35), that the number

$$\langle \nu' + \nu_o - {}^{(o)}r(\nu' + \nu_o),\ H + \nu_o \rangle = \langle \nu' - r'\nu',\ H' \rangle$$

is negative. This completes the proof of the lemma. $\square$

§8. RELATIONS WITH THE INTERTWINING OPERATORS

Once again we fix a standard cuspidal subgroup P of G. Suppose that λ is an element in $\underline{a}_{\mathbb{C}}$ whose real part belongs to $\underline{a}^{+}$. Suppose that τ is a double representation of K and that ψ is in $C_o(M,\tau_M)$. Then Harish-Chandra has shown that for $m \in M$,

$$(c(1 : \lambda)\psi)(m) = \gamma(P)\int_V \psi(m \cdot v_M) \cdot \tau(v_k) \cdot e^{\langle\lambda+\rho,\ H(v)\rangle}\, dv\ ,$$

([5(f)], Lemma 9). The proof involves combining Lemma 7.2, the formula

$$E(\psi : \lambda : m\ a) = \int_{K_M\backslash K} \tau(k^{-1})\ \psi\ (k\ a\ m) \cdot e^{\langle\lambda+\rho,\ H(k\ a\ m)\rangle}\, dk$$

and the change of variables formula

$$\int_{K_M\backslash K} h(k)\ d\ k = \gamma(P)\cdot\int_V h(v_K) \cdot e^{\langle 2\rho,\ H(v)\rangle}\, dv, \quad h \in C_c^{\infty}\ (K_M\backslash K)\ .$$

The reader should be able to reproduce this result.

Now suppose that F is a finite subset of $\underline{E}(K)$, and $\tau = \rho_F$. Suppose also that $\omega \in E_2(M)$, that σ is a representation in the class ω, and that

$$\psi = \psi_T\ , \quad T \in \text{End}\ (\underline{H}_F(\sigma)).$$

For $k_1,\ k_2 \in K$,

$$\int_V (\psi_T\ (m\ v_M)\ \rho_F(v_K))(k_1,\ k_2) \cdot e^{\langle\lambda+\rho,\ H(v)\rangle}\, dv$$

$$= \gamma(P)\int_V \psi_T(m\ v_M)(k_1,\ v_K k) \cdot e^{\langle\lambda+\rho,\ H(v)\rangle}\, dv$$

$$= \gamma(P)\int_V \text{Tr}\ \{\sigma(m\ v_M) \cdot K_T(k_2^{-1}\ v_K^{-1},\ k_1^{-1})\} \cdot e^{\langle\lambda+\rho,\ H(v)\rangle}\, dv\ .$$

Let $s_\ell \in \Omega(\underline{a}, \hat{\underline{a}})$ be the unique element in $\Omega(\underline{a})$ of greatest length. Then $s_\ell^{-1}(\underline{a}'')^+ = -\underline{a}^+$, and $V_{s_\ell} = V$. Let w_ℓ be a fixed representative of s_ℓ in ${}^{(o)}\widetilde{M}$. Recall that w_ℓ can also be regarded as a map from $\underline{H}(\sigma)$ to $\underline{H}(w_\ell\sigma)$.

$w_\ell^{-1} R(w_\ell : \lambda)T$ is a new operator in End $(\underline{H}_F(\sigma))$. It is clear from the definition of K_T that

$$\int_V \operatorname{Tr}\{\sigma(m\, v_M)\quad K_T(k_2^{-1}\, v_K^{-1},\, k_1^{-1})\}\quad e^{\langle\lambda+\rho, H(v)\rangle}dv$$

$$= \operatorname{Tr}\{\sigma(m)\cdot K_{w_\ell^{-1} R(w_\ell:\lambda)T}(k_2^{-1},\, k_1^{-1})\}$$

$$= \psi_{w_\ell^{-1} R(w_\ell:\lambda)T}^{(m)}(k_1, k_2)$$

We have shown that

$$(c(1 : \lambda)\psi_T)(m) = \gamma(P)\cdot \psi_{w_\ell^{-1}R(w_\ell:\lambda)T}(m).$$

We shall generalize this formula. Let $s \in \Omega(\underline{a}, \underline{a}')$. Suppose that $s_\ell' \in \Omega(a', \hat{\underline{a}}')$ is the element in $\Omega(\underline{a}')$ of greatest length. Define $\bar{s} = s_\ell' s$. Then the length of s_ℓ' is the sum of the lengths of $\bar{s}$ and s^{-1}. Choose representatives w, w_ℓ' and $\bar{w}$ in ${}^{(o)}\widetilde{M}$ of these three maps so that

$$\bar{w} = w_\ell' w.$$

<u>LEMMA 8.1</u>: For $\lambda \in C(\underline{a}_c, \delta, s)$, and $T \in$ End $(\underline{H}_F(\sigma))$,

$$(8.1)\qquad c(s:\lambda)\psi_T \;=\; \gamma(P)\cdot\psi_{(w'_\ell)^{-1}R(\bar{w}\,:\,\lambda)T\,R\,(w^{-1}:\,s\,\lambda)}\,.$$

PROOF: Suppose that there is a $T_1 \in \text{End}\ \underline{H}_F(\sigma)$ such that

$$T = R(w^{-1}: s\lambda)\ w\ T_1 .$$

Then for any elements $k_1, k_2 \in K$,

$$E(\psi_T : \lambda : x)_{(k_1,k_2)}$$

$$= E(\psi_{R(w^{-1}: s\lambda)\ w\ T_1} : \lambda : k_1 \times k_2)_{(1,1)}$$

$$= \text{Tr}\ \{\pi(\sigma, \lambda : k_1 \times k_2) \cdot R(w^{-1} : s\lambda)\ w\ T_1\}$$

$$= \text{Tr}\ \{R(w^{-1}: s\lambda)\ \pi(w\sigma, s\lambda : k_1 \times k_2) w\ T_1\}$$

$$= \text{Tr}\ \{\pi(w\sigma, s\lambda : k_1 \times k_2)\ w\ T_1\ R(w^{-1}: s\lambda)\}$$

$$= E(\psi_{w\ T_1 R(w^{-1}: s\lambda)} : s\lambda : x)_{(k_1,k_2)} .$$

Let c be the Weyl chamber $s^{-1}(\underline{a}')^+$. It follows from Lemma 7.2 that for $m \in M$,

$$(s^{-1}c(s : \lambda)\psi_T)(m)$$

$$= \lim_{(a \xrightarrow{c} \infty)} e^{\langle -\lambda + s^{-1}\rho', H(a)\rangle}\ E(\psi_T : \lambda : a\ m)$$

$$= \lim_{(a \xrightarrow{c} \infty)} e^{\langle -\rangle + s^{-1}\rho', H(a)\rangle}\ E(\psi_{wT_1R(w-1: s\lambda)} : s\lambda : a\ m) .$$

Notice that $m' = w\ m\ w^{-1}$ belongs to M' and $a' = w\ a\ w^{-1}$ belongs to $\exp(\underline{a}')^+$. Furthermore, $s\lambda$ belongs to $(\underline{a}')^+$. Our expression equals

$$\lim_{\substack{(\underline{a}')^+ \\ (a' \longrightarrow \infty)}} e^{\langle -\lambda + s^{-1}\rho', s^{-1}H'(a')\rangle} E(\psi_{w\, T_1 R(w^{-1}: s\lambda)} : s\lambda : w^{-1}a'm'w)$$

$$= \rho(w)^{-1}\{\lim_{\substack{(\underline{a}')^+ \\ (a' \longrightarrow \infty)}} e^{\langle -s\lambda + \rho', H'(a')\rangle} E(\psi_{w\, T_1 R(w^{-1}: s\lambda)} : s\lambda : a'm')\} \rho(w) .$$

By Lemma 7.2, this equals

$$\rho(w_1)^{-1}(c(1 : s\lambda)\psi_{w\, T_1 R(w^{-1}: s\lambda)})(m') \cdot \rho(w) ;$$

which by the discussion preceding this lemma is the same as

$$\gamma(P) \cdot \rho(w)^{-1} \cdot \psi_{(w_\ell')^{-1} R(w_\ell' : s\lambda)\, w\, T_1 R(w^{-1}: s\lambda)}(m') \quad \rho(w)$$

$$= \gamma(P)\, (s^{-1}\psi_{(w_\ell')^{-1} R(w^{-1}: s\lambda) w\, T_1 R(w^{-1} : s\lambda)})(m) .$$

By Corollary 4.5,

$$R(w_\ell' : s\lambda) = R(\overline{w} : \lambda)\, R(w^{-1} : s\lambda) .$$

This completes the verification of (8.1) in the case that there is a T_1 such that

$$T = R(w^{-1} : s\lambda)\, w\, T_1 = w^{-1} \cdot r(s^{-1}: s\lambda) \cdot w\, T_1 .$$

By Corollary 4.7, any $T \in \mathrm{End}\ \underline{H}_F(\sigma)$ may be written this way for almost all λ. By continuity, (8.1) is true for all $T \in \mathrm{End}\ \underline{H}_F(\sigma)$, and all $\lambda \in C(\underline{a}_c, \delta, s)$.

□

COROLLARY 6.2: Under the hypotheses of the lemma,

$$s^{-1}\, c(s : \lambda)\psi = \gamma(P)\psi_{r(\bar{s} : \lambda)T\, r(s : -\bar{\lambda})^*}.$$

PROOF: It is a simple matter to check that if $T' \in \text{End}\ \underline{H}_F(\sigma)$,

$$s^{-1}\,\psi_{T'} = \psi_{w^{-1}\, T'\, w}$$

Therefore, by the lemma,

$$s^{-1}\, c(s : \lambda)\, \psi_T = \gamma(P)\psi_{w^{-1}(w'_\ell)^{-1}\, R(\bar{w} : \lambda)T\, R(w^{-1} : s\lambda)w}.$$

Now

$$w^{-1}(w'_\ell)^{-1}\, R(\bar{w} : \lambda) = \bar{w}^{-1}\, R(\bar{w} : \lambda) = r(\bar{s} : \lambda),$$

while by (4.6)

$$R(w^{-1} : s\lambda)w = R(w : -\bar{\lambda})^*(w^{-1})^* = r(s : -\bar{\lambda})^*.$$

This proves the corollary. □

Harish-Chandra has shown that $c(s : \lambda)$ can be analytically continued as a meromorphic function of λ on $\underline{a}_{\mathbb{C}}$. He has also shown that the restriction of the operator

$$d(s : \lambda) = c(s : -\bar{\lambda})^*\ c(s : \lambda) : \underline{C}(M, \tau_M) \longrightarrow \underline{C}(M', \tau_{M'})$$

to $\underline{C}_\omega(M, \tau_M)$ is a scalar multiple of the identity which is independent of s, ([5(f)]). Here

$$c(s : \lambda)^* : \underline{C}(M', \tau_{M'}) \longrightarrow \underline{C}(M, \tau_M)$$

is the adjoint of $c(s : \lambda)$. His first step in carrying out the proof of these two facts is to deal with the case that P is a maximal parabolic subgroup of G. We shall assume this first step here. We will then be able to obtain the general case from the work of §4. Specifically we shall assume that the above two results are valid if (G,P) is replaced by the pair (M^{β}, P_{β}), for β a reduced root of (P, A).

Suppose then that P is of arbitrary rank and that β is an element in $\bar{\Sigma}$. If F is a finite subset of $\underline{E}(K^{\beta})$ and $T \in$ End $\underline{H}_{\beta,F}(\sigma)$, the function

$$\nu \longrightarrow c_{\beta}(1 : \nu)\, \psi_T \;, \; \nu \in C((\underline{a}_{\beta})_{\mathbb{C}}, \delta, 1),$$

can be analytically continued to $(\underline{a}_{\beta})_{\mathbb{C}}$. Here, as usual, the subscripts β connote objects associated to the group M^{β}. Recalling the definition of the map $r_{\beta}(\nu)$, we see from Corollary 6.2 that

$$(6.2) \qquad c_{\beta}(1 : \nu)\, \psi_T = \gamma_{\beta}(P_{\beta}) \cdot \psi_{r_{\beta}(\nu)T} \;.$$

By Theorem 5.1, the map

$$T' \longrightarrow \psi_{T'} \;, \; T' \in \text{End } \underline{H}_{\beta,F}(\sigma),$$

is an isomorphism. Therefore the function

$$\nu \longrightarrow r_{\beta}(\nu) \quad , \; \nu \in (\underline{a}_{\beta})_c \;, \; \langle \nu, \beta \rangle_R > 0,$$

can be analytically continued to $(\underline{a}_{\beta})_{\mathbb{C}}$. It follows that the map

$$\lambda \longrightarrow r_{\beta}^{G}(\lambda) \;, \; \nu \in \underline{a}_{\mathbb{C}}, \langle \lambda, \beta \rangle_R > 0,$$

can be analytically continued to $\underline{a}_{\mathbb{C}}$.

Suppose that $\underline{a}'$ is another distinguished subspace of ${}^{(o)}\underline{a}$, and that s belongs to $\Omega(\underline{a}, \underline{a}')$. Let w be a representative of s

in ${}^{(o)}\tilde{M}$. Then it follows from Lemma 4.6, Corollary 8.2, and the definition of $R(w : \lambda)$ that the functions

$$\lambda \longrightarrow r(s : \lambda),\ \lambda \in T(s),$$

$$\lambda \longrightarrow c(s : \lambda),\ \lambda \in C(\underline{a}_c, \delta, s),$$

$$\lambda \longrightarrow R(w : \lambda),\ \lambda \in T(s),$$

can all be analytically continued as meromorphic functions to $\underline{a}_{\mathbb{C}}$. If F is a finite subset of $\underline{E}(K)$ and A and B are operators on the Hilbert space $\underline{H}_F(\sigma)$, then the adjoint of the map

$$\psi_T \longrightarrow \psi_{A\,T\,B},\quad T \in \text{End } \underline{H}_F(\sigma),$$

is the map

$$\psi_T \longrightarrow \psi_{A^*\,T\,B^*},\quad T \in \text{End } \underline{H}_F(\sigma).$$

Combining this fact, the equation

$$c(s : -\bar{\lambda})^* c(s : \lambda) = (s^{-1}c(s : -\bar{\lambda}))^* (s^{-1}c(s : \lambda)),$$

and Corollary 8.2, we see that

$$(8.3)\qquad d(s : \lambda)\ \psi_T = \gamma(P)^2 \cdot \psi_{r(\bar{s} : -\bar{\lambda})^* r(\bar{s} : \lambda) T\, r(s : -\bar{\lambda})^* r(s : \lambda)}$$

for all $T \in \text{End } \underline{H}_F(\sigma)$.

Suppose again that β is as above. Then for $T \in \text{End } \underline{H}_{\beta,F}(\sigma)$, we see from (8.2) that

$$(8.4)\qquad d_\beta(1 : \lambda_\beta)\ \psi_T = c_\beta(1 : -\bar{\lambda}_\beta)^*\ c(1 : \lambda_\beta)\ \psi_T$$
$$= \gamma_\beta(P_\beta)^2 \cdot \psi_{r_\beta(-\bar{\lambda}_\beta)^* \cdot r_\beta(\lambda_\beta) T}.$$

According to our earlier remarks, there is a complex valued meromorphic function

$$\nu \longrightarrow \delta_\beta(\omega, \nu),\quad \nu \in (\underline{a}_\beta)_{\mathbb{C}},$$

such that

$$d_\beta(1 : \lambda_\beta)\ \psi_T = \delta_\beta(\omega, \lambda_\beta)\ \psi_T.$$

It follows easily from (8.4) and the definition of the operator r^G_β that for any $\phi \in \underline{H}_F(\sigma)$,

$$r^G_\beta(-\bar{\lambda})^* \; r^G_\beta(\lambda)\phi = \gamma_\beta(P_\beta)^{-2} \cdot \delta_\beta(\omega,\lambda_\beta)\ \phi .$$

If s is as above, we examine (8.3). In the notation of Lemma 4.6 the operator

$$r(s : -\bar{\lambda})^* \; r(s : \lambda)$$

on $\underline{H}_F(\sigma)$ equals

$$r^G_{\beta_n}(-\bar{\lambda})^* \cdots\cdots r^G_{\beta_1}(-\bar{\lambda})^* \cdot r^G_{\beta_1}(\lambda)\cdots\cdots r^G_{\beta_n}(\lambda).$$

By Lemma 3.2 and our above results, this is just

$$\prod_{\beta\in\bar{\Sigma}_s} \gamma_\beta(P_\beta)^{-2} \cdot \delta_\beta(\omega,\lambda_\beta) \ I .$$

Similarly on $\underline{H}_F(\sigma)$ we have

$$r(\bar{s} : -\bar{\lambda})^* \; r(\bar{s} : \lambda) = \prod_{\beta\in\bar{\Sigma}_{\bar{s}}} \gamma_\beta(P_\beta)^{-2} \cdot \delta_\beta(\omega,\lambda_\beta) \cdot I .$$

However, $\bar{\Sigma}$ is the disjoint union of $\bar{\Sigma}_s$ and $\bar{\Sigma}_{\bar{s}}$. We have thus shown that the restriction of $d(s : \lambda)$ to $\underline{C}_\omega(M, \rho_{F,M})$ is the product of the identity operator and the scalar

$$\gamma(P)^2 \cdot \prod_{\beta\in\bar{\Sigma}} \gamma_\beta(P_\beta)^{-2} \ \delta_\beta(\omega,\lambda_\beta) .$$

Let us denote this scalar by $\delta(\omega,\lambda)$. Then $\delta(\omega,\lambda)$ is a meromorphic function which is independent of s. Recalling the formula (4.5), we obtain

$$(8.5) \quad \delta(\omega,\lambda) = \prod_{\beta\in\bar{\Sigma}} \delta_\beta(\omega,\lambda_\beta) .$$

Harish-Chandra has obtained this product formula in [5(f)] by essentially the above method. According to our remarks in §5, we can replace ρ_F in the above discussion by any representation τ in $F(K,K)$, and still obtain the same formulas.

For harmonic analysis it is convenient to define

$$\mu(\omega,\lambda) = \gamma(P)^{-1} \cdot \delta(\omega,\lambda)^{-1} .$$

This is a meromorphic function of λ which is nonnegative if λ is purely imaginary. For $\beta \in \bar{\Sigma}$ and $\nu \in (\underline{a}_\beta)_{\mathbb{C}}$, we define $\mu_\beta(\omega,\nu)$ the same way. Harish-Chandra has calculated $\mu_\beta(\omega,\nu)$ explicitly. In view of the above product formula, this gives an explicit formula for $\mu(\omega,\lambda)$. In particular it establishes certain properties of $\mu(\omega,\lambda)$ which we now list.

<u>LEMMA 8.3</u>: (i) For $s \in \Omega(\underline{a}, \underline{a}')$

$$\mu(\omega,\lambda) = \mu(s\omega, s\lambda).$$

(ii) There is a constant $\delta > 0$, independent of ω, such that $\mu(\omega,\lambda)$ is analytic for $\lambda \in C(\underline{a}_{\mathbb{C}},\delta)$. In fact, given any differential operator D_λ with constant coefficients on $\underline{a}_{\mathbb{C}}$, there is a polynomial p such that for $\omega \in \underline{E}_2(M)$ and $\lambda \in C(\underline{a}_{\mathbb{C}},\delta)$,

$$|D_\lambda \mu(\omega,\lambda)| \le p(|\omega| + |\lambda|).$$

□

<u>LEMMA 8.4</u>: For $\omega \in \underline{E}_2(M)$, σ a representation in the class ω, and λ a P-regular point in $i\underline{a}$, the representation $\pi(\sigma, \lambda)$ is irreducible.

PROOF: Assume the contrary. Then we can find two unit vectors ϕ_1 and ϕ_2 in $\underline{H}^o(\sigma)$ such that for each $x \in G$,

$$(\pi(\sigma, \lambda : x)\phi_1, \phi_2) = 0 .$$

Let T be the operator on $\underline{H}(\sigma)$ which maps ϕ_2 onto ϕ_1 and which vanishes on the orthogonal complement of ϕ_2 in $\underline{H}(\sigma)$. Then for $x \in G$, and $k_1, k_2 \in K$,

$$E(\psi_T : \lambda : x)_{(k_1,k_2)}$$

$$= E(\psi_T : \lambda : k_1 \; x \; k_2)_{(1,1)}$$

$$= \operatorname{Tr}\{\pi \; (\sigma, \lambda : k_1 \; x \; k_2)T\}$$

$$= 0 .$$

We apply Lemma 6.1 with ${}^{(u)}P = P$. Fix a small number $\varepsilon > 0$. Given $m \in M$ we can find positive numbers D and d such that for $H \in {}^{(u)}S\,(\varepsilon)$, $m \in M$, and $t \geq 0$,

$$\left|\Sigma_{s \in \Omega(\underline{a}\;,\underline{a}\;)}\;(c(s : \lambda)\psi_T)(m) \cdot e^{t<s\lambda,H>}\right| \leq D \cdot e^{-dt}.$$

We are of course using the fact that

$$E(\psi_T : \lambda : m \exp t H) = 0.$$

The vectors

$$\{s\,\lambda : s \in \Omega(\underline{a}, \underline{a})\}$$

are all distinct and imaginary. In addition, H is allowed to vary over an open set in the unit sphere of $\underline{a}$. It follows that for any s,

$$(c(s : \lambda)\psi_T)(m) = 0 .$$

On the other hand, the map

$$\zeta \longrightarrow c(s : -\bar{\zeta})^* , \quad \zeta \in \underline{a}_{\mathbb{C}},$$

is regular at $\zeta = \lambda$. In particular

$$\delta(\omega, \lambda)\psi_T = c(s : -\bar{\lambda})^* c(s : \lambda)\psi_T = 0 .$$

But in view of Lemma 6.3, the function

$$\delta(\omega,\zeta) = \gamma(P)^{-1} \cdot \mu(\omega,\zeta)^{-1} , \quad \zeta \in \underline{a}_{\mathbb{C}},$$

cannot vanish on $i\,\underline{a}$. It follows that $\psi_T = 0$. This contradicts the assertion in Lemma 5.1 that the map

$$T \longrightarrow \psi_T , \quad T \in \mathrm{End}^o(\underline{H}(\sigma)),$$

is injective. □

Suppose that $s \in \Omega(\underline{a}, \underline{a}')$ as above, and that $s' \in \Omega(\underline{a}', \underline{a}'')$ for some other distinguished subspace $\underline{a}''$ of ${}^{(o)}\underline{a}$. Let w' and w be representatives in ${}^{(o)}\widetilde{M}$ of s' and s respectively. Then

$$\lambda \longrightarrow \rho_{w',w}(\sigma,\lambda) = R(w'w : \lambda)^{-1} R(w' : s\lambda) R(s' : \lambda), \quad \lambda \in \underline{a}_{\mathbb{C}} ,$$

is a nontrivial meromorphic function which for any finite subset F of $\underline{E}(K)$ takes values in $\mathrm{End}\,\underline{H}_F(\sigma)$. Corollary 4.7 has permitted us to define $R(w'w : \lambda)^{-1}$. If f is a K-finite function in $C_c^{\infty}(G)$, we have

$$\rho_{w',w}(\sigma, \lambda)\, \pi(\sigma, \lambda : f) = \pi(\sigma, \lambda : f)\, \rho_{w',w}(\sigma, \lambda),$$

by (4.4). It follows from the last lemma that $\rho_{w',w}(\sigma,\lambda)$ must be a scalar multiple of the identity operator when λ is a P-regular point in $i\underline{a}$. Therefore by analytic continuation we can regard

$$\lambda \longrightarrow \rho_{w',w}(\sigma, \lambda), \ \lambda \in \underline{a}_{\mathbb{C}},$$

as a complex valued meromorphic function such that

(8.6) $\quad \rho_{w',w}(\sigma, \lambda)\, R(w'w : \lambda) = R(w' : s\lambda)\, R(w : \lambda).$

Of course if the length of $s's$ is the sum of the lengths of s' and s,

$$\rho_{w',w}(\sigma, \lambda) = 1 .$$

Following Harish-Chandra, we define

$$M(s : \lambda) = c(1 : s\lambda)^{-1}\, c(s : \lambda), \ \lambda \in \underline{a}_{\mathbb{C}} .$$

(Harish-Chandra denotes: $M(s : \lambda)$ by ${}^{\circ}c(s : \lambda)$.)

This gives a meromorphic function which for any F takes values in the space of linear maps from $\underline{C}_{\omega}(M, \rho_{F,M})$ to $\underline{C}_{s\omega}(M', \rho_{F,M'})$. In the notation of Lemma 8.1 we observe that for $T \in \mathrm{End}^{\circ}(\underline{H}(\sigma))$,

$$M(s : \lambda)\psi_T$$

$$= c(1 : s\lambda)^{-1}\ \psi_{(w_\ell^1)^{-1} R(\bar{w} : \lambda) T R(w^{-1} : s\lambda)}$$

$$= \psi_{R(w_\ell' : s\lambda)^{-1} R(\bar{w} : \lambda) T R(w^{-1} : s\lambda)} .$$

However, by Corollary 4.5,

$$R(w_\ell' : s\lambda)^{-1} R(\bar{w} : \lambda)$$

$$= R(\bar{w}\, w^{-1} : s\lambda)^{-1} R(\bar{w} : \lambda)$$

$$= (R(\bar{w} : \lambda)\, R(w^{-1} : s\lambda))^{-1}\ R(\bar{w} : \lambda)$$

$$= R(w^{-1} : s\lambda)^{-1}.$$

Finally, by (8.6),

$$R(w^{-1} : s\lambda) = \rho_{w,w^{-1}}(\sigma, \lambda) \cdot R(w : \lambda)^{-1} .$$

We have shown that

$$(8.7) \quad M(s : \lambda)\psi_T = \psi_{R(w : \lambda)T\, R(w : \lambda)^{-1}} .$$

Harish-Chandra's functional equations for $M(s : \lambda)$ can now be proved easily.

<u>Theorem 8.5</u>: For $s \in \Omega(\underline{a}, \underline{a}')$ and $T \in \mathrm{End}^{\circ}(\underline{H}(\sigma))$,

$$E(\psi_T : \lambda : x) = E(M(s : \lambda)\psi_T : s\lambda : x), \ \lambda \in \underline{a}_{\mathbb{C}}.$$

If $s' \in \Omega(\underline{a}', \underline{a}'')$,

$$M(s's : \lambda) = M(s' : s\lambda)\, M(s : \lambda), \ \lambda \in \underline{a}_{\mathbb{C}} .$$

Furthermore, we have

$$M(s : \lambda)^* = M(s^{-1} : - s\bar{\lambda}) = M(s : - \bar{\lambda})^{-1}, \ \lambda \in \underline{a}_{\mathbb{C}} .$$

In particular $M(s : \lambda)$ is analytic on $i\,\underline{a}$.

PROOF: The first statement of the theorem follows from (8.7), Lemma 5.2, and the intertwining property (4.4). The second statement is an immediate consequence of (8.7) and (8.6). For the third statement we note that if F is a finite subset of $\underline{E}(K)$, the adjoint of the map

$$\psi_T \longrightarrow \psi_{R(w : \lambda)\, T\, R(w : \lambda)^{-1}}, \ T \in \mathrm{End}\, \underline{H}_F(\sigma),$$

is the map

$$\psi_{T'} \longrightarrow \psi_{R(w : \lambda)^*\, T'\, R(w : \lambda)^{*-1}}, \ T' \in \mathrm{End}\, H_F(w\sigma) .$$

But by (4.6) and (8.7),

$$ {}^{\psi}R(w : \lambda)^{*} \; T' \; R(w : \lambda)^{*\,-1} = M(s^{-1} : -\, s\,\bar{\lambda}) \; \psi_{T'} \, , $$

which by the first statement of the lemma equals

$$ M(s : -\bar{\lambda})^{-1} \; \psi_{T'} \quad . $$

If λ is imaginary, the norm $||M(s : \lambda)||$ equals one. It follows that $M(s : \lambda)$ is regular.

$\square$

BIBLIOGRAPHY

1. Arthur, J., Harmonic Analysis of Tempered Distributions on Semisimple Lie Groups of Real Rank One, Thesis, Yale, 1970.
2. Bourbaki, N., Éléments de mathématique XXXIV, Groupes et algébres de Lie, Hermann, Paris, 1968.
3. Chevalley, C., Invariants of finite groups generated by reflections, Amer. J. Math. 77 (1955), 778-782.
4. Gindikin, S.G., and Karpelevic, F.I., Plancherel measure of Riemannian symmetric spaces of non-positive curvature, Doklady Akad. Nauk. SSSR, vol.145 (1962), 252-255.
5. Harish-Chandra,

   (a) Representations of a semisimple Lie group on a Banach space I, T.A.M.S. 75 (1953), 185-243.

   (b) The characters of semisimple Lie groups, T.A.M.S. 83 (1956), 98-163.

   (c) Spherical functions on a semisimple Lie group I, Amer. J. Math. 80 (1958), 241-310.

   (d) Two theorems on semisimple Lie groups, Ann. of Math. 83 (1965), 241-318.

   (e) Discrete series for semisimple Lie groups II, Acta. Math. 116 116 (1966), 1-111.

   (f) On the theory of the Eisenstein integral, Lecture Notes in Mathematics, vol. 266 (1970), 529-551.
6. Knapp, A.W., and Stein, E.M., Intertwining operators for semisimple groups, Ann. of Math., 93 (1971), 489-578.

7. Langlands, R.P.,

(a) On the functional equations satisfied by Eisenstein series, mimeographed notes.

(b) On the classification of irreducible representations of real algebraic groups, preprint.

8. Nelson, E., Analytic vectors, Ann. of Math. 70 (1959), 572-615.

9. Schiffmann, G., Integrales d'entrelacement et fonctions de Whittaker, Bull. Soc. Math. France 99 (1971), 3-72.

10. Trombi, P.C., and Varadarajan, V. S., Asymptotic behaviour of eigenfunctions on a semisimple Lie group: the discrete spectrum, Acta. Math. 129 (1972), 237-280.

11. Warner, G., Harmonic analysis on semisimple Lie groups I and II, Springer, Berlin, 1972.

12. Wolf, J. A., Unitary Representations on Partially Holomorphic Cohomology Spaces, Memoirs of A.M.S., 138.

HARMONIC ANALYSIS OF THE SCHWARTZ SPACE ON A REDUCTIVE LIE GROUP II

by

James Arthur

INTRODUCTION

This paper is the second half of [1(b)]. In the introduction to [1(b)], we alluded to a theorem on the Fourier transform of the Schwartz space on a reductive Lie group G. Without formally stating the theorem in that paper, we laid the groundwork for proving it. In this paper we shall state the theorem (Theorem 3.1), and complete its proof (§6).

Theorem 3.1 is already known in some special cases. For G = PSL(2, ℝ), the result was established by Ehrenpreis and Mautner, ([3]). In [1(a)], the case of real rank one case was dealt with. In [2], Eguchi establishes the result in case G has one conjugacy class of Cartan subgroups. Analogues of Theorem 3.1 for certain subspaces of $\underline{C}(G)$ have been proved by Harish-Chandra, in his second paper on spherical functions, by Eguchi and Okomoto, and by Trombi and Varadarajan ([6]).

The techniques for proving any result similar to Theorem 3.1 were introduced by Harish-Chandra, first, in his second paper on spherical functions, and later, in his second paper on discrete series ([4(c)]). In fact most of the results of both this paper and [1(b)] are simply restatements of results of Harish-Chandra. The author learned a great deal from Harish-Chandra's lectures at the Institute for Advanced Study, 1970-1971, in which much of the work summarized in [4(d)] was presented in detail.

CONTENTS

§1. DEFINITION OF $\hat{f}$

Suppose that G is a reductive Lie group with Lie algebra $\underline{g}$. Suppose that K is a maximal compact subgroup of G, that Θ is an involution on $\underline{g}$, and that B is a bilinear form on $\underline{g}$ such that (G, K, Θ, B) satisfies all the assumptions of [1 (b)], §1. We shall use the definitions and notation of [1 (b)], usually without further comment.

Let $P = NAM$ be a standard cuspidal subgroup of G, and choose $\omega \in \underline{E}_2(M)$. Recall that ρ is a double representation of K on $L^2(K \times K)$, and that ρ_M is the restriction of ρ to $K_M = K \cap M$. We let $L^2_\omega(M, \rho_M)$ be the set of functions

$$\psi: (m, (k_1,k_2)) \longrightarrow \psi(k_1 : m : k_2),\ k_1, k_2 \in K,\ m \in M,$$

in $L^2_\omega(M) \otimes L^2(K \times K)$ which are ρ_M-spherical. This means that for $k_1', k_2' \in K_M$,

$$\psi(k_1 : k_1' m k_2' : k_2) = \psi(k_1 k_1' : m : k_2' k_2),\ k_1, k_2 \in K,\ m \in M.$$

$L^2_\omega(M, \rho_M)$ is a Hilbert space under the norm

$$\|\psi\| = \int_M \int_{K \times K} |\psi(k_1 : m : k_2)|^2 d\,k_1 d\,k_2 \ d\,m \quad , \ \psi \in L^2_\omega(M, \rho_M) .$$

For $\psi_1, \psi_2 \in L^2_\omega(M, \rho_M)$, $k_1, k_2 \in K$, and $m \in M$, we define

$$(\psi_1 \psi_2)(k_1 : m : k_2) = \int_M \int_K \psi_1(k_1 : \tilde{m} : k^{-1})\, \psi_2(k : \tilde{m}^{-1} m : k_2)\, d\,k\, d\,\tilde{m} .$$

This integral converges for almost all k_1 and k_2. It makes $L^2_\omega(M, \rho_M)$ a Hilbert algebra.

Suppose that σ is a representation of M in the class ω, acting on the Hilbert space H_σ. The map

$$T \longrightarrow d_\omega^{\frac{1}{2}} \quad \psi_T \quad , \quad T \in \mathrm{End}^\circ\,(\underline{H}(\sigma)),$$

defined in [1(b)] extends to an isometry from the space of Hilbert-Schmidt operators on $\underline{H}(\sigma)$ onto $L^2_\omega\,(M,\rho_M)$.

It is easy to interpret $L^2_\omega\,(M,\rho_M)$ directly as a space of Hilbert-Schmidt operators. Let ξ be a fixed vector of norm $d_\omega^{\frac{1}{2}}$ in H_σ. For $\varphi \in \underline{H}(\sigma)$, define

$$\varphi_\xi(m\,k) = (\varphi(m\,k),\, \xi), \quad m \in M,\ k \in K.$$

Then the map

$$\varphi \longrightarrow \varphi_\xi \ , \ \varphi \in \underline{H}(\sigma) \ ,$$

is a linear isometry from $\underline{H}(\sigma)$ onto a closed subspace $\underline{H}\,(\sigma)_\xi$ of $L^2(M \times K)$. For $T \in \mathrm{End}^\circ(\underline{H}(\sigma))$, and $\varphi \in \underline{H}(\sigma)$, we have

$$\int_K \int_M \varphi_\xi(m\,k)\ \psi_T(k^{-1} : m^{-1} m' : k')\ d\,m\,d\,k$$

$$= \int_K \int_M (\sigma(m)\varphi(k),\ \xi) \cdot \underline{\mathrm{Tr}\ \{\sigma(m^{-1})\sigma(m')K_T(k'^{-1},\ k)\}}\ d\,m\,d\,k$$

$$= \Sigma_i \int_K \int_M (\sigma(m)\varphi(k),\ \xi)\ (\sigma(m)\xi_i,\ \sigma(m')\ K_T(k'^{-1},\ k)\xi_i)\, d\,m\,d\,k\ ,$$

where $\{\xi_i\}$ is an orthonormal basis of H_σ. By the Schur orthogonality relations, this expression equals

$$d_\omega^{-1} \int_K (\varphi(k),\ \xi_i)\ (\sigma(m')K_T\ (k'^{-1},\ k)\ \xi_i,\ \xi)\ d\ k$$

$$= d_\omega^{-1} \int_K (\sigma(m')\ K_T(k'^{-1}, k)\ \varphi(k),\ \xi)\ d\ k$$

$$= d_\omega^{-1}\ (\sigma(m')\ (T\ \varphi)\ (k'),\ \xi)\ .$$

We have shown that

$$(1.1) \qquad (T\ \varphi)_\xi(m'k') = d_\omega \int_K \int_M \varphi_\xi(m\ k) \cdot \psi_T(k^{-1} : m^{-1}m' : k')\ d\ m\ d\ k\ .$$

Suppose that $S,\ T \in \mathrm{End}^{\circ}\ \underline{H}(\sigma)$, and that φ and ξ are as above. Then

$$\int_K \int_M \varphi_\xi(m\ k) \cdot \psi_{T\ S}(k^{-1} : m^{-1}m'' : k'')\ d\ m\ d\ k$$

$$= d_\omega^{-1}\ (T\ S\ \varphi)_\xi\ (m''\ k'')$$

$$= \int_K \int_M (S\ \varphi)_\xi\ (m'\ k') \cdot \psi_T(k'^{-1} : m'^{-1}m'' : k'')\ d\ m'\ d\ k'$$

$$= d_\omega \int_K \int_M \int_K \int_M \varphi_\xi(m\ k)\ \psi_S(k^{-1} : m^{-1}\ m' : k')\ \psi_T(k'^{-1} : m'^{-1}m'' : k'')\ d\ m\ d\ k\ d\ m'\ d\ k'$$

$$= d_\omega \int_K \int_M \varphi_\xi(m\ k)\ \left\{\int_K \int_M \psi_S(k^{-1} : m' : k')\ \psi_T(k'^{-1} : m'^{-1}\ m'\ m'' : k'')\ d\ m'\ d\ k'\right\}\ d\ m\ d\ k$$

$$= d_\omega \int_K \int_M \varphi_\xi(m\ k)\ (\psi_S\ \ \psi_T)\ (k^{-1} : m^{-1}m'' : k'')\ d\ m\ d\ k\ .$$

It follows that

$$(1.2) \qquad \psi_{T\ S} = d_\omega\ (\psi_S\ \ \psi_T)\ .$$

By continuity, this equation holds if S and T are arbitrary Hilbert-Schmidt operators on $\underline{H}(\sigma)$.

For $\psi \in L^2_\omega(M, \rho_M)$ define

$$\psi^*(k_1 : m : k_2) = \overline{\psi(k_2^{-1} : m^{-1} : k_1^{-1})}$$

If $\psi(k_1 : m : k_2)$ is continuous, we also define

$$\tau(\psi) = \int_K \psi(k^{-1} : 1 : k)\, dk .$$

Notice that if $\psi = \psi_1 \psi_2^*$,

$$\tau(\psi) = \int_K \int_M \int_K \psi_1(k_1^{-1} : m : k^{-1})\, \psi_2^*(k : m^{-1} : k_1)\, dk\, dm\, dk_1$$

$$= \int_{K \times K} \int_M \psi_1(k_1^{-1} : m : k^{-1}) \cdot \overline{\psi_2(k_1^{-1} : m : k^{-1})}\, dm \cdot dk\, dk_1$$

$$= (\psi_1, \psi_2) .$$

Suppose that σ is a representation in the class ω. If T is a Hilbert-Schmidt operator on $\underline{H}(\sigma)$,

$$\psi_T^* = \psi_{T^*} .$$

Suppose in addition that T is of trace class and that $\psi_T(k_1 : m : k_2)$ is continuous. Then we can write

$$T = S_2^* S_1$$

for Hilbert-Schmidt operators S_1 and S_2. By (1.2) and the above remark, $\tau(\psi_T)$ equals

$$d_\omega \cdot \tau(\psi_{S_1} \psi_{S_2}^*) = d_\omega (\psi_{S_1}, \psi_{S_2}) .$$

Since the map

$$S \longrightarrow d_\omega^{\frac{1}{2}} \cdot \psi_S$$

is an isometry, this expression is just

$$\mathrm{Tr}\ (S_1\ S_2^*) = \mathrm{Tr}\ (S_2^*\ S_1) = \mathrm{Tr}\ (T)\ .$$

We have shown that

$$(1.3) \qquad \tau(\psi_T) = \mathrm{Tr}\ (T)\ .$$

For $\lambda \in i\ \underline{a}$ and $f \in C_c^\infty(G)$,

define

$$\hat{f}(\omega,\lambda) = d_\omega \cdot \psi_{\pi(\sigma,\lambda\ :\ \tilde{f})}\ ,$$

where

$$\tilde{f}\ (x) = f(x^{-1}).$$

$\hat{f}(\omega,\lambda)$ is an element of $L^2_\omega(M,\rho_M)$. It is independent of the representative σ of ω. If g is another function in $C_c^\infty(G)$,

$$\hat{f}(\omega,\lambda)\ \hat{g}(\omega,\lambda) = (d_\omega)^2 \cdot \psi_{\pi(\sigma,\lambda:\tilde{f})} \cdot \psi_{\pi(\sigma,\lambda:\tilde{g})}$$

$$= d_\omega \cdot \psi_{\pi(\sigma,\lambda\ :\ \tilde{g})\pi(\sigma,\lambda\ :\ \tilde{f})}$$

$$= d_\omega \cdot \psi_{\pi(\sigma,\lambda\ :\ \widetilde{f*g})}\ .$$

In other words,

$$(1.4) \qquad (\widehat{f * g})\ (\omega,\lambda) = \hat{f}(\omega,\lambda)\ \hat{g}(\omega,\lambda)\ .$$

Define a two sided action $\pi(\omega,\lambda)$ of G on $L^2_\omega(M,\rho_M)$ by

$$\pi(\omega,\lambda\ :\ y_1) \cdot \psi_T \cdot \pi(\omega,\lambda\ :\ y_2) = \psi_{\pi(\sigma,\lambda:y_2^{-1})\ T\ \pi(\sigma;\lambda:y_1^{-1})}\ ,$$

for y_1, $y_2 \in G$ and T a Hilbert-Schmidt operator on $\underline{H}(\sigma)$. Let

r denote the regular two-sided action

$$(r(y_1)f\, r(y_2))(x) = f(y_1^{-1}\, x\, y_2^{-1}), \quad f \in L^2(G),\ x \in G,$$

of G on $L^2(G)$. For $f \in C_c^\infty(G)$ we have

$$\pi(\sigma, \lambda : \widetilde{r(y_1)\, f\, r(y_2)})$$

$$= \int_G f(y_1^{-1}\, x\, y_2^{-1})\, \pi(\sigma, \lambda : x)\, d\,x$$

$$= \int_G f(x^{-1})\, \pi(\sigma, \lambda : y_2^{-1})\, \pi(\sigma, \lambda : x)\, \pi(\sigma, \lambda : y_1^{-1})\, d\,x$$

$$= \pi(\sigma, \lambda : y_2^{-1})\, \pi(\sigma, \lambda : \tilde{f})\, \pi(\sigma, \lambda : y_1^{-1})\ .$$

It follows that

$$(r(y_1)\, f\, r(y_2))^\wedge\ (\omega,\lambda)$$

$$= d_\omega \cdot \psi\, \pi(\sigma,\lambda : y_2^{-1})\pi(\sigma, \lambda : \tilde{f})\pi(\sigma, \lambda : y_1^{-1})$$

$$= \pi(\omega,\lambda : y_1)\ \hat{f}(\omega,\lambda)\ \pi(\omega,\lambda : y_2)\ .$$

Suppose that T is an arbitrary operator in $\mathrm{End}^o(\underline{H}(\sigma))$. Then

$$(\hat{f}(\omega,\lambda),\ \psi_T) = d_\omega\ (\psi_{\pi(\sigma,\, \lambda\, :\, \hat{f})},\ \psi_T)$$

$$= \mathrm{Tr}\ \{\pi(\sigma, \lambda : \hat{f})\ T^*\}$$

$$= \int_G f(x^{-1})\ \mathrm{Tr}\ \{T^*\ \pi(\sigma, \lambda : x)\}\ d\,x$$

$$= \int_G f(x)\quad \mathrm{Tr}\ \{\pi(\sigma, \lambda : x)T\}^*\ d\,x$$

$$= \int_G f(x)\quad \overline{E(\psi_T : \lambda : x)}_{(1,1)}\ d\,x,.$$

by [1(b)], Lemma 5.2. This last expression equals

$$\int_G f(x) \cdot \int_K \overline{(\rho_F(k_1^{-1})\ \psi_T(k_1 x))_{(1,1)} \cdot e^{\langle\lambda+\rho,\ H(k_1 x)\rangle}}\ d\,k_1 \cdot d\,x$$

$$= \int_K \int_G f(k_1 x) \cdot \overline{\psi_T(x)_{(k_1,1)} \cdot e^{\langle\lambda+\rho, H(x)\rangle}}\ d\,x\,d\,k_1$$

$$= \int_{K\times K} \int_M \int_A \int_N f(k_1 a\ n\ m\ k_2)\ \overline{\psi_T(a\ n\ m\ k_2)}_{(k_1,1)} \cdot \overline{e^{\langle\lambda+\rho, H(a)\rangle}}\ dndadm\ dk_1 dk_2$$

$$= \int_{K\times K} \int_M \left\{ \int_A \int_N f(k_1 a\ n\ m\ k_2)\ e^{\langle-\lambda+\rho, H(a)\rangle}\ d\,n\,d\,a \right\}$$

$$\overline{\psi_T(k_1 : m : k_2)}\ d\,m\,d\,k_1\,d\,k_2\ ,$$

since λ is purely imaginary. If α is any function in $\underline{C}(M)$, and Θ_ω is the character of ω, the function

$$\alpha_\omega(m) = d_\omega \cdot \int_M \alpha(m'^{-1} m)\ \Theta_\omega(m')\ d\,m'\ ,\ m \in M,$$

is the projection of α onto $\underline{C}_\omega(M)$. It follows that $\hat{f}(\omega,\lambda)_{(k_1 : m : k_2)}$ equals

$$(1.5) \quad d_\omega \cdot \int_A \int_N \int_M f(k_1 a\ n\ m'\ k_2)\ \Theta_\omega(m'^{-1} m)\ e^{\langle-\lambda+\rho, H(a)\rangle}\ d\,m'\,d\,a\,d\,n.$$

We can combine this result with (1.3) to obtain the formula for the character,

$\Theta_{\omega,\lambda}$, of the representation $\pi(\sigma, \lambda)$, [4(d)], §11). For $f \in C_c^\infty(G)$, and $\lambda \in i\,\underline{a}$, the trace of $\pi(\sigma, \lambda : f)$ equals

$$d_\omega^{-1} \cdot \tau(\hat{\hat{f}}(\omega,\lambda)) = d_\omega^{-1} \int_K \hat{\hat{f}}(\omega,\lambda)_{(k^{-1} : 1 : k)}\, d\,k$$

$$= \int_K \int_A \int_N \int_M \tilde{f}(k^{-1}\, a\, n\, m\, k) \cdot \Theta_\omega(m^{-1})\, e^{\langle -\lambda+\rho,\, H(a)\rangle}\, dm\, dn\, da\, k$$

$$= \int_K \int_A \int_N \int_M f(k^{-1} m\, n\, a\, k) \cdot \Theta_\omega(m) \cdot e^{\langle \lambda-\rho,\, H(a)\rangle}\, dm\, dn\, da\, dk.$$

This proves that $\Theta_{\omega,\lambda}(f)$ equals

$$(1.6) \quad \int_K \int_A \int_N \int_M f(k^{-1}\, a\, m\, n\, k) \cdot \Theta_\omega(m) \cdot e^{\langle \lambda+\rho,\, H(a)\rangle} dm\, dn\, da\, dk.$$

Suppose P' is another standard cuspidal subgroup of G. Suppose that $\omega' \in \underline{E}_2(M')$, $\lambda' \in i\,\underline{a}'$, and that σ is a representation in the class ω. The representations $\pi(\sigma, \lambda)$ and $\pi(\sigma', \lambda')$ are equivalent if and only if $\Theta_{\omega,\lambda}(f)$ equals $\Theta_{\omega',\lambda'}(f)$ for each $f \in C_c^\infty(G)$. It is a straight forward matter to show that this is the case if and only if P is associatedto P', and there is an $s \in \Omega(\underline{a}, \underline{a}')$ such that $\omega' = s_\omega$ and $\lambda' = s\,\lambda$, (see [4(d)], Lemma 12).

§2. THE PLANCHEREL THEOREM

As a prelude to stating our main theorem, we must discuss the theory of L^2-harmonic analysis on G. This leads to a version of the Plancherel formula on G.

Let Cl (G) be the collection of associativity classes of standard cuspidal subgroups of G. Suppose that $\underline{P} \in \mathrm{Cl}(G)$ and that $P \in \underline{P}$. We shall write $n(\underline{P})$ for the number of elements in $\Omega(\underline{a})$. By [1(b)], Lemma 3.1, this is the same as the number of Weyl chambers in $\underline{a}$. Recall that we denote the dimension of $\underline{a}$ by q. Recall also that our canonical Haar measure on $\underline{a}$ is determined by the Euclidean norm $|\ |$. If we multiply this measure by $(\frac{1}{2\pi})^q$, we obtain the dual measure.

For any $P \in \underline{P}$, we defined in [1(b)] the function

$$(\omega,\lambda) \longrightarrow \mu(\omega,\lambda),\ \omega \in \underline{E}_2(M),\ \lambda \in \underline{a}_c .$$

Harish-Chandra's version of the Plancherl theorem, ([4(d)], Theorem 11 and Lemma 15), states that for any $f \in C_c^\infty(G)$, the series

$$\Sigma_{P \in \underline{P}} \ \Sigma_{\omega \in \underline{E}_2(M)} \int_{i\underline{a}} |\Theta_{\omega,\lambda}(f)| \cdot d_\omega \cdot \mu(\omega,\lambda)\ d|\lambda|$$

is finite, and that

$$f(1) = \sum_{\underline{P} \in \mathrm{Cl}(G)} n(\underline{P})^{-1} \sum_{P \in \underline{P}} \ \sum_{\omega \in \underline{E}_2(M)} \left(\frac{1}{2\pi i}\right)^q \int_{i\underline{a}} \Theta_{\omega,\lambda}(f) \cdot d_\omega \cdot \mu(\omega,\lambda) d\lambda$$

Since $f(1) = \tilde{f}(1)$, and

$$d_\omega \cdot \Theta_{\omega,\lambda}(\tilde{f}) = \tau(\hat{f}(\omega,\lambda)),$$

the above formula becomes

$$(2.1)\quad f(1) = \sum_{\underline{P}\in Cl(G)} n(\underline{P})^{-1} \cdot \sum_{P\in\underline{P}} \sum_{\omega\in\underline{E}_2(M)} \left(\frac{1}{2\pi i}\right)^q \int_{i\underline{a}} \tau(\hat{f}(\omega,\lambda))\,\mu(\omega,\lambda)\,d\lambda\ .$$

Suppose that F is a finite subset of $\underline{E}(K)$. Suppose that P and P' are standard cuspidal subgroups of G, and that $s \in \Omega(\underline{a},\underline{a}')$. We saw in [1(b)], Theorem 8.5, that for $\lambda \in i\,\underline{a}$ there was a unitary map $M(s:\lambda)$ from $\underline{C}_{\omega}(M,\rho_{F,M})$ to $\underline{C}_{s\omega}(M',\rho_{F,M'})$. Since $L^2_{\omega}(M,\rho_M)$ is the closure of the union over F of the spaces $\underline{C}_{\omega}(M,\rho_{F,M})$, $M(s:\lambda)$ can be extended to a unitary operator from $L^2_{\omega}(M,\rho_M)$ to $L^2_{s\omega}(M',\rho_{M'})$. Suppose that σ and σ' are representations of M and M' belonging to the classes ω and $s\omega$, respectively, and that $\Gamma(s:\lambda)$ is any unitary intertwining operator from $\pi(\sigma,\lambda)$ and $\pi(\sigma', s\lambda)$. It follows from [1(b)], (8.7), that for any Hilbert-Schmidt operator T on $\underline{H}(\sigma)$,

$$(2.2)\quad M(s:\lambda)\,\psi_T = \psi_{\Gamma(s:\lambda)T\Gamma(s:\lambda)^{-1}}\ .$$

For any $\underline{P} \in Cl(G)$, define $L^2_{\underline{P}}(\hat{G})$ to be the space of measurable functions

$$(\omega,\lambda) \longrightarrow a_P(\omega,\lambda),\quad P\in\underline{P},\ \omega\in\underline{E}_2(M),\ \lambda\in i\,\underline{a},$$

with values in $L^2_{\omega}(M,\rho_M)$ which satisfy the following two conditions:

(i). If $P, P' \in \underline{P}$, $s \in \Omega(\underline{a},\underline{a}')$, $\omega \in \underline{E}_2(M)$, and $\lambda \in i\,\underline{a}$, then

(2.3) $a_{\underline{P}}(s\,\omega, s\,\lambda) = M(s : \lambda)\, a_{\underline{P}}(\omega, \lambda)$.

(ii). The expression

(2.4) $\|a_{\underline{P}}\|^2 = (\frac{1}{2\pi i})^q \cdot n(\underline{P})^{-1} \cdot \sum_{P \in \underline{P}} \sum_{\omega \in \underline{E}_2(M)} \int_{i\underline{a}} \|a_{\underline{P}}(\omega, \lambda)\|^2 \cdot \mu(\omega, \lambda) d\lambda$

is finite.

Let $L^2(\hat{G})$ be the direct sum over all $\underline{P} \in Cl(G)$ of the spaces $L^2_{\underline{P}}(\hat{G})$. Suppose $\hat{f} \in C_c^\infty(G)$. Define $\hat{f}_{\underline{P}}$ to be the function whose value at $P \in \underline{P}$, $\omega \in \underline{E}_2(M)$ and $\lambda \in i\,\underline{a}$ is the vector

$$\hat{f}(\omega, \lambda) = \hat{f}_{\underline{P}}(\omega, \lambda)$$

in $L^2_\omega(M, \rho_M)$ introduced in the last section. Define

$$\hat{f} = \oplus_{\underline{P}} \quad \hat{f}_{\underline{P}} \ .$$

<u>THEOREM 2.1</u>: The map

$$f \longrightarrow \hat{f} \ , \quad f \in C_c^\infty(G),$$

extends to an isometry from $L^2(G)$ onto $L^2(\hat{G})$.

PROOF: Suppose $f \in C_c^\infty(G)$. It follows from the formula (2.2) that $\hat{f}_{\underline{P}}$ satisfies the condition (2.3). Suppose that

$$g(x) = (f * f^*)(x) = \int_G \overline{f(y)\ f(x^{-1}y)}\ dy \ .$$

Then given P, ω, and λ, it follows from (1.4) that

$$\tau(\hat{g}_{\underline{P}}(\omega, \lambda)) = \tau(\hat{f}_{\underline{P}}(\omega, \lambda)\ \hat{f}_{\underline{P}}(\omega, \lambda)^*) = \|\hat{f}_{\underline{P}}(\omega, \lambda)\|^2 \ .$$

On the other hand,

$$g(1) = \int_G |f(x)|^2 d\,x \ .$$

Applying formula (2.1) to g, we see that the map

$$f \longrightarrow \hat{f} \quad , \quad f \in C_c^\infty(G),$$

extends to an isometry from $L^2(G)$ onto a closed subspace of $L^2(\hat{G})$. We have only to show that the range is $L^2(\hat{G})$.

Fix a class $\underline{P}$. Let $\pi_{\underline{P}}$ be the two sided representation of G on $L^2_{\underline{P}}(\hat{G})$ given by

$$(\pi_{\underline{P}}(y_1)\cdot a_{\underline{P}}\cdot \pi_{\underline{P}}(y_2))(\omega,\lambda) = \pi(\omega,\lambda : y_1)\cdot a_{\underline{P}}(\omega,\lambda)\cdot \pi(\omega,\lambda : y_2),$$

for $y_1, y_2 \in G$, $P \in \underline{P}$, $\omega \in \underline{E}_2(M)$, and $\lambda \in i\,\underline{a}$. $\pi_{\underline{P}}$ can be regarded as a representation of $G \times G$. The same can be said of each of the double representations $\pi(\omega,\lambda)$. Using [4(a)], Pg. 230, one can show that $G \times G$ is of type I. Therefore the representation $\pi_{\underline{P}}$ is of type I.

Fix $P \in \underline{P}$, and let S be the Cartesian product of $\underline{E}_2(M)$ with $i\,\underline{a}^+$, the positive Weyl chamber in $i\,\underline{a}$. Let C be the measure class on S defined by the discrete measure on $\underline{E}_2(M)$ and the Euclidean measure on $i\,\underline{a}^+$. It is a consequence of the results of [1(b)], §8, that $\mu(\omega,\lambda)$ does not vanish for any (ω,λ) in S. From this fact and condition (2.3), it follows that $\pi_{\underline{P}}$ is isomorphic to the direct integral of the representations

$$\pi(\omega,\lambda) \quad , \quad \omega \in \underline{E}_2(M),\ \lambda \in i\,\underline{a}^+ ,$$

with respect to the measure class C. By the remarks at the end of the last section and [1(b)], Lemma 8.4, these representations are irreducible and mutually inequivalent. Therefore $\pi_{\underline{P}}$ is multiplicity free ([5(b)], Theorem 5). It follows that $R(\pi_{\underline{P}}, \pi_{\underline{P}})$, the algebra of intertwining operators of $\pi_{\underline{P}}$, is commutative.

Let r be the two-sided regular representation of $G \times G$ on $L^2(G)$. Then the map

$$f \longrightarrow \hat{f}_{\underline{P}}\ ,\ f \in L^2(G)\ ,$$

is an intertwining operator between r and $\pi_{\underline{P}}$. Thus if L is the closed set

$$\{\hat{f}_{\underline{P}} : f \in L^2(G)\},$$

the orthogonal projection, P, of $L^2_{\underline{P}}(\hat{G})$ onto L is in $R(\pi_{\underline{P}}, \pi_{\underline{P}})$. Since $R(\pi_{\underline{P}}, \pi_{\underline{P}})$ is commutative P is of the form P_E where E is a Borel subset of S and the range of P_E is

$$\{a_{\underline{P}} \in L^2_{\underline{P}}(\hat{G}) : a_{\underline{P}} \text{ vanishes on } E\}.$$

In order to conclude that the map

$$f \longrightarrow \hat{f}_{\underline{P}}$$

is surjective, we need to know that E is a null set. This fact is an easy consequence of the formula (1.5).

□

Choose $\underline{P} \in Cl(G)$, $P \in \underline{P}$, $\omega \in \underline{E}_2(M)$ and $\lambda \in i\,\underline{a}$. Suppose that F is a finite subset of $\underline{E}(K)$ and that $\psi \in \underline{C}_{\omega}(M, \rho_{F,M})$. Since we can express ψ in the form ψ_T, we have the formula

$$(2.5)\qquad (\hat{f}_{\underline{P}}(\omega,\lambda),\psi) = \int_G f(x)\ E(\psi : \lambda : x)_{(1,1)} dx\ ,\quad f \in C_c^{\infty}(G),$$

from §1. From this result we will obtain our version of the

Fourier inversion formula.

Let $\underline{C}_{\omega}(M, \rho_M)$ be the space of functions

$$(k_1, m, k_2) \longrightarrow \psi(k_1 : m : k_2),\ m \in M,\ k_1\ k_2 \in K,$$

in $L^2_{\omega}(M, \rho_M)$ which are infinitely differentiable in k_1 and k_2, and which, for fixed k_1 and k_2, are Schwartz functions in m. Given $\psi \in \underline{C}_{\omega}(M, \rho_M)$, and $k_1, k_2 \in K$, we define

$$\psi(n\ a\ m\ k)_{(k_1,k_2)} = \psi(m)_{(k_1,k\ k_2)},\ n \in N,\ a \in A,\ m \in M,\ k \in K,$$

and

$$E(\psi : \lambda : x)_{(k_1,k_2)} = \int_K \psi(k\ x)_{(k_1 k^{-1},\ k_2)} \cdot e^{\langle\lambda+\rho,\ H(k\ x)\rangle}\, dk,$$

$$x \in G,\ \lambda \in \underline{a}_c,$$

in the usual way. For $\eta_1, \eta_2 \in \underline{E}(K)$ and $x \in G$, we denote the smooth function

$$\int_{K \times K} d_{\eta_1}\ \Theta_{\eta_1}(k_1)\ \rho(k_1) \cdot \psi(x) \cdot \rho(k_2)\ \Theta_{\eta_2}(k_2) \cdot d_{\eta_2} \cdot dk_1\ d\ k_2$$

on $K \times K$ by $\psi_{\eta_1,\eta_2}(x)$. Here d_η and Θ_η stand for the degree and character of η. It is evident that for $k_1, k_2 \in K$,

$$\psi(x)_{(k_1,k_2)} = \Sigma_{\eta_1,\eta_2 \in \underline{E}(K)}\ \psi_{\eta_1,\eta_2}(x)_{(k_1,k_2)},$$

and

$$E(\psi : \lambda : x)_{(k_1,k_2)} = \Sigma_{\eta_1,\eta_2 \in \underline{E}(K)}\ E(\psi_{\eta_1,\eta_2} : \lambda : x)_{(k_1,k_2)}$$

where the convergence is uniform for x belonging to any compact subset of G. From these two formulas it follows that (2.5) is valid for any $\psi \in \underline{C}_{\omega}(M, \rho_M)$.

Suppose that $a = a_P$ is a function in $\underline{C}_{\underline{P}}(\hat{G})$ such that for any $P \in \underline{P}$, $\omega \in \underline{E}_2(M)$ and $\lambda \in i\,\underline{a}$, $a(\omega,\lambda)$ belongs to $\underline{C}_{\omega}(M,\rho_M)$, and such that

$$\Sigma_{P\in\underline{P}} \quad \Sigma_{\omega\in E_2(M)} \int_{i\underline{a}} |E(a(\omega,\lambda):\lambda:x)_{(1,1)} \quad \mu(\omega,\lambda)\,|d|\lambda|$$

is a locally bounded function of $x \in G$. Let f_a be the unique function in $L^2(G)$ such that

$$\hat{f}_a = a .$$

Then for any $h \in C_c^{\infty}(G)$,

$$\int_G h(x) \quad \overline{f_a(x)} \quad dx$$

$$= \quad n(\underline{P})^{-1} \quad \Sigma_{P\in\underline{P}} \quad \Sigma_{\omega\in E_2(M)} \quad (\frac{1}{2\pi i})^q \cdot \int_{i\underline{a}} (\hat{h}_{\underline{P}}(\omega,\lambda),\ a(\omega,\lambda))\mu(\omega,\lambda)d\,\lambda.$$

From (2.5) we obtain

$$n(\underline{P})^{-1}\ \Sigma_P\ \Sigma_{\omega}\ (\frac{1}{2\pi i})^q \cdot \int_{i\underline{a}} \int_G h(x) \quad \overline{E(a(\omega,\lambda):\lambda:x)_{(1,1)}} \cdot \mu(\omega,\lambda)\ dx\ d\lambda ,$$

which by Fubini's theorem equals

$$\int_G h(x) \quad \overline{\{n(\underline{P})^{-1}\ \Sigma_p\Sigma_{\omega}(\frac{1}{2\pi i})^q \int_{i\underline{a}} E(a(\omega,\lambda):\lambda:x)_{(1,1)} \quad \mu(\omega,\lambda)d\,\lambda\}} .$$

Since h is arbitrary, we obtain

$$(2.6) \quad f_a(x) = n(\underline{P})^{-1}\ \Sigma_{P\in\underline{P}}\ \Sigma_{\omega\in\underline{E}_2(M)} (\frac{1}{2\pi i})^q \int_{i\underline{a}} E(a(\omega,\lambda):\lambda:x)_{(1,1)} \mu(\omega,\lambda)d\,\lambda .$$

§3. STATEMENT OF OUR MAIN THEOREM

Our aim is to describe the image in $L^2(\hat{G})$ of $\underline{C}(G)$ under the isometry defined in Theorem 2.1. For each $\underline{P}$ we shall define a topological vector space $\underline{C}_P(\hat{G})$ such that

$$\underline{C}_P(\hat{G}) \subset L^2_{\underline{P}}(\hat{G}),$$

$\underline{C}_{\underline{P}}(\hat{G})$ will be defined by a set of seminorms.

Choose $P \in \underline{P}$, $\omega \in \underline{E}_2(M)$, and let σ be a representation in the class ω. Suppose that η_1 and η_2 are in $\underline{E}(K)$. Define

$$F_i = \{\eta_i\},\ i = 1,\ 2.$$

Suppose that T is an operator on $\underline{H}(\sigma)$ which maps $\underline{H}_{F_1}(\sigma)$ into $\underline{H}_{F_2}(\sigma)$, but which vanishes on the orthogonal complement of $\underline{H}_{F_1}(\sigma)$. Define

$$\psi = \psi_T .$$

Then the representations $\eta_1 = \eta_1(\psi)$ and $\eta_2 = \eta_2(\psi)$ depend only on the class ω. Let $U(\omega)$ denote the set of unit vectors $\psi \in L^2_\omega(M,\rho_M)$ obtained in this manner. If $\psi \in U(\omega)$, and

$$F(\psi) = \{\eta_1(\psi),\ \eta_2(\psi)\} ,$$

the function

$$m \longrightarrow \psi(k_1 : m : k_2),\ m \in M,\ k_1,\ k_2 \in K,$$

is $\rho_{F(\psi),M}$-spherical function from M to $V_{F(\psi)}$ which is an eigenfunction of the center of the universal enveloping algebra of $\underline{m}_c$. Therefore ([4(c)], Lemma 67) it belongs to $\underline{C}_\omega(M,\ \rho_{F(\psi),M})$

Suppose that n is a positive integer and that $D = D_\lambda$ belongs to $\underline{D}(i\,\underline{a})$, the space of differential operators with constant coefficients on $i\,\underline{a}$. For $a_{\underline{P}} \in L^2_{\underline{P}}(\hat{G})$, we set $\|a_{\underline{P}}\|_{D,n} = \infty$ if for some $\omega \in \underline{E}_2(M)$, and for some $\psi \in U(\omega)$, the function

$$\lambda \longrightarrow (a_{\underline{P}}(\omega,\lambda),\psi)$$

is not differentiable. Otherwise we define $\|a_{\underline{P}}\|_{D,n}$ to be supremum over all $\lambda \in i\,\underline{a}$, all $\omega \in \underline{E}_2(M)$, and all vectors ψ in $U(\omega)$, of

$$|D_\lambda(a_{\underline{P}}(\omega,\lambda),\ \psi)| \quad (1 + |\lambda|^2)^n \cdot (1 + |\gamma_1(\psi)|^2)^n \cdot (1 + |\gamma_2(\psi)|^2)^n .$$

Let $\underline{C}_{\underline{P}}(\hat{G})$ be the set of those $a_{\underline{P}} \in L^2_{\underline{P}}(\hat{G})$ such that for all $P \in \underline{P}$, and all D and n, $\|a_{\underline{P}}\|_{D;n} < \infty$. $\underline{C}_{\underline{P}}(\hat{G})$, together with our collection of semi-norms, becomes a topological vector space. Define $\underline{C}(\hat{G})$ to be the direct sum over all $\underline{P}$ of the spaces $\underline{C}_{\underline{P}}(\hat{G})$. Our main result is

<u>THEOREM 3.1.</u> The map

$$f \longrightarrow \hat{f} \quad , \quad f \in C_c^\infty(G),$$

extends to a topological isomorphism, $\underline{F}$, from $\underline{C}(G)$ onto $\underline{C}(\hat{G})$.

The proof of this theorem has two parts. The first half requires us to show that for any fixed $\underline{P}$, the map

$$f \longrightarrow \hat{f}_{\underline{P}} \quad , \quad f \in C_c^\infty(G),$$

extends to a continuous injection from $\underline{C}(G)$ into $\underline{C}_P(\hat{G})$. We shall conclude this section with a proof of this fact.

For $P \in \underline{P}$, we fix a continuous seminorm $\| \|_{D,n}$ on $\underline{C}_P(\hat{G})$ of the type described above. Choose a function $f \in C_c^\infty(G)$. Suppose that ω is in $\underline{E}_2(M)$ and that ψ is in $U(w)$. If

$$F(\psi) = \{\eta_1(\psi), \eta_2(\psi)\}$$

then ψ belongs to $\underline{C}_\omega(M, \rho_{F(\psi),M})$. By (2.5)

$$(\hat{f}_{\underline{P}}(\omega,\lambda), \psi)$$

equals

$$\int_G f(x) \overline{E(\psi : \lambda : x)}_{(1,1)} \, dx .$$

Let Z_G and Z_K be the elements in $\underline{G}$ introduced in [1(b)], §1. By formula (2.2) of [1(b)],

$$\eta(Z_K) = |\eta|^2 ,$$

for any $\eta \in \underline{E}(K)$. It follows from Lemma 5.2 of [1(b)] that

$$E(\psi : \lambda : Z_K ; x)_{(1,1)} = |\eta_1(\psi)|^2 \cdot E(\psi : \lambda : x)_{(1,1)} ,$$

and

$$E(\psi : \lambda : x ; Z_K)_{(1,1)} = |\eta_2(\psi)|^2 \cdot E(\psi : \lambda : x)_{(1,1)} .$$

Notice also that from [1(b)], (formula (2.8) and Lemma 5.2), we obtain

$$E(\psi : \lambda : Z_G ; x)_{(1,1)} = (|\omega|^2 - |\lambda|^2) \cdot E(\psi : \lambda : x)_{(1,1)} .$$

By Lemma 2.3 of [1(b)] there is a constant C_M, depending only on M, such that

$$1 + |\lambda|^2 \leq 1 - (|\omega|^2 - |\lambda|^2) + (|\eta_1(\psi)|^2 + C_M) .$$

The right hand expression suggests that we should look at the element

$$(C_M + 1) I - Z_G + Z_K$$

in $\underline{G}$. In fact we define the elements

$$Y_1 = (I + Z_K)^s \cdot ((1 + C_M) I - Z_G + Z_K)^s$$

and

$$Y_2 = (I + Z_K)^s ,$$

where s is a positive integer to be chosen. Then

(3.1) $\quad |D_\lambda(\hat{f}_{\underline{P}}(\omega,\lambda),\psi)| \cdot (1 + |\lambda|^2)^n \cdot (1 + |\eta_1(\psi)|^2)^n \cdot (1 + |\eta_2(\psi)|^2)^n$

is bounded by the product of

(3.2) $\quad (1 + |\lambda|^2)^{n-s} \cdot (1 + |\eta_1(\psi)|^2)^{n-s} \cdot (1 + |\eta_2(\psi)|^2)^{n-s}$

and

$$|D_\lambda \cdot \int_G f(x) \cdot \overline{E(\psi : \lambda : Y_1 ; x ; Y_2)_{(1,1)}} \, dx| .$$

Since Y_1 and Y_2 are real symmetric elements in $\underline{G}$, this last expression equals

$$|D_\lambda \int_G f(Y_1 ; x ; Y_2) \, \overline{E(\psi : \lambda : x)_{(1,1)}} \, dx| ,$$

which is bounded by

$$\int_G |f(Y_1 ; x ; Y_2)| \, |D_\lambda E(\psi : \lambda : x)_{(1,1)}| \, dx .$$

By formula (5.1) of [1(b)], there is a polynomial p_1 such that

$$|D_\lambda E(\psi : \lambda : x)_{(1,1)}| \leq p_1(|\rho_1(|\rho_{F(\psi)}|)\,|D_\lambda E(\psi : \lambda : x)|,$$

where the right hand norm is that of the finite dimensional Hilbert space $\underline{C}_\omega(M, \rho_{F(\psi),M})$. Corollary 5.5 of [1(b)] allows us to choose a polynomial p and a constant m such that the right hand term is bounded by

$$p(|\omega| + |\lambda| + |\rho_{F(\psi)}|) \cdot \|\psi\| \cdot (1 + \sigma(x))^m \cdot \Xi(x).$$

Since

$$|\omega|^2 \leq |\eta_1(\psi)|^2 + C_M ,$$

we can choose the integer s, introduced above, so that the polynomial

$$p(|\omega| + |\lambda| + |\rho_{F(\psi)}|) = p(|\omega| + |\lambda| + (|\eta_1(\psi)| + |\eta_2(\psi)|)^2)$$

is bounded by a constant multiple of (3.2). Then there is a constant c, independent of f, ω, λ and ψ such that (3.1) is bounded by

$$c \;\|\psi\| \cdot \int_G |f(Y_1 : x ; Y_2)| \cdot (1 + \sigma(x))^m \cdot \Xi(x)\, dx .$$

Since $\|\psi\| = 1$, this in turn is bounded by

$$\|f\|_d = c_d \cdot \sup_{x\in G} \{|f(Y_1 : x ; Y_2)| \cdot (1 + \sigma(x))^d \cdot \Xi(x)^{-1}\} ,$$

where

$$c_d = c \int_G \Xi(x)^2 \cdot (1 + \sigma(x))^{m-d}\, dx .$$

It is known ([5(e)], Lemma 11) that for sufficiently large d, c_d is finite. For any such d, $\|f\|_d$ is a continuous seminorm on $\underline{C}(G)$ and is independent of ω, λ, and ϕ. We have shown that

$$\|\hat{f}_{\underline{P}}\|_{D,n} \leq \|f\|_d \quad , \quad f \in C_c^\infty(G) \; .$$

This completes the first half of the proof of Theorem 3.1. The second half of the proof requires some more preparation.

§4. SOME FURTHER ESTIMATES

We resume our discussion of the c-functions begun in [1(b)]. Fix a standard cuspidal subgroup P of G and let τ be a unitary double representation of K on a finite dimensional Hilbert space V. Let $\mu \in \Lambda'(M)$, and $\psi \in \underline{C}_{\{\mu\}}(M, \tau_M)$. Suppose that ${}^{(u)}P = P'$ is another standard parabolic subgroup which is associated to P. In [1(b)], §6 and §7, we have studied the function

$$ {}^{(u)}\Theta(m\ a) = \sum_{s \in \Omega(\underline{a},\underline{a}')} (c(s:\lambda)\psi)(m) \cdot e^{\langle s\lambda,\ \log a\rangle} $$

on $M' \times A'$. This function also depends holomorphically on the points $\lambda \in C_r(\underline{a}_c, \delta)$, the set of P-regular elements in the cylinder in $\underline{a}_{\mathbb{C}}$ over the open ball of radius δ in $\underline{a}$. δ is a small positive number which depends only on P. For each $s \in \Omega(\underline{a}, \underline{a}')$, $c(s:\lambda)$ can be continued as a meromorphic function on $\underline{a}_c$ with values in the space of linear maps from $\underline{C}_{\{\mu\}}(M, \tau_M)$ to $\underline{C}_{\{s\mu\}}(M', \tau_{M'})$.

The function $c(s:\lambda)$ is analytic on $C_r(\underline{a}_{\mathbb{C}}, \delta)$. In particular, the only singularities of $c(s:\lambda)$ which meet the cylinder $C(\underline{a}_{\mathbb{C}}, \delta)$ are along hyperplanes of the form

$$ \langle\beta,\lambda\rangle = 0, $$

for β a root of (P,A). In [1(b)], §8, we saw that for each $\omega \in \underline{E}_2(M)$ there was a meromorphic complex valued function $\mu(\omega,\lambda)$ such that

(4.1) $c\,(s : -\bar{\lambda})^* \, c(s : \lambda)\psi = \gamma(P)^{-1} \mu(\omega,\lambda)^{-1}\psi,$

for $\psi \in \underline{C}_\omega(M, \tau_M)$. From [1(b)], (4.5) and (8.5), we have the product formula

(4.2) $\mu(\omega,\lambda) = \prod_{\beta\in\bar{\Sigma}} \mu_\beta(\omega,\lambda_\beta),$

which expresses $\mu(\omega,\lambda)$ in terms of the corresponding functions associated to the pairs (M^β, P_β) .

Let $c_\omega(s : \lambda)$ be the restriction of $c(s : \lambda)$ to $\underline{C}_\omega(M, \tau_M)$. For each $\beta \in \bar{\Sigma}$, let $n^\beta(\omega)$ be the multiplicity of the pole of $c_\omega(s : \lambda)$ along the hyperplane $\langle\beta,\lambda\rangle = 0$. By (4.1) this integer is independent of s. Combining (4.1) with (4.2), we see that the function

$$\nu \longrightarrow \mu_\beta(\omega,\nu) , \quad \nu \in \mathbb{C} ,$$

has a zero of order $2n^\beta(\omega)$ at the origin. Define a polynomial

$$q(\omega,\lambda) = \prod_{\beta\in\bar{\Sigma}} \langle\beta,\lambda\rangle^{n^\beta(\omega)} .$$

For any element $s \in \Omega(\underline{a},\underline{a}')$, we have

(4.3) $$q(s\omega, s\lambda) = q(\omega,\lambda).$$

The functions

$$\lambda \longrightarrow q(\omega,\lambda) \quad c_\omega(s : \lambda),$$

$$\lambda \longrightarrow q(\omega,\lambda)^{-2} \cdot \mu(\omega,\lambda),$$

are both holomorphic on $C(\underline{a}_{\mathbb{C}}, \delta)$.

Our first task in this section is to convert Lemma 6.7 of [1(b)] into an estimate of $q(\omega,\lambda)\, c_{\omega}(s : \lambda)$. Suppose that $\mu = \mu(\omega)$ is an element in $\Lambda'(M)$ corresponding to ω. Define $\Lambda = \mu + \lambda$, a vector in ${}^{(o)}\underline{h}_c$. From [1(b)], §7, we have the formula

$$(c(s : \lambda)\, \psi)(m) = t_1(F^{(i)}_{\Lambda} \cdot {}^{(u)}\circledH(m))$$

In the notation of [1(b)], (6, 3), we have

$$(c(s : \lambda)\psi)(m) = \Sigma_j\, \tau^j(s_i\Lambda)\, v_1(s_i\Lambda)(u)\varpi(s_i\Lambda)^{-1}\, t_j({}^{(u)}\circledH(m)).$$

Recall that τ^j and v_1 were polynomial functions on ${}^{(o)}\underline{h}_c$.

Notice that

$${}^{(u)}\varpi(s_i\Lambda) = \varepsilon(s_i)\cdot\varpi_P(\Lambda),$$

where $\varepsilon(s_i)$ equals 1 or -1. For any $\mu \in \Lambda'(M)$, let Δ_+^{μ} be the set of elements α in Δ_+, the set of positive roots of $(\underline{g}_c, {}^{(o)}\underline{h}_c)$ which do not vanish on $\underline{a}$, such that $\langle\alpha,\mu\rangle \neq 0$. By virtue of [1(b)], Lemma 2.1, we may assume that for any $\mu \in \Lambda'(M)$, and $\alpha \in \Delta_+^{\mu}$, the function

$$\lambda \longrightarrow \langle\alpha, \mu + \lambda\rangle$$

does not vanish on $C(\underline{a}_c, \delta)$. Define

$$\varpi_P^1(\Lambda) = \prod_{\alpha\in\Delta_+^{\mu}} \langle\alpha,\Lambda\rangle$$

and

$$\varpi_P^2(\Lambda) = \prod_{\alpha\in\Delta_+-\Delta_+^{\mu}} \langle\alpha,\lambda\rangle .$$

Then $\varpi_P(\Lambda) = \varpi_P^1(\Lambda)\cdot\varpi_P^2(\lambda)$. Given $\psi \in \underline{C}_{\{\mu\}}(M, \tau_M)$, we have

$$(c(s:\lambda)\psi)(m) = \varpi_P^2(\lambda)^{-1}\cdot \Sigma_j\ r_{ij}(\Lambda)\cdot t_j({}^{(u)}\textcircled{H}(m)) .$$

Here,

$$r_{ij}(\Lambda) = \varpi_P^1(\Lambda)^{-1}\cdot \varepsilon(s_i)\cdot v_j(s_i\Lambda)\cdot \tau^1(s_i\Lambda)$$

is a rational function on $\underline{a}_c$, no singularities of which meet

$C(\underline{a}_c, \delta)$.

By [1(b)], Lemma 6.7, ${}^{(u)}\textcircled{H}(m)$ is bounded if λ ranges over any bounded subset of $C_r(\underline{a}_{\mathbb{C}}, \delta)$. It follows that for each j, $t_j({}^{(u)}\textcircled{H}(m))$ extends to a holomorphic function on $C(\underline{a}_{\mathbb{C}}, \delta)$. In particular, the function

$$\lambda \longrightarrow \varpi_P^2(\lambda) \cdot (c(s : \lambda)\psi)(m)$$

is regular on $C(\underline{a}_{\mathbb{C}}, \delta)$. Therefore $q(\omega,\lambda)^{-1} \cdot \varpi_P^2(\lambda)$ is a product of linear functions of the form

$$\lambda \longrightarrow \langle\beta,\lambda\rangle \ , \ \beta \in \bar{\Sigma}^{\cdot}.$$

<u>LEMMA 4.1</u>: Suppose that D_λ is a differential operator in $\underline{D}(a_{\mathbb{C}})$ and that $\varepsilon > 0$. Then we can find a polynomial p such that for any $s \in \Omega(\underline{a}, \underline{a}')$, $m \in M'$, $\omega \in \underline{E}_2(M)$, $\psi \in \underline{C}_\omega(M, \tau_M)$, and $\lambda \in i\,\underline{a}$,

$$|D_\lambda \{q(\omega,\lambda)\ (c(s : \lambda)\ \psi)(m)\}| \leq p(|\omega|+|\lambda|+|\tau|) \cdot \|\psi\| \cdot \Xi'(m) \cdot e^{\varepsilon \cdot \sigma(m)}.$$

The proof of this result requires another lemma.

<u>LEMMA 4.2</u>: For γ and n positive, let $H_n(\underline{a}_{\mathbb{C}}, \gamma)$ be the set of holomorphic complex valued functions on $\underline{a}_{\mathbb{C}}$ such that

$$\|f\|_{n,\gamma} = \sup_{\lambda \in C(\underline{a}_{\mathbb{C}},\gamma)} |f(\lambda)| \cdot (1 + |\lambda|)^n < \infty .$$

Let X be any unit vector in $\underline{a}$. Then there is a constant c_o with the following property: if f is an element in $H_n(\underline{a}_{\mathbb{C}}, \gamma)$ such that the function

$$f_X(\lambda) = \langle X,\lambda\rangle^{-1} f(\lambda)$$

is holomorphic on $C(\underline{a}_{\mathbb{C}}, \gamma)$, then

$$\|f_X\|_{n,\gamma/2} \leq c_o \|f\|_{n,\gamma}\,\lambda.$$

PROOF: Suppose that λ is a point in $C(\underline{a}_{\mathbb{C}}, \frac{\gamma}{2})$ such that

$$|\langle X, \lambda\rangle| \leq \frac{\gamma}{4} .$$

Then

$$f_X(\lambda) = \frac{1}{2\pi} \cdot 2\gamma^{-1} \cdot \int_0^{2\pi} \frac{f(\lambda + \frac{1}{2}\gamma e^{i\theta}X)}{\langle\alpha, \lambda + \frac{1}{2}\gamma e^{i\theta}X\rangle} \, d\theta .$$

Since

$$|\langle X, \lambda + \tfrac{1}{2}\gamma e^{i\theta} X\rangle| \geq \tfrac{1}{2}\gamma\langle X, X\rangle - |\langle X, \lambda\rangle| \geq \tfrac{1}{4}\gamma,$$

we have

$$|f_X(\lambda)| \leq 8\gamma^{-2} \sup_{\theta} |f(\lambda + \tfrac{1}{2}\gamma e^{i\theta}X)|$$

$$\leq 8\gamma^{-2} (1 + |\lambda| + \tfrac{1}{2}\gamma)^n \cdot \|f\|_{n,\gamma}$$

$$\leq 8\gamma^{-2} \cdot (1 + \tfrac{1}{2}\gamma)^n \cdot (1 + |\lambda|)^n \cdot \|f\|_{n,\gamma} .$$

Lemma 4.2 follows with

$$c_o = \max\{8\gamma^{-2}(1 + \tfrac{1}{2}\gamma)^n , 4\gamma^{-1}\} .$$

□

We now prove Lemma 4.1. As usual, let $\mu = \mu(\omega)$ and define $\Lambda = \mu + \lambda$. Then $|\Lambda| = |\omega| + |\lambda|$. We have

$$[illegible](m) = q(\omega,\lambda) \cdot \omega_P^2(\lambda)^{-1} \cdot \Sigma_j \, r_{ij}(\Lambda) \cdot t_j(\,^{(u)}\textcircled{H}(m)).$$

We can use [1(b)], Lemma 6.7, to estimate the [illegible]

$$f(\lambda) = \Sigma_j \; r_{ij}(\Lambda) \cdot t_j({}^{(u)}\textcircled{H}(m)),$$

for $\lambda \in C(\underline{a}_{\mathbb{C}}, \delta)$. We set $\gamma = \delta$ and then apply Lemma 4.1 a number of times, once for each linear factor of $\varpi_P^2(\lambda) \cdot q(\omega,\lambda)^{-1}$. The number of such linear factors is certainly bounded by n_P, the number of elements in $\overline{\Sigma}$.

As a result, we can find a polynomial p, and constants N and d such that for m in ${}^{(u)}M = M'$, and $\lambda \in C(\underline{a}_{\mathbb{C}}, 2^{-n_P} \cdot \delta)$,

$$|q(\omega,\lambda)(c(s:\lambda)\psi)(m)| \leq p(|\Lambda| + |\tau|) \cdot \|\psi\| \; {}^{(u)}\Xi(m) \cdot (1 + \sigma(m))^d \cdot e^{N|\lambda_R| \cdot \sigma(m)}.$$

Our lemma now follows from Cauchy's integral formula. □

The same methods also yield

<u>LEMMA 4.3</u>. Suppose that D_λ is an element in $\underline{D}(\underline{a}_c)$. Then there is a polynomial p such that for $\omega \in \underline{E}_2(M)$ and $\lambda \in i\,\underline{a}$,

$$|D_\lambda\{q(\omega,\lambda)^{-2} \cdot \gamma(\omega,\lambda)\}| \leq p(|\omega| + |\lambda|).$$

PROOF: The function

$$\lambda \longrightarrow q(\omega,\lambda)^{-2} \cdot \gamma(\omega,\lambda)$$

is analytic on $C(\underline{a}_{\mathbb{C}}, \delta)$. Our lemma follows from Lemma 4.1 and Lemma 8.3 of [1(b)].

□

Lemma 4.1 is not quite in the form that we need it. Before attending to this, however, we must restate the main estimate of [1(b)], §6.

In the notation of [1(b)], §1, ${}^{(1)}A$ is the split component of the parabolic subgroup $P = G$ of G. ${}^{(1)}\underline{a}$ is the Lie algebra of ${}^{(1)}A$, and ${}^{(o)}_{(1)}\underline{a}$ is the orthogonal complement of ${}^{(1)}\underline{a}$ in ${}^{(o)}\underline{a}$. Let $\alpha_1, \ldots, \alpha_{{}^{(o)}r}$ be the simple roots of $({}^{(o)}P, {}^{(o)}A)$. For $1 \leq k \leq {}^{(o)}r$, let $H^{(k)}$ be the unit vector in ${}^{(o)}_{(1)}\underline{a}^+$ such that

$$\langle\alpha_j , H^{(k)}\rangle = 0, \; j \neq k .$$

Then

$$\underline{a}^{(k)} = \mathbb{R}\, H^{(k)} \oplus {}^{(1)}\underline{a}$$

is a distinguished subspace of ${}^{(o)}\underline{a}$. It is the Lie algebra of the split component of a maximal parabolic subgroup $P^{(k)} = N^{(k)}L^{(k)}$ of G. We shall apply the results of [1(b)], §6, to the case that ${}^{(u)}P = P^{(k)}$.

If ε is any positive number, let $A_k(\varepsilon)$ be the set of all h in ${}^{(o)}_{(1)}A^+$ such that

$$\langle\alpha_k, H^{(k)}\rangle^{-1} \cdot \langle\alpha_k, \log h\rangle \geq \varepsilon\, \sigma(h) .$$

Fix ε_o so small that

$$\exp ({}^{(o)}_{(1)}a^+) = {}^{(o)}_{(1)}A^+ = \bigcup_{k=1}^{{}^{(o)}r} A_k (\varepsilon_o) .$$

In stating the following lemma, we adopt the notation of [1(b)], §6. In particular, $\Phi^{(k)}$, $\Theta^{(k)}$, $d^{(k)}$, and $\Xi^{(k)}$ are functions defined on the group ${}^{(u)}L = L^{(k)}$.

<u>LEMMA 4.4</u>: There is a positive number m and a polynomial p such that if $\lambda \in i\,\underline{a}_r$, and $h \in A_k(\varepsilon_o)$ for some k,

$$|d^{(k)}(h) \cdot \Xi^{(k)}(h)^{-1} \cdot \phi(h) - \Xi^{(k)}(h)^{-1} \cdot \theta^{(k)}(h)| \leq \|\psi\| \cdot p(|\Lambda| + |\tau|) \cdot e^{-m\,\sigma(h)} .$$

PROOF: We may write h uniquely in the form

$$h = y \cdot \exp t\, H^{(k)},$$

where $t \geq 0$ and y is a point in $\binom{o}{1}A^+$ such that

$$\langle \alpha_k, \log y \rangle = 0.$$

By Corollary 6.6 of [1(b)], we can find numbers m_k and d, and a polynomial p, such that if λ is in $i\,\underline{a}_r$, and h is as above,

$$|d^{(k)}(h)\, \phi(h) - \theta^{(k)}(h)| \leq \|\psi\| \cdot p(|\Lambda| + |\tau|) \cdot \Xi^{(k)}(y) \cdot e^{-t\, m_k} \cdot (1 + \sigma(y))^d .$$

However, since $h \in A_k(\varepsilon_o)$, we have

$$e^{-t m_k} \leq e^{-m_k \cdot \varepsilon_o \cdot \sigma(h)} .$$

Since $\Xi^{(k)}(y) = \Xi^{(k)}(h)$, our lemma follows for

$$m = \min_{1 \leq k \leq (o)_r} \{\tfrac{1}{2}\, \varepsilon_o\, m_k\} .$$

□

Let $\Delta^{(k)}_+$ be the set of positive roots of $(\underline{g}_c, \underline{h}_c)$ which do not vanish on $\underline{a}^{(k)}$. Then

$$d^{(k)}(h) = \prod_{\beta \in \Delta_+^{(k)}} \left(e^{\frac{1}{2}\langle \beta, \log h \rangle} - e^{-\frac{1}{2}\langle \beta, \log h \rangle}\right).$$

This is the product of $e^{\langle\rho^{(k)}, \log h\rangle}$ and $\prod_{\beta \in \Delta_+^{(k)}} (1 - e^{-\langle\beta, \log h\rangle})$. This second function is bounded away from 0 on $A_k(\varepsilon_o)$. It follows from [4(b)], (Theorem 3 and Lemma 36), that there are constants c_1 and c_2 such that for all $h \in A_k(\varepsilon_o)$

$$\Xi(h)^{-1} \leq d^{(k)}(h) \cdot \Xi^{(k)}(h)^{-1} \leq c_1 \cdot \Xi(h)^{-1} \cdot (1 + \sigma(h))^{c_2} .$$

Suppose for the moment that $P = G$. Then $\underline{a} = {}^{(1)}\underline{a}$. Given $\lambda \in {}^{(1)}\underline{a}_{\mathbb{C}}$,

$$\phi(x) = \psi(x) \cdot e^{\langle\Lambda, {}^{(1)}H(x)\rangle}, \quad x \in G,$$

for $\psi \in \underline{C}_o(G^1, \tau)$. It is a consequence of [5(c)], Lemma 43, that

$$\Theta^{(k)}(h) = 0, \quad 1 \leq k \leq {}^{(o)}r .$$

It follows from the above inequality, Lemma 4.4, and the fact that

$$G^1 = K \cdot {}^{(o)}_{(1)}A^+ \cdot K ,$$

that there is a polynomial p, and a positive number m such that for ϕ of this form,

$$(4.4) \qquad |\Xi(x)^{-1} \phi(x)| \leq \|\psi\| \cdot p(|\Lambda| + |\tau|) \cdot e^{-m\sigma(x)}, \quad x \in G^1 .$$

This is a special case of a result of Trombi and Varadarajan ([6], Theorem 7.3).

We return to the situation in which $\underline{a}$ and $\underline{a}'$ are arbitrary distinguished subspaces of ${}^{(o)}\underline{a}$, and

$$\phi(x) = E(\psi : \lambda : x) , \quad \psi \in \underline{C}_\omega(M, \tau_M), \ \lambda \in C_r(\underline{a}_{\mathbb{C}}, \delta).$$

LEMMA 4.5: If D_λ is any differential operator in $\underline{D}(i\,\underline{a})$ there is a polynomial p such that for each $s \in \Omega(\underline{a}, \underline{a}')$, $\omega \in \underline{E}_2(M)$, and $\lambda \in i\,\underline{a}$,

$$\|D_\lambda \; q(\omega,\lambda)\; c_\omega(s : \lambda)\| \leq p(|\Lambda| + |\tau|) .$$

PROOF: Suppose that $\psi \in \underline{C}_\omega(M, \tau_M)$, and $\psi' \in \underline{C}_{s\omega}(M', \tau_{M'})$. We must estimate the inner product

$$(D_\lambda\{q(\omega,\lambda)\; c(s : \lambda)\; \psi\} , \psi') .$$

We apply (4.4) with G replaced by M' and ϕ replaced by ψ'. Note that $M' = (M')^1$. By Lemma 4.1 and the fact that

$$\int_{M'} \Xi'(m)\; e^{-(m-\varepsilon)\sigma(m)}\; d\,m < \infty$$

if $0 < \varepsilon < m$, we may choose a polynomial p such that

$$|(D_\lambda\{q(\omega,\lambda)\; c\;(s : \lambda)\; \psi\}, \psi')| \leq p(|\Lambda| + |\tau|) \;\; \|\psi\|\cdot\|\psi'\|.$$

The lemma follows.

$\square$

§5. THE CONSTANT TERM FOR ARBITRARY ${}^{(u)}\underline{a}$

Fix P and τ as in the last section. For $s \in \Omega(\underline{a}, \underline{a}')$ we follow Harish-Chandra and define meromorphic functions of λ by

$$c^{o}(s : \lambda) = c(s : \lambda) \cdot c(1 : \lambda)^{-1} ,$$

$$E^{o}(\psi : \lambda : x) = E(c(1 : \lambda)^{-1} \psi : \lambda : x), \quad \psi \in \underline{C}_{o}(M, \tau_M).$$

It is clear that

$$c^{o}(s : \lambda) = c(1 : s\lambda)\ M(s : \lambda)\ c(1 : -\lambda)^{-1} .$$

For $s' \in \Omega(\underline{a}', \underline{a}'')$, it follows from this formula and [1(b)], Theorem 8.5, that

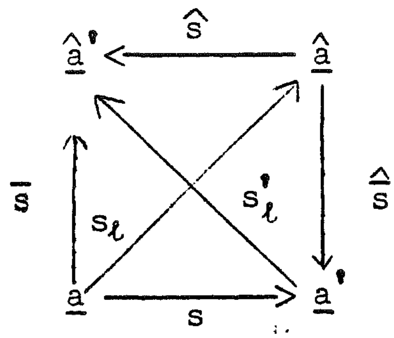
$$(5.1) \quad c^{o}(s's : \lambda) = c^{o}(s' : s\lambda) \quad c^{o}(s : \lambda) .$$

We also obtain from [1(b)], Theorem 8.5, the formula

$$(5.2) \quad E^{o}(\psi : \lambda : x) = E^{o}(c^{o}(s : \lambda)\psi : s\lambda : x).$$

These functional equations are due to Harish-Chandra, ([4(d)], Theorem 6). He proves them without first introducing the function $M(s : \lambda)$.

Suppose $s \in \Omega(\underline{a}, \underline{a}')$. Define $s_{\ell} \in \Omega(\underline{a}, \hat{a})$ and $s_{\ell}' \in \Omega(\underline{a}', \hat{\underline{a}}')$ to be the elements of greatest length in $\Omega(\underline{a})$ and $\Omega(\underline{a}')$ respectively. In addition, define $\hat{s} = (s_{\ell}') \ s \ (s_{\ell})^{-1}$, $\bar{s} = s_{\ell}' s$, and $\hat{\bar{s}} = (s_{\ell}')^{-1} \ \bar{s} \ (s_{\ell})^{-1}$. We have the commutative diagram



Let

$$w\ ,\ w_\ell,\ w_\ell'\ ,\ \ldots .$$

be representatives in ${}^{(o)}\widetilde{M}$ of these various mappings. We may assume that

$$w_\ell = \hat{\bar{w}}^{-1} w\ ,\quad w_\ell = \hat{w}\ \hat{\bar{w}}^{-1}\ .$$

Suppose now that F is a finite subset of $\underline{E}(K)$ and that $\tau = \rho_F$. From [1(b)], (Lemma 8.1, and formula (8.7)), we observe that for $w \in \underline{E}_2(M)$, $\sigma \in \omega$, and $T \in \mathrm{End}\ (\underline{H}_F(\sigma)$,

$$c^{o}(s : \lambda)\psi_T = c(1 : s\lambda)M(s : \lambda)\ c(1 : \lambda)^{-1}\psi_T$$

$$= \psi_{(w_\ell')^{-1}R(w_\ell' : s\lambda)R(w:\lambda)R(w_\ell:\lambda)^{-1}w_\ell T \cdot R(w:\lambda)^{-1}}\ .$$

The length of w_ℓ is the sum of the lengths of $(\hat{\bar{s}})^{-1}$ and s. It follows from [1(b)], Corollary 4.5, that

$$R(w_\ell : \lambda) = R(\hat{\bar{w}} : s\lambda)\ R(w : \lambda).$$

Similarly

$$R(w_\ell' : s\lambda) = R(\hat{w} : s_\ell\lambda)\ R(\hat{\bar{w}}^{-1} : s\lambda)\ .$$

We have shown that

$$(5.3)\quad c^{o}(s : \lambda)\, \psi_T = \psi_{(w_\ell')^{-1}R(\hat{w} : s_\ell\lambda)w_\ell \cdot T \cdot R(w : \lambda)^{-1}} .$$

Combining this formula with [1(b)], (4.5), one can derive the formula

$$(5.4)\quad c^{o}(s : \lambda)^{*} = c^{o}(s^{-1} : - s\,\bar{\lambda}) .$$

Suppose that ${}^{(u)}\underline{a}$ is a distinguished subspace of both $\underline{a}$ and $\underline{a}'$. If ${}^{(u)}P = {}^{(u)}N \cdot {}^{(u)}A \cdot {}^{(u)}M$ is the standard parabolic subroup of G corresponding to ${}^{(u)}\underline{a}$,

$${}^{(o)}_{(u)}P = {}^{(o)}P \cap {}^{(u)}M$$

is a minimal parabolic subgroup of ${}^{(u)}M$. Both

$${}_{(u)}\underline{a} = \underline{a} \cap {}^{(u)}\underline{m}$$

and

$${}_{(u)}\underline{a}' = \underline{a}' \cap {}^{(u)}\underline{m}$$

are distinguished subspaces of ${}^{(o)}_{(u)}\underline{a}$. Denote the set of mappings in $\Omega(\underline{a}, \underline{a}')$ which leave ${}_{(u)}\underline{a}$ pointwise fixed by ${}_{(u)}\Omega(\underline{a}, \underline{a}')$. We identify ${}_{(u)}\Omega(\underline{a}, \underline{a}')$ with the set of mappings from ${}_{(u)}\underline{a}$ to ${}_{(u)}\underline{a}'$ obtained by restricting to ${}_{(u)}\underline{a}$ elements of the restricted Weyl group of ${}^{(u)}M$. We shall also write ${}_{(u)}\Omega(\underline{a})$ for the union, over all distinguished subspaces $\underline{a}'$ of ${}^{(o)}\underline{a}$ which contain ${}_{(u)}\underline{a}$, of the sets ${}_{(u)}\Omega(\underline{a}, \underline{a}')$.

For $\lambda \in \underline{a}_{\mathbb{C}}$, denote the projection of λ onto ${}_{(u)}\underline{a}_{\mathbb{C}}$ by ${}_{(u)}\lambda$. We shall write ${}_{(u)}c$, ${}_{(u)}c^{o}$, ${}_{(u)}E$ and ${}_{(u)}E^{o}$ for the various functions discussed above, but associated to the pair

$({}^{(u)}M,\ {}^{(u)}M \cap P)$ instead of to (G, P).

<u>LEMMA 5.1</u>: If $s \in {}_{(u)}\Omega(\underline{a}, \underline{a}')$ and $\lambda \in \underline{a}_c$,

$$c^o(s : \lambda) = {}_{(u)}c^o(s : \lambda) .$$

PROOF: We may assume that $\tau = \rho_F$, for a finite subset F of $\underline{E}(K)$. For $\omega \in \underline{E}_2(M)$, $\sigma \in \omega$, and $T \in \mathrm{End}\,(\underline{H}_F(\sigma))$, $c^o(s : \lambda)\psi_T$ equals

$$\psi_{(w_\ell')^{-1} R(\hat{w} : s_\ell\lambda) w_\ell \, T \, R(w : \lambda)^{-1}} .$$

It follows, essentially from [1(b)], Lemma 4.3, that

$$R(w : \lambda)^{-1} = {}_{(u)}R(w : {}_{(u)}\lambda)^{-1} .$$

Notice that the restrictions of s_ℓ and s_ℓ' to ${}^{(u)}\underline{a}$ are the same. In fact they both define the unique element in $\Omega({}^{(u)}\underline{a})$ of greatest length. Let ${}_{(u)}s_\ell$ and ${}_{(u)}s_\ell'$ be the elements of greatest length in ${}_{(u)}\Omega(\underline{a})$ and ${}_{(u)}\Omega(\underline{a}')$ respectively. It is a consequence of [1(b)], Lemma 3.1, that

$$s_\ell\,({}_{(u)}s_\ell)^{-1} = s_\ell' \cdot ({}_{(u)}s_\ell')^{-1} .$$

We denote this element by t. Choose representatives ${}_{(u)}w_\ell$ and ${}_{(u)}w_\ell'$ of these elements in ${}^{(o)}\widetilde{M}$ such that

$$w_t = w_\ell \cdot ({}_{(u)}w_\ell)^{-1} = w_\ell'({}_{(u)}w_\ell')^{-1} .$$

This element is a representative of t in ${}^{(o)}\widetilde{M}$.

Finally, define

$${}_{(u)}\hat{s} = {}_{(u)}s_\ell' \cdot s \cdot ({}_{(u)}s_\ell)^{-1}$$

and

$$ {}_{(u)}\hat{w} = {}_{(u)}w_\ell' \cdot w \cdot ({}_{(u)}w_\ell)^{-1} \quad . $$

Then

$$ (w_\ell')^{-1} \cdot R(\hat{w} : s_\ell\lambda)\, w_\ell $$

equals

$$ ({}_{(u)}w_\ell')^{-1} \cdot (w_t)^{-1} \cdot R((w_t) \cdot {}_{(u)}\hat{w} \cdot (w_t)^{-1} : s_\ell\lambda) \cdot (w_t) \cdot {}_{(u)}w_\ell \quad . $$

By a simple change of variables in the integral used to define the intertwining operators, this last expression equals

$$ ({}_{(u)}w_\ell')^{-1} \cdot R({}_{(u)}\hat{w} : {}_{(u)}s_\ell\lambda) \cdot ({}_{(u)}w_\ell) \ . $$

Again appealing to [1(b)], Lemma 4.3, we see that

$$ R({}_{(u)}\hat{w}_\ell : {}_{(u)}s_\ell\, \lambda) = {}_{(u)}R({}_{(u)}\hat{w}_\ell : {}_{(u)}({}_{(u)}s_\ell\lambda)). $$

Putting these formulas together, we obtain

$$ c^o(s : \lambda) = {}_{(u)}c^o(s : {}_{(u)}\lambda), $$

which is what we were required to prove.

□

Suppose now that ${}^{(u)}\underline{a}$ is any distinguished subspace of ${}^{(o)}\underline{a}$. We shall say that two maps s and t in $\Omega(\underline{a})$ are ${}^{(u)}\underline{a}$-equivalent if $s\,\underline{a}$ and $t\,\underline{a}$ both contain ${}^{(u)}\underline{a}$, and the map $t\,s^{-1}$ leaves ${}^{(u)}\underline{a}$ pointwise fixed. Let $\Omega\,(\underline{a}\,/\,{}^{(u)}\underline{a})$ be a fixed set of representatives of the equivalence classes. Of course this set is empty unless ${}^{(u)}\underline{a}$ is contained in some distinguished subspace a' of ${}^{(o)}\underline{a}$ which is associated to $\underline{a}$.

LEMMA 5.2: For $\psi \in \underline{C}_o(M, \tau_M)$ and $\lambda \in i\, \mathfrak{a}_r$, define

$$\phi(x) = E^o(\psi : \lambda : x) .$$

Then for $m \in {}^{(u)}M$ and $H \in {}^{(u)}\underline{a}$, ${}^{(u)}\theta(m \exp H)$ equals

$$\sum_{s \in \Omega(\underline{a}/{}^{(u)}\underline{a})} {}_{(u)}E^o(c^o(s : \lambda)\psi : {}_{(u)}(s\lambda) : m) \cdot e^{\langle H, s\lambda\rangle} .$$

PROOF: Define

$${}^{(u)}\hat{\theta}(m \exp H) = \sum_{s \in \Omega(\underline{a}/{}^{(u)}\underline{a})} {}_{(u)}E^o(c^o(s : \lambda)\psi : {}_{(u)}(s\lambda) : m) \cdot e^{\langle H, s\lambda\rangle}, \quad m \in {}^{(u)}M, \quad H \in {}^{(u)}\underline{a} .$$

By Lemma 5.1, and the functional equations (5.1) and (5.2), the summand depends only on the equivalence class of s in $\Omega(\underline{a}/{}^{(u)}\underline{a})$. For fixed $H \in {}^{(u)}\underline{a}$ define

$$f(m) = {}^{(u)}\theta(m \exp H) - {}^{(u)}\hat{\theta}(m \exp H), \; m \in {}^{(u)}M .$$

Any standard cuspidal subgroup of ${}^{(u)}M$ is of the form

$$ {}^{(v)}_{(u)}P = {}^{(u)}M \cap {}^{(v)}P \ , \ v \leq u \ . $$

Suppose, first of all, that $v \leq u$ and that ${}^{(v)}\underline{a}$ is not associated to $\underline{a}$. Then for any H_1 in ${}^{(v)}_{(u)}\underline{a}$, the orthogonal complement of ${}^{(u)}\underline{a}$ in ${}^{(v)}\underline{a}$, the function

$$ m_1 \longrightarrow {}^{(v)}_{(u)}({}^{(u)}\Theta)(m_1 \exp H_1 \exp H), \ m_1 \in {}^{(v)}M \ , $$

is orthogonal to $\underline{C}_o \ ({}^{(v)}M, \ {}^{(v)}\tau)$. This follows from [4(d)], Lemma 8, and the fact that ${}^{(v)}_{(u)}({}^{(u)}\Theta) = {}^{(v)}\Theta$. Again by [4(d)], Lemma 8, we have the same result if ${}^{(u)}\Theta$ is replaced by ${}^{(u)}\hat{\Theta}$. Therefore the function

$$ m_1 \longrightarrow {}^{(v)}_{(u)}f \ (m_1 \exp H_1), \ m_1 \in {}^{(v)}M, $$

is orthogonal to $\underline{C}_o \ ({}^{(v)}M, \ {}^{(v)}\tau)$.

On the other hand, suppose that $v \leq u$, but that ${}^{(v)}\underline{a}$ is associated to $\underline{a}$. Then for $H_1 \in {}^{(v)}_{(u)}\underline{a}$ and $m_1 \in {}^{(v)}M$, ${}^{(v)}_{(u)}({}^{(u)}\hat{\Theta}) \ (m_1 \exp H_1 \exp H)$ equals

$$ \sum_{s \in \Omega(\underline{a}/{}^{(u)}\underline{a})} \ \sum_{t \in {}_{(u)}\Omega(s\,\underline{a}, \ {}^{(v)}\underline{a})} \ ({}_{(u)}c \ ({}_{(u)}t : {}_{(u)}(s\lambda))^{-1} $$

$$ c^o(s:\lambda)\psi)(m_1) \quad e^{\langle H_1, \ t_{(u)}(s\lambda)\rangle} \ e^{\langle H, s\lambda\rangle} $$

$$ = \sum_{s \in \Omega(\underline{a}/{}^{(u)}\underline{a})} \ \sum_{t \in {}_{(u)}\Omega(s\,\underline{a}, \ {}^{(v)}\underline{a})} (c^o(t: s\lambda) \ c^o(s : \lambda)\psi)(m_1) \cdot $$

$$ e^{\langle H_1 + H, t\,s\,\lambda\rangle} . $$

We use the functional equation for c^o and the fact that any element in $\Omega(\underline{a}, {}^{(v)}a)$ can be represented uniquely in the form

$$ t\,s \ , \ s \in \Omega(\underline{a} \ / {}^{(u)}\underline{a}), \ t \in {}_{(u)}\Omega(s\,\underline{a}, \ {}^{(v)}\underline{a}), $$

to rewrite the above expression as

$$\sum_{s \in \Omega(\underline{a},\ {}^{(v)}\underline{a})} (c^{o}(s : \lambda)\psi)(m_1) \cdot e^{\langle H_1 + H,\ s\lambda \rangle}.$$

This expression is the same as

$${}^{(v)}\theta\ (m_1 \exp (H_1 + H)) = {}^{(v)}_{(u)}({}^{(u)}\theta)\ (m_1 \exp H_1 \cdot \exp H)\ .$$

It follows that

$${}^{(v)}_{(u)}f\ (m_1 \exp H_1) = 0\ .$$

We conclude from [4(c)], Lemma 43, that $f \in \underline{C}_o({}^{(u)}M,\ {}^{(u)}\tau)$. But f is orthogonal to $\underline{C}_o({}^{(u)}M,\ {}^{(u)}\tau)$. Therefore $f = 0$, which is what we wanted to prove. □

In the next section we shall apply this result to the case that ${}^{(u)}P = P^{(k)}$, one of the maximal standard parabolic subgroups defined in §4. In this case we shall write E_k, c_k, and λ_k instead of ${}_{(u)}E$, ${}_{(u)}c$ and ${}_{(u)}\lambda$ respectively.

<u>COROLLARY 5.3</u>: Suppose

$$\phi(x) = E(\psi : \lambda : x)\ ,$$

for $\psi \in \underline{C}_o(M,\ \tau_M)$, and $\lambda \in i\ \underline{a}_r$.
Then for $m \in M^{(k)}$ and $H \in \underline{a}^{(k)}$, $\theta^{(k)}(m \exp H)$ equals

$$\sum_{s \in \Omega(\underline{a}/\underline{a}^{(k)})} E_k(c_k(1 : (s\lambda)_k)^{-1} c(s : \lambda)\psi : (s\lambda)_k : m)\ e^{\langle H,\ s\lambda\rangle}.$$

PROOF: In the lemma, replace ψ by $c(1 : \lambda)\ \psi$. The corollary follows immediately.

□

§6. COMPLETION OF THE PROOF OF THEOREM 3.1

Suppose that P is a standard cuspidal subgroup of G. If V is any finite dimensional Hilbert space there is the usual topology on $\underline{S}(i\,\underline{a}) \otimes V$, the space of Schwartz functions on $i\,\underline{a}$ with values in V. Suppose that N is any continuous semi-norm on $\underline{S}(i\,\underline{a})$. For $a \in \underline{S}(i\,\underline{a}) \otimes V$, let $N(a)$ be the supremum over all unit vectors ξ in V of the numbers $N(\langle a, \xi\rangle)$ where $\langle a,\xi\rangle$ is the function

$$\lambda \longrightarrow \langle a(\lambda), \xi\rangle , \lambda \in i\,\underline{a} .$$

This defines a continuous semi-norm on $\underline{S}(i\,\underline{a}) \otimes V$.

THEOREM 6.1: For any positive number r there is a continuous semi-norm N on $\underline{S}(i\,\underline{a})$ and polynomials p_1 and p_2 such that for any $\tau \in F(K, K)$, $\omega \in \underline{E}_2(M)$, and $a \in \underline{S}(i\,\underline{a}) \otimes \underline{C}_{\omega}(M, \tau_M)$,

$$\sup_{x\in G} \{\Xi(x)^{-1} (1 + \sigma(x))^r \; |\int_{i\,\underline{a}} E(a(\lambda) : \lambda : x)\, \mu(\omega, \lambda)\, d\lambda|\}$$

is bounded by

$$N(a) \cdot p_1(|\omega|) \cdot p_2(|\tau|).$$

PROOF: For any x in G, we can write

$$x = y \cdot \exp H, \quad y \in G^1, \; H \in {}^{(1)}\underline{a} .$$

Then $\Xi(x) = \Xi(y)$. Furthermore, by Lemma 10 of [5(c)],

$$(1 + \sigma(x))^r \leq (1 + \sigma(y))^r \cdot (1 + |H|)^r .$$

Suppose that

$$\lambda = \nu + \zeta ,$$

for $\lambda \in i\,\underline{a}$, $\zeta \in i^{(1)}\underline{a}$, and $\nu \in i_{(1)}\underline{a}$, the orthogonal complement of $i^{(1)}\underline{a}$ in $i\,\underline{a}$. Then

$$E(a(\lambda) : \lambda : x)\ \mu(\omega,\lambda)$$

equals

$$E(a(\lambda) : \nu : y)\ \mu(\omega,\nu)\ e^{\langle\zeta,H\rangle} .$$

It follows that we can find a differential operator D_ζ in $\underline{D}(i^{(1)}\underline{a})$ such that for any $x = y \exp H$ as above,

$$\Xi(x)^{-1}\cdot (1 + \sigma(x))^r \cdot \left| \int_{i\,\underline{a}} E(a(\lambda) : \lambda : x)\cdot \mu(\omega,\lambda) d\lambda \right|$$

is bounded by

$$\Xi(y)^{-1} (1 + \sigma(y))^r \cdot \left| \int_{i_{(1)}\underline{a}} E(a_H(\nu) : \nu : y)\ \mu(\omega, \nu)\, d\nu \right|,$$

where

$$a_H(\nu) = \int_{i^{(1)}\underline{a}} \{D_\zeta a(\nu + \zeta)\}\ e^{\langle\zeta,H\rangle}\, d\zeta.$$

If N is any continuous seminorm on $\underline{S}(i_{(1)}\underline{a})$,

$$\alpha \longrightarrow \sup_{H\in {}^{(1)}\underline{a}} N(\alpha_H) , \quad \alpha \in \underline{S}(i\,\underline{a}) ,$$

is a continuous seminorm on $\underline{S}(i\,\underline{a})$. Therefore to prove our theorem, we may assume that ${}^{(1)}\underline{a}$ is trivial.

As in §4, let $P^{(1)},\ldots, P^{({}^{(o)}r)}$ be the maximal standard parabolic subgroup of G. Proceeding by induction, we assume the theorem is true for each of the groups $M^{(k)}$. Since

$$G = G^1 = K \cdot {}^{(o)}_{(1)}A^+ \cdot K ,$$

we have only to find p_1, p_2 and N such that for fixed k and an $h \in A_k(\varepsilon_o)$,

$$(6.1) \qquad \Xi(h)^{-1} \cdot (1 + \sigma(h))^r \; \left| \int_{i\underline{a}} E(a(\lambda) : \lambda : h) \cdot \mu(\omega,\lambda) \, d\lambda \right|$$

is bounded by $N(a) \cdot p_1(|\omega|) \cdot p_2(|\tau|)$. Recalling that

$$\Xi(h)^{-1} \leq d^{(k)}(h) \;\; \Xi^{(k)}(h)^{-1} ,$$

we apply Lemma 4.4 and Corollary 5.3.

Suppose

$$h = h_k \cdot \exp t\, H^{(k)} \;\; , \; h_k \in M^{(k)}, \; t \geq \varepsilon_o \, \sigma(h) ,$$

where $H^{(k)}$ is the unit vector in $(\underline{a}^{(k)})^+$. Then there are polynomials p_o, p_1 and p_2 such that (6.1) is bounded by the sum of

$$(6.2) \qquad \Xi^{(k)}(h_k)^{-1} \;\; (1 + \sigma(h_k \cdot \exp t\, H^{(k)}))^r \cdot \sum_{s \in \Omega(\underline{a}/\underline{a}^{(k)})}$$

$$\left| \int_{i\underline{a}} E_k(c_k(1 : (s\lambda)_k)^{-1} \;\; c(s : \lambda)\; a(\lambda) \; : (s\lambda)_k : h_k)\; e^{t\langle H^{(k)}, s\lambda\rangle} \cdot \mu(\omega,\lambda) d\lambda \right|$$

and

$$(6.3) \qquad (1 + \sigma(h))^r \cdot e^{-m\,\sigma(h)} \{ \int_{i\underline{a}} \|a(\lambda)\| \cdot \mu(\omega,\lambda) \cdot p_o(|\lambda|) d\lambda \} \; p_1(|\omega|) \; p_2((\tau)).$$

We deal with (6.3) first.

Certainly

$$C_r = \sup_{h \in {}^{(o)}A_k^+(d_o)} (1 + \sigma(h))^r \cdot e^{-m\sigma(h)}$$

is finite. By Lemma 8.3 of [1(b)], there are polynomials p_o' and p_1' such that

$$\mu(\omega,\lambda) \leq p_o'(|\lambda|) \cdot p_1'(|\omega|) .$$

If N is the semi-norm on $\underline{S}$ (i $\underline{a}$) given by

$$N(\alpha) = \int_{i\underline{a}} |\alpha(\lambda)| \cdot p_o(|\lambda|) \cdot p_o'(|\lambda|) \, d\lambda, \quad \alpha \in \underline{S}(i\,\underline{a}) ,$$

(6.3) is bounded by

$$C_r \cdot N(a) \cdot p_1(|\omega|) \cdot p_1'(|\omega|) \cdot p_2(|\tau|) .$$

In light of the first assertion of Lemma 8.3 of [1(b)], the summand in (6.2) can be written

$$\Big|\int_{i\underline{a}'} E_k(c_k(1 : \lambda_k)^{-1} \cdot c(s : s^{-1}\lambda)\, a^s(\lambda) : \lambda_k : h_k) \cdot e^{t\langle H^{(k)},\lambda\rangle} \cdot \mu(s\omega,\lambda) d\lambda\Big| ,$$

where $s \in \Omega(\underline{a}, \underline{a}')$ for some $\underline{a}'$, and

$$a^s(\lambda) = a(s^{-1}\lambda) .$$

For any $\psi \in \underline{C}_{s\omega}(M, \tau_M)$, it follows from (4.1) that

$$c_k(1 : \lambda_k)^{-1}\psi = \gamma_k(P_k) \cdot c_k(1 : \lambda_k)^* \cdot \mu_k(s\omega, \lambda_k)\psi ,$$

where μ_k is the Plancherel distribution corresponding to the standard cuspidal subgroup $P_k = M^{(k)} \cap P$ of $M^{(k)}$. If $\underline{a}_k' = \underline{a}^{(k)} \cap \underline{a}'$ then $\underline{a}' = \underline{a}^{(k)} \oplus \underline{a}_k'$. We write

$$\lambda = \nu + z H^{(k)} , \quad \nu \in \underline{a}_k' , \quad z \in i\mathbb{R},$$

and express the above integral as an iterated integral over $i\,\underline{a}_k'$

and $i\mathbb{R}$. The result is

(6.4) $\gamma_k(P_k) | \int_{i\underline{a}_k'} E_k\ (\tilde{a}_t^s\ (\nu)\ :\ \nu\ :\ K\ h_k)\ \ \mu_k(\underline{s}\,\omega,\ \nu)\ d\ \nu |$,

where $\tilde{a}_t^s(\nu)$ equals

$$\int_{i\mathbb{R}} \mu(s\ \omega\ ,\ \nu + z\ H^{(k)}) \cdot c_k(1\ :\ \nu)^* c(s\ :\ s^{-1}(\nu + z\ H^{(k)}))a^s(\nu + zH^{(k)})\, e^{t\ z} d_z .$$

Let D_z be the differential operator $(1 - \frac{d}{d\,z})^r$. Then

$$\tilde{a}_t^s(\nu) = (1 + t)^{-r}\ \ a_t^s\ (\nu)\ ,$$

Where $a_t^s\ (\nu)$ equals

$$\gamma_k(P_k) \cdot \int_{i\mathbb{R}} D_z\ \ \{\mu(s\ \omega,\ \nu + z\ H^{(k)})\ c_k\ (1\ :\ \nu)^*\ c(s\ :\ s^{-1}(\nu + z\ H^{(k)}))$$

$$\cdot\ a^s(\nu + z\ H^{(k)})\}\ e^{t\ z} d\ z\ .$$

Appealing to the induction hypothesis we choose polynomials p_1' and p_2' and a continuous seminorm N' on $\underline{S}(i\ \underline{a}_k')$ such that (6.4) is bounded by

$$(1 + t)^{-r} \cdot N'(a_t^s) \cdot p_1'(|\omega|) \cdot p_2'\ (|\tau|) \cdot \Xi^{(k)}(h_k) \cdot (1 + \sigma(h_k))^{-r}\ .$$

We have used (2.6) of [1(b)] as well as the fact that $|\omega| = |s\,\omega|$. Since

$$(1 + \sigma(h_k \cdot \exp\ t\ H^{(k)}))^r \leq (1 + t)^r \cdot (1 + \sigma(h_k))^r\ ,$$

(6.2) is bounded by

$$N' \cdot (a_t^s) \cdot p_1'(|\omega|)\ \ p_2'(|\tau|)\ .$$

In the expression for $a_t^s\ (\nu)$ we write

$$\mu(s\ \ \omega\ ,\ \ \nu + z\ H^{(k)})\ c_k(1\ :\ \nu)^*\ c(s\ :\ s^{-1}(\nu + z\ H^{(k)}))$$

as the product of four terms:

(6.5) $q(s\,\omega,\ \nu + z\,H^{(k)}) \cdot q_k(s\,\omega,\ \nu)^{-1}$,

(6.6) $q(s\,\omega,\ \nu + z\,H^{(k)})^{-2} \cdot \mu(s\,\omega,\ \nu + z\,H^{(k)})$,

(6.7) $q_k(s\,\omega,\ \nu)\ \ c_{k,s\,\omega}(1 : \nu)^*$,

(6.8) $q(\omega,\ s^{-1}(\nu + z\,H^{(k)})) \cdot c_\omega(s : s^{-1}(\nu + z\,H^{(k)}))$.

Here $q(\omega, \ )$ is the polynomial function on $\underline{a}_{\mathbb{C}}$ defined in §4, and $q_k(s\,\omega, \ )$ is the corresponding function associated to the pair $(M^{(k)}, P_k)$. We have used the fact that

$$q(s\,\omega,\ \nu + z\,H^{(k)}) = q(\omega,\ s^{-1}(\nu + z\,H^{(k)})) .$$

The expression (6.5) is a polynomial. In fact by (4.2) and the definition of q, it equals

$$\prod_\beta <\beta,\ \nu + z\,H^{(k)}>^{n^\beta(s\,\omega)} ,$$

where the product is taken over those reduced roots β of (P', A') which do not vanish on $\underline{a}^{(k)}$. Suppose that $D = D_{z,\nu}$ in any differential operator in $\underline{D}(i\,\underline{a}')$, and that $H(\omega, \nu, z)$ is any one of the above four terms. Then from Lemmas 4.3 and 4.5 we observe that there is a polynomial p_D such that

$$\| D_{z,\nu} \cdot H(\omega, \nu, z) \| \le p_D(|\omega| + |\nu| + |z| + |\tau|) .$$

We can certainly find a polynomial p_o' and a finite set $\{D_\nu^j\}$ of differential operators in $\underline{D}(i\,\underline{a}_k')$ such that

$$N'(a_t^s) \le \sup_{\nu \in i\,\underline{a}_k'} \{\Sigma_j\ p_o'(|\nu|) \cdot |D_\nu^j\ a_t^s(\nu)|\} .$$

It follows from Leibnitz' rule and the above remarks that we can find polynomials p_o'', p_1'', and p_2'', and a finite set $\{\tilde{D}^j_{z,\nu}\}$ of operators in $\underline{D}(i\ \underline{a}')$ such that $N'(a^s_t)$ is bounded by

$$p_1''(|\omega|)\cdot p_2''(|\tau|)\cdot \sup_{\nu\in i\ \underline{a}_k'}\{\Sigma_j \int_{i\mathbb{R}} p_o''(|\nu|+|z|)\,|\tilde{D}^j_{z,\nu}\ a^s(\nu+z\ H^{(k)})|\cdot d|z|\}\ .$$

But

$$N''(\alpha) = \sup_{\nu\in i\ \underline{a}_k'}\{\Sigma_j\ \{\int_{i\mathbb{R}} p_o''(|\nu|)\cdot|\tilde{D}^j_{z,\nu}\ \alpha^s(\nu+z\ H^{(k)})|\cdot d|z|\}\},$$

$$\alpha\in\underline{S}(i\ \underline{a}),$$

is a continuous seminorm on $\underline{S}(i\ \underline{a})$.

We have shown that (5.2) is bounded by

$$N''(a)\cdot p_1'(|\omega|)\cdot p_1''(|\omega|)\cdot p_2'(|\tau|)\cdot p_2''(|\tau|)\ .$$

This completes the proof of the theorem.

□

<u>COROLLARY 6.2</u>. For any elements Y_1 and Y_2 in $\underline{G}$, and any real r, there is a continuous seminorm N on $\underline{S}(i\ \underline{a})$ and polynomials p_1 and p_2 such that for any τ and ω, and any $a\in\underline{S}(i\ \underline{a})\otimes\underline{C}_\omega(M,\ \tau_M)$,

$$\sup_{x\in G}\{\Xi(x)^{-1}\cdot(1+\sigma(x))^r\cdot|\int_{i\ \underline{a}} E(a(\lambda):\lambda\ :\ Y_1;\ x;\ Y_2)\ \mu(\omega,\lambda)d\ \lambda|\}$$

is bounded by

$$N(a)\cdot p_1(|\omega|)\cdot p_2(|\tau|).$$

PROOF: This corollary follows immediately from the theorem and Lemma

5.3 of [1(b)]. □

Let us now complete the second half of the proof of Theorem 3.1. We shall establish a theorem which is a slightly more general version of what remains to be proved. Fix $\underline{P} \in \mathfrak{O}(G)$. Let $\underline{S}_{\underline{P}}(\hat{G})$ be the space of functions which is defined analagously to $\underline{C}_{\underline{P}}(\hat{G})$ but <u>without</u> the symmetry condition (2.3).

<u>THEOREM 6.3</u>: Suppose $a \in \underline{S}_{\underline{P}}(\hat{G})$. Then for $P \in \underline{P}$, $\omega \in \underline{E}_2(M)$ and $\lambda \in i\,\underline{a}$, $a(\omega,\lambda)$ belongs to $\underline{C}_{\omega}(M, \rho_M)$.

The function

$$f_a(x) = n(\underline{P})^{-1} \cdot \sum_{P\in\underline{P}} \sum_{\omega\in\underline{E}_2(M)} \left(\frac{1}{2\pi i}\right)^q \int_{i\,\underline{a}} E(a(\omega,\lambda):\lambda:x)_{(1,1)} \mu(\omega,\lambda)d\lambda,$$

is defined by an expression which converges absolutely uniformly on $x \in G$. Finally, the map

$$a \longrightarrow f_a\ , \quad a \in \underline{S}_{\underline{P}}(\hat{G}),$$

is a continuous linear transformation from $\underline{S}_{\underline{P}}(\hat{G})$ to $\underline{C}(G)$.

PROOF: Fix $P \in \underline{P}$. For each $\omega \in \underline{E}_2(M)$, let

$$\{\psi^k : k \in I_2(\omega)\}$$

be an orthonormal basis of $L^2_\omega(M, \rho_M)$ consisting of vectors in $U(\omega)$. For each $k \in I_2(\omega)$, and $i = 1, 2$, we shall denote the representation $\eta_i(\psi^k)$ in $\underline{E}(K)$ by $\eta_i^{\ k}$. Given $k \in I_2(\omega)$,

$a \in \underline{S}_P(\hat{G})$, and $\lambda \in i\,\underline{a}$, define

$$a_\omega^k(\lambda) = (a(\omega,\lambda),\ \psi^k)\ \psi^k$$

and

$$F^k = \{\eta_1^k\ ,\ \eta_2^k\}\ .$$

Then a_ω^k is a function in $\underline{S}(i\,\underline{a}) \otimes \underline{C}_\omega(M,\ \rho_{F^k}\ ,\ M)$.

It is a consequence of the results of [1(b)] (Formula (5.1), Lemmas 5.4 and 8.3) that there is a polynomial p such that for $\lambda \in i\,\underline{a}$, $\omega \in \underline{E}_2(M)$, $k \in I_2(\omega)$ and $x \in G$,

$$(6.9) \qquad |E(a_\omega^k(\lambda) : \lambda : x)_{(1,1)} \quad \mu(\omega,\lambda)|$$

is bounded by

$$p(|\omega| + |\lambda| + |\rho_{F^k}|) \cdot |(a(\omega,\lambda),\ \psi^k)|\ .$$

Notice that

$$|\rho_{F^k}| = (|\eta_1^k| + |\eta_2^k|)^2\ .$$

It follows from Lemma 2.3 of [1(b)] that there is a positive integer m and a constant d_o such that (6.9) is dominated by

$$d_o \cdot |(a(\omega,\lambda),\ \psi^k)| \cdot (1 + |\lambda|^2)^m \cdot (1 + |\eta_1^k|^2)^m \cdot (1 + |\eta_2^k|^2)^m\ .$$

Choose an integer n such that

$$c_n = \Sigma_{\omega \in \underline{E}_2(M)} \quad \Sigma_{k \in I_2(\omega)}\ (1 + |\eta_1^k|^2)^{-n} \cdot (1 + |\eta_2^k|^2)^{-n}$$

is finite. This is possible by [1(b)], Corollary 2.5. Then

$$\Sigma_{\omega \in \underline{E}_2(M)} \quad \Sigma_{k \in I_2(\omega)} \quad \int_{i\underline{a}} |E(a_\omega^k(\lambda) : \lambda : x)_{(1,1)}| \; \mu(\omega,\lambda) \; d|\lambda|$$

is bounded by

$$c_n \cdot \sup_{\lambda,\, \omega,\, k} \{ |(a(\omega,\lambda), \psi^k)| \cdot (1 + |\lambda|^2)^m \cdot (1 + |\eta_1^k|^2)^{m+n} \cdot (1 + |\eta_2^k|^2)^{m+n} \}$$

This quantity is finite. Therefore the function

$$f_{a,P}(x) = n(\underline{P})^{-1} \cdot \sum_{\omega \in \underline{E}_2(M)} \Sigma_{k \in I_2(\omega)} \left(\frac{1}{2\pi i}\right)^q \int_{i\underline{a}} E(a_\omega^k(\lambda) : \lambda : x)_{(1,1)}$$

$$\mu(\omega,\lambda) \; d\lambda \, , \qquad x \in G,$$

is defined by an expression which converges absolutely uniformly in x.

Fix $Y_1, Y_2 \in \underline{G}$. It is clear that in the expression

$$\int_{i\underline{a}} E(a_\omega^k(\lambda) : \lambda : Y_1 ; x ; Y_2)_{(1,1)} \cdot \mu(\omega,\lambda) d\lambda$$

the differential operators defined by Y_1 and Y_2 can be taken outside the integral sign. On the other hand, the absolute value of

$$n(\underline{P})^{-1} \cdot \sum_{\omega \in \underline{E}_2(M)} \sum_{k \in I_2(\omega)} \left(\frac{1}{2\pi i}\right)^q \cdot \int_{i\underline{a}} E(a_\omega^k(\lambda) : \lambda : Y_1 ; x ; Y_2)_{(1,1)}$$

$$\cdot \mu(\omega,\lambda) d\lambda$$

is dominated by

$$\Sigma_\omega \cdot \Sigma_k \cdot \left| \int_{i\underline{a}} E(a_\omega^k(\lambda) : \lambda : Y_1 ; x ; Y_2)_{(1,1)} \cdot \mu(\omega,\lambda) \; d\lambda \right| .$$

Fix a real number r. According to Corollary 6.2, we can find a continuous seminorm N on $\underline{S}(i\,\underline{a})$ and polynomials p_1 and p_2 such that this expression is no greater than

$$\{\Sigma_{\omega\in\underline{E}_2(M)}\ \Sigma_{k\in I_2(\omega)}\ N(a^k_\omega)\cdot p_1(|\omega|)\cdot p_2(|\rho_{F^k}|)\}$$
$$\Xi(x)\cdot(1+\sigma(x))^{-r}\ .$$

Remembering [1(b)], Lemma 2.3, we choose a positive integer m and a constant d_1 such that the term in the brackets is bounded by

$$\Sigma_\omega\ \Sigma_k\ \ N(a^k_\omega)\cdot(1+|\eta^k_1|^2)^m\ \ (1+|\eta^k_2|)^m\ .$$

Select a constant d_2, a positive integer n and a differential operator $D = D_\lambda$ in $\underline{D}(i\,\underline{a})$ such that

$$N(a^k_\omega)\le d_2\cdot\sup_{\lambda\in i\underline{a}}\ \{(1+|\lambda|^2)^n\cdot|D_\lambda(a(\omega,\lambda),\ \psi^k)|\}\ .$$

We may assume that n is large enough that the constant c_{n-m}, defined above, is finite. We have shown that $f_{a,P}(x)$ is a smooth function, and that

$$\sup_{x\in G}\{|f_{a,P}(Y_1;\ x\ ;\ Y_2)|\ \ \Xi(x)^{-1}\cdot(1+\sigma(x))^r\}\le d_1\ d_2\ c_{n-m}\|a\|_{D,n}\ .$$

Here $\|\ \|_{D,n}$ is the continuous seminorm on $\underline{S}_P(\hat{G})$ defined in §3. In particular, $f_{a,P}$ belongs to $\underline{C}(G)$.

Suppose that in the above discussion we replace (G,P) by the pair $(K\times M\times K,\ K\times M\times K)$. Then we obtain the statement that for each ω and λ, the series

$$\Sigma_{k\in I_2(\omega)}\qquad a^k_\omega$$

converges pointwise to a function in $\underline{C}_{\omega}(M, \rho_M)$. Since we already know that the series converges in $L^2(K \times M \times K)$ to $a(\omega,\lambda)$, we see that $a(\omega,\lambda)$ belongs to $\underline{C}_{\omega}(M, \rho_M)$. In particular, $E(a(\omega,\lambda) : \lambda : x)$ is well defined, and

$$f_{a,P}(x) = n(\underline{P})^{-1} \sum_{\omega \in \underline{E}_2(M)} \left(\frac{1}{2\pi i}\right)^q \int_{i\underline{a}} E(a(\omega,\lambda) : \lambda : x)_{(1,1)} \cdot \mu(\omega,\lambda) d\lambda .$$

The right hand expression converges absolutely uniformly in x. In addition,

$$f_a(x) = \Sigma_{P \in \underline{P}} \; f_{a,\underline{P}}(x).$$

Therefore ~~the~~ map

$$a \longrightarrow f_a , \; a \in \underline{S}_{\underline{P}}(\hat{G}),$$

is continuous. Our proof is complete. □

Suppose that $a \in \underline{C}_P(\hat{G})$. Then, as we have just seen, a has the properties that we demanded in our derivation of formula (2.6). It follows that the function f_a, defined by the formula in Theorem 6.3, is also the unique function in $L^2(g)$ such that

$$\hat{f}_a = a .$$

The proof of Theorem 3.1 is, at ~~the~~ last, complete.

§7. TEMPERED DISTRIBUTIONS

The main reason for proving Theorem 3.1 is to allow us to define the Fourier transform of a tempered distribution. A distribution of G is said to be *tempered* if it extends to a continuous linear functional from $\underline{C}(G)$ to $\mathbb{C}$. Since $C_c^\infty(G)$ is dense in $\underline{C}(G)$, and since the inclusion map is continuous, we regard the space of tempered distributions as the topological dual space, $\underline{C}'(G)$, of G. It becomes a locally convex topological vector space when endowed with the weak topology.

For any $\underline{P} \in \mathrm{Cl}(G)$, let $\underline{C}'_{\underline{P}}(\hat{G})$ be the topological dual space of $\underline{C}_{\underline{P}}(G)$. Let $\underline{C}'(\hat{G})$ be the topological dual space of $\underline{C}(\hat{G})$. Then

$$\underline{C}'(G) = \bigoplus_{\underline{P}\in \mathrm{Cl}(G)} \underline{C}'_{\underline{P}}(\hat{G}).$$

Theorem 3.1 defines the topological isomorphism

$$\underline{F} : \underline{C}(G) \longrightarrow \underline{C}(\hat{G}) .$$

An immediate consequence of this theorem is

THEOREM 7.1: The transpose

$$\underline{F}' : \underline{C}'(\hat{G}) \longrightarrow \underline{C}'(G)$$

of $\underline{F}$ is a topological isomorphism.

□

It is useful to obtain a slightly different characterization of the space $\underline{C}'_{\underline{P}}(\hat{G})$. Fix $\underline{P} \in \mathrm{Cl}(G)$ and $a \in \underline{S}_{\underline{P}}(G)$. For each P in $\underline{P}$, each ω in $\underline{E}_2(M)$, and every point λ in $i\,\underline{a}$, define

$$(7.1)\quad (\underline{M}\,a)(\omega,\lambda) = n(\underline{P})^{-1} \sum_{P'\in\underline{P}} \sum_{s\in\Omega(\underline{a},\underline{a}')} M(s^{-1}: s\lambda)\, a(s\omega, s\lambda).$$

LEMMA 7.2: The map

$$a \longrightarrow \underline{M}\,a, \qquad a \in \underline{S}_{\underline{P}}(\hat{G}),$$

is a continuous projection on $\underline{S}_{\underline{P}}(\hat{G})$ whose range is $\underline{C}_{\underline{P}}(\hat{G})$.

PROOF: Let $S^{\infty}_{c,\underline{P}}(\hat{G})$ be the space of functions a in $\underline{S}_{\underline{P}}(\hat{G})$ such that for each $P \in \underline{P}$, $a(\omega,\lambda)$ vanishes for all but finitely many $\omega \in \underline{E}_2(M)$, and such that for any $\omega \in \underline{E}_2(M)$ there is a finite subset F of $\underline{E}(K)$ such that the function

$$\lambda \longrightarrow a(\omega,\lambda), \quad \lambda \in i\,\underline{a},$$

belongs to $C^{\infty}_{c}(i\,\underline{a}) \otimes \underline{C}_{\omega}(M, \rho_{F,M})$. Then $S^{\infty}_{c,\underline{P}}(\hat{G})$ is a dense subspace of $S_{\underline{P}}(\hat{G})$.

Suppose that a is in $S^{\infty}_{c,\underline{P}}(\hat{G})$. By the functional equations for $M(s^{-1}: s\lambda)$, ([1(b)], Theorem 8.5), $\underline{M}\,a$ satisfies the symmetry conditions (2.3). It follows that $\underline{M}\,a$ belongs to $\underline{C}_{\underline{P}}(\hat{G})$. Apply formula (4.1). Then $f_{\underline{M}\,a}$, the unique function in $L^2(G)$ such that

$$\hat{f}_{\underline{M}\,a} = \underline{M}\,a,$$

is given by the formula

$$n(\underline{P})^{-1} \cdot \sum_{P\in\underline{P}} \sum_{\omega\in\underline{E}_2(M)} \left(\frac{1}{2\pi i}\right)^q \cdot \int_{i\,\underline{a}} E((\underline{M}\,a)(\omega,\lambda) : \lambda : x)_{(1,1)} \cdot \mu(\omega,\lambda)\, d\lambda.$$

Substitute the formula (7.1) in this expression. We note that

$$E(M(s^{-1} : s\lambda)\, a(s\omega, s\lambda) : \lambda : x)$$
$$= E(M(s : \lambda)^{-1}\, a(s\omega, s\lambda) : \lambda : x)$$
$$= E(a(s\omega, s\lambda) : s\lambda : x).$$

After a change of variables, we arrive at the formula for $f_a(x)$ defined in Theorem 6.3. It follows that

$$\underline{F}\, f_a = \underline{M}\, a\ .$$

We have shown that the restriction of $\underline{M}$ to the dense subspace $S^{\infty}_{c,\underline{P}}(\hat{G})$ is the composition of the two maps

$$a \longrightarrow f_a \longrightarrow \underline{F}\, f_a\ .$$

By Theorem 3.1 and Theorem 6.3, this composition is continuous. We can extend it to a continuous linear map from $\underline{S}_{\underline{P}}(\hat{G})$ to $\underline{C}_{\underline{P}}(\hat{G})$.

We must show that an <u>arbitrary</u> function a in $\underline{S}_{\underline{P}}(\hat{G})$ is sent by this map to the function $\underline{M}\, a$ defined by (7.1). It is enough to check that if $a \in \underline{S}_{\underline{P}}(\hat{G})$, and $\{a_n\}$ is a sequence of functions in $S^{\infty}_{c,\underline{P}}(\hat{G})$ which converges in $\underline{S}_{\underline{P}}(\hat{G})$ to a, then

$$\underline{M}\, a = \lim_{n \to \infty} \underline{M}\, a_n\ .$$

We have just observed that $\{\underline{M}\, a_n\}$ converges in $\underline{C}_{\underline{P}}(\hat{G})$ to some function, say a'. Then $\underline{M}\, a_n$ also converges to a' in the topology of $L^2_{\underline{P}}(\hat{G})$. But it is evident from (7.1) that $\underline{M}\, a_n$ converges to $\underline{M}\, a$ in $L^2_{\underline{P}}(\hat{G})$. It follows that for $P \in \underline{P}$,

$$(\underline{M}\, a)\ (\omega, \lambda) = a'(\omega,\lambda)$$

for almost all $(\omega,\lambda) \in \underline{E}_2(M) \times i\, \underline{a}$. Since both functions are continuous, they are equal everywhere.

If a is already in $\underline{C}_P(\hat{G})$, it is clear that $\underline{M}\, a = a$. Therefore $\underline{M}$ is a projection whose range is $\underline{C}_P(\hat{G})$.

□

We shall regard $\underline{M}$ as an operator on $\underline{S}_P(\hat{G})$. If we replace the co-domain by $\underline{C}_P(\hat{G})$, we denote the resulting map by $\tilde{\underline{M}}$. If $\underline{S}_P'(\hat{G})$ is the topological dual space of $\underline{S}_P(\hat{G})$, let

$$\underline{M}' : \underline{S}_P'(\hat{G}) \longrightarrow \underline{S}_P'(\hat{G})$$

and

$$\tilde{\underline{M}}' : \underline{C}_P'(\hat{G}) \longrightarrow \underline{S}_P'(\hat{G})$$

be the transposes of the above maps.

THEOREM 7.3: $\tilde{\underline{M}}'$ is a topological isomorphism from $\underline{C}_P'(\hat{G})$ onto the closed subspace

$$\tilde{\underline{S}}_P'(\hat{G}) = \{A \in \underline{S}_P'(\hat{G}) : \underline{M}'\, A = A\}$$

of $\underline{S}_P'(\hat{G})$.

PROOF: Clearly $\tilde{\underline{M}}'$ is an injection into $\tilde{\underline{S}}_P'(\hat{G})$. Suppose that $A \in \tilde{\underline{S}}_P'(\hat{G})$. Define a distribution $\tilde{A}$ in $\underline{C}_P'(\hat{G})$ to be the restriction of A to the closed subspace $\underline{C}_P(\hat{G})$ of $\underline{S}_P(\hat{G})$. For $a \in \underline{S}_P(\hat{G})$,

$$\langle \tilde{\underline{M}}'\, \tilde{A}, a\rangle = \langle \tilde{A}, \underline{M}\, a\rangle = \langle A, \underline{M}\, a\rangle$$
$$= \langle \underline{M}'\, A, a\rangle = \langle A, a\rangle .$$

Therefore the range of $\underline{\widetilde{M}}'$ is $\widetilde{\underline{S}}'_{\underline{P}}(G)$. The theorem is proved.

□

From now on, we identify the spaces $\underline{C}'_{\underline{P}}(\hat{G})$ and $\widetilde{\underline{S}}'_{\underline{P}}(\hat{G})$. Then if $F \in \underline{C}'(G)$, $(\underline{F}')^{-1}(F)$ is a collection of Euclidean distributions, which satisfies certain symmetry conditions. We call it the Fourier transform of F.

BIBLIOGRAPHY

1. Arthur, J.

(a) Harmonic Analysis of Tempered Distributions on Semisimple Lie Groups of Real Rank One, Thesis, Yale, 1970.

(b) Harmonic analysis of the Schwartz space on a reductive Lie group I, (preprint).

2. Eguchi, M., Harmonic analysis on some types of semisimple Lie groups, (to appear).

3. Ehrenpreis, L., and Mautner, F.I., Some properties of the Fourier-transforms on semi-simple Lie groups. III. T.A.M.S. 90 (1959), 431-484.

4. Harish-Chandra,

(a) Representations of a semisimple Lie group on a Banach space I, T.A.M.S. 75 (1953), 185-243.

(b) Spherical functions on a semisimple Lie group I, Amer. J. Math. 80 (1958), 241-310.

(c) Discrete series for semisimple Lie groups II, Acta. Math. 116 (1966), 1-111.

(d) On the theory of the Eisenstein integral, Lecture notes in Mathematics, vol. 266 (1970), 529-551.

5. Mackey, G. W.,

(a) Induced representations on locally compact groups I, Ann. of Math. (2) 55 (1952), 101-139.

(b) Borel Structure in groups and their duals, T.A.M.S. 85 (1957), 134-165.

6. Trombi, P.C. and Varadarajan, V.S., Asymptotic behaviour of eigenfunctions on a semisimple Lie group: the discrete spectrum Acta. Math. 129 (1972), 237-280.