(1)

Duality & Hecke operators: Orsay, Arith. Geometry, May 30, 1995

2 problems: 1 specific, 1 general: G (split) / F (p-adic).

Packets: (i) $G \rightsquigarrow \hat{G}$ complex L-gp.

G	$\hat{G}$
$Sp(2n)$	$SO(2n+1, \mathbb{C})$
$SO(2n+1)$	$Sp(2n, \mathbb{C})$
$SO(2n)$	$SO(2n, \mathbb{C})$.

(ii) $L_F = W_F \times SL(2, \mathbb{C})$

Conj: (Del.-Langl.): Irred. rep^s of $G(F)$ occur in finite packets Π_ϕ param. by $\hat{G}$-orbits of homo^{sm} $\phi: L_F \to \hat{G}$.

The rep^s in Π_ϕ should satisfy char. rel^{ns} relating $S_\phi = \mathrm{Cent}(\mathrm{Im}(\phi), \hat{G})$ with the Langl.-Shelstad transfer map $f \to f^{G'}$ (maps f^{ns} on $G(F)$ to f^{ns} on $G'(F)$, G' end. gp for G) through a mapping $\Pi_\phi \hookrightarrow (\pi_0(S_\phi))^\wedge$, $\pi \to \langle \cdot, \pi \rangle$.

Shelstad: $F = \mathbb{R}$.

[Hope to construct Π_ϕ for G classical (as above) by global means (trace formula) + their existence at one place (Shelstad)]

[What rep^s of $G(F)$ have global significance – i.e. are local components of aut. rep^s? I introduced a subset of $\{\phi\}$ which should classify aut. forms]

Take $\hat{G}$-orbits of homo^{sm} $\psi: L_F \times SL(2, \mathbb{C}) \to \hat{G}$ ∋ image of $W_F \subset L_F$ is b^{dd}. If $\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right)$ ($|\cdot| : L_F \to W_F \to F^\times \to \mathbb{R}^+$), then $\psi \to \phi_\psi$ is an injection.

I conjectured existence of finite packets Π_ψ sat. char. rel^{ns} as

(of not nec. irred. rep^s)

above, but for $S_\psi = \mathrm{Cent}(\mathrm{Im}(\psi), \hat{G})$. [J. may do construct Π_ψ for classical G by global means, but need their existence at one place for suff. many $\{\psi\}$ + to have many $\pi \in \Pi_\psi$ cusp'd.]

2 signif. advances.

(i) Moeglin: F-padic, residues of Eisenstein series: need clas. rel$^{\underline{ns}}$

(ii) A. B. V: F-real, perverse sheaves on $\hat{G}$: clas. rel$^{\underline{ns}}$, but need irreducibility of π.

Duality: A. M. Aubert (Curtis, Bernstein): If π is virtual character on $G(F)$, set

$$\hat{\pi} = \sum_{P \supset P_0} (-1)^{\dim(A_{P_0}/A_P)} \, i_P^G \, (r_P^G (\pi))$$

induced module   Jacquet module

$\pi \rightsquigarrow$ locally integ. f$^{\underline{c}}$ of $\gamma \in G(F)_{reg}$. Can define duality for a very wide class of class f$^{\underline{ns}}$ on $G(F)_{reg}$.

Formula collapses: Any $\gamma \in G(F)_{reg} \rightsquigarrow (M, Q)$. For $\gamma \in T(F)_{reg}$, T a max. torus; M is the unique Levi $\ni$. $A_M = A_T$ (split cmp$^{\underline{ts}}$) + $Q \supset M$ is the unique psgp. $\ni$. $H_M(\gamma) \in \mathfrak{a}_Q^+$.

[elliptic v$^{\underline{s}}$ bdd elt$^{\underline{s}}$ in $G(F)$: Clozel, Howe conjecture]

PROPOSITION: $\hat{\pi}(\gamma) = \sum_{\{P : M \subset P \subset \overline{Q}\}} (-1)^{\dim(A_{P_0}/A_P)} \, r_P^G (\pi, \gamma)$.

G' end. gp for G; $f \to f^{G'}$ - transfer of orbital integrals on $G(F)$ to stable orbital integrals on $G'(F)$ - + is dual to a map $\Phi \to \Phi^G$ from stable class f$^{\underline{ns}}$ on $G'(F)$ to class f$^{\underline{ns}}$ on $G(F)$.

THEOREM: (i) If Φ is a stable class f^{n} on $G'(F)$, so is $\hat{\Phi}$

(ii) $(\Phi^G)^\wedge = (\hat{\Phi})^G (-1)^{(\text{s.rk } G)-(\text{s.rk } G')}$

Should also work for twisted characters. In partic., $GL(N)$ has outer aut. $\alpha(g) = {}^t g^{-1}$. Look at a parameter

$$\psi : W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow GL(N,\mathbb{C})$$

with $\psi^\alpha \cong \psi$. Then ψ factors through a classical gp $\hat{G}$. So does the dual parameter $\hat{\psi}$ (obtained by switching $SL(2,\mathbb{C})$ factors), & $S_\psi = S_{\hat{\psi}}$.

COROLLARY(?): Assume the packet Π_ψ for G has been constructed (as a family of class f$^{\underline{ns}}$). Then $\Pi_{\hat{\psi}}$ is given as $\pm\{\hat{\pi} : \pi \in \Pi_\psi\}$, with $\langle s, \hat{\pi}\rangle = \langle s_\psi^1 s_\psi^2 s, \pi\rangle$, ($s_\psi^i = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $i^{\underline{th}}$ $SL(2,\mathbb{C})$-factor).

Difficulty: If π is irred. rep., hard to show that $\pm\hat{\pi}$ is also.

Fix Δ (Iwahori) $\subseteq G(F)$.

Borel: $\exists$ bijection $\theta \in \Pi(\mathcal{H}_\Delta)$ (irred. rep$^{\underline{s}}$ of Hecke alg. $\mathcal{H}_\Delta = C_c(\Delta\backslash G(F)/\Delta)$) $\longleftrightarrow \pi_\theta \in \Pi_\Delta$ (irred. rep$^{\underline{s}}$ of $G(F)$ with Δ-fixed vector).

Iwahori-Matsumoto: Involution on $\mathcal{H}_\Delta$ ⇝ involution $\theta \to \hat{\theta}$ on $\Pi(\mathcal{H}_\Delta)$

Kato: $\hat{\pi}_\theta = \pi_{\hat{\theta}}$. In partic., if $\pi \in \Pi_\Delta$, then $\hat{\pi}$ is irred. & also lies in Π_Δ.

Suppose G classical, & that ψ vanishes on the inertia gp. $I_F \subset W_F$ and on the second $SL(2, \mathbb{C})$. Then ϕ vanishes on the first $SL(2, \mathbb{C})$ factor. Expect to use trace formula to construct Π_ψ as a family of irred. tempered ~~chars~~ rep^s of $G(F)$. Assume so. Then using Kazh.-Luszt., one gets

PROPOSITION: $\Pi_\psi \cap \Pi_\Delta$ consists of those π s.t. $\xi(z) = \langle z, \pi \rangle$ lies in image of Springer corresp.

COROLL: $\Pi_\psi^\wedge \cap \Pi_\Delta$ has same property. In partic., there are a large number of irred. rep^s in $\Pi_{\hat\psi}$.

Extensions: (i) Allow ψ to be tamely ramified on W_F. (Morris)
(ii) Deduce intertwining op. identities for rep^s in Π_ϕ from those for Π_ψ.

Now suppose $G/\mathbb{Q}$, & defines Shimura variety S_K, $K \subset G(\mathbb{A}_f)$. Can take global parameter $\psi = \bigotimes_v \psi_v$, which contributes a subsp. $H^*_{(2)}(S_K)_\psi$ to the L^2-cohom. of S_K. Suppose
(i) $\psi_{\mathbb{R}}$ has $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$ cohom,
(ii) Away from ~~fixed~~ finite place v, ψ^v & K^v are unram.

Take $F = \mathbb{Q}_v$. Then

$$H^*_{(2)}(S_K)_\psi = \bigoplus_{\xi \in \hat{S}_\psi} \bigoplus_{\{\pi_v \in \Pi_{\psi_v} : \langle \cdot, \pi_v \rangle = \xi\}} \left(V_{\xi \varepsilon_\psi} \otimes \pi_v^{K_v} \right),$$

where $r : \hat{G} \to V$ is the finite dim. rep. of $\hat{G}$ assoc. to S_K,
$\varepsilon_\psi : S_\psi \to \{\pm 1\}$ is canonical character + $V_{\xi\varepsilon_\psi}$ is the $\xi\varepsilon_\psi$-equiv. part of V (i.e. $S_\psi \hookrightarrow \hat{G} \xrightarrow{r} V$). There is canonical rep.

$$r \circ \psi_v : W_{\mathbb{Q}_v} \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow V$$

that commutes with S_ψ.

Expectation (deep): $r \circ \psi_v$ is the rep. of $W_F \times \{\text{monodromy}\} \times \{\text{Lefschetz}\}$ defined by ℓ-adic cohom. of S_K.

Choose ψ' satisfying above cond^{ns}, with $\psi'_v = \hat{\psi}_v$. Choose K_v compatible with Hecke alg. defined by $\psi_v|_{I_{\mathbb{Q}_v}}$.
(Special case: ψ_v vanishes on $I_{\mathbb{Q}_v}$, $K_v = \Delta$)
Then duality defines iso $H^{(*)}_{(2)}(S_K)_\psi \xrightarrow{\sim} H^{*}_{(2)}(S_K)_{\psi'}$.

Problem: (Kottwitz) Varieties over $\mathbb{C}$?

mirror symmetry?