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The Trace Paley Wiener Theorem for Schwartz Functions

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Suppose that G is a connected reductive algebraic group over a local field F of characteristic 0. If f is a function in the Schwartz space $\mathcal{C}(G(F))$, and $\pi \in \Pi_{\mathrm{temp}}(G(F))$ is an irreducible tempered representation of $G(F)$, the operator

$$f_G(\pi) = \int_{G(F)} f(x)\pi(x)dx$$

is of trace class. We can therefore map f to the function

$$f_G(\pi) = \mathrm{tr}(\pi(f))$$

on $\Pi_{\mathrm{temp}}(G(F))$. The object of this note is to characterize the image of the map.

Results of this nature are well known. The case of the Hecke algebra on $G(F)$, which is in fact more difficult, was established in [3] and [5]. A variant of the problem for the smooth functions of compact support on a real group was solved in [4]. For the Schwartz space, one has a choice of several possible approaches. We shall use the characterization of the operator valued Fourier transform

$$f \to \pi(f), \qquad f \in \mathcal{C}(G(F)),$$

which was solved separately for real and p-adic groups [2], [9, Part B]. (See also [6, Lemma 5.2].)

Irreducible tempered representations occur as constituents of induced representations

$$\mathcal{I}_P(\sigma):\ G(F) \to \mathrm{End}(\mathcal{H}_P(\sigma)), \qquad \sigma \in \Pi_2(M(F)).$$

Here M belongs to the finite subset $\mathcal{L}$ of Levi subgroups of G which contain a fixed minimal Levi subgroup, P belongs to the set $\mathcal{P}(M)$ of parabolic subgroups

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with Levi component M, and $\Pi_2(M(F))$ is the set of (equivalence classes of) irreducible unitary representations of $M(F)$ which are square integrable modulo the center. The irreducible constituents of $\mathcal{I}_P(\sigma)$ in general are determined by projective representations of the R-group R_σ of σ [7], [8]. To convert projective representations to ordinary representations, one takes a finite central extension

$$1 \longrightarrow Z_\sigma \longrightarrow \widetilde{R}_\sigma \longrightarrow R_\sigma \longrightarrow 1$$

of the R-group. The usual intertwining operators then give rise to a representation

$$r \longrightarrow \widetilde{R}(r,\sigma), \qquad r \in \widetilde{R}_\sigma,$$

of $\widetilde{R}_\sigma = \widetilde{R}_\sigma^G$ on $\mathcal{H}_P(\sigma)$ which commutes with $\mathcal{I}_P(\sigma)$. (See [1, §2].) The process singles out a character χ_σ of Z_σ; following [1], we write $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ for the set of irreducible representations of $\widetilde{R}_\sigma$ whose central character on Z_σ equals χ_σ. Then there is a bijection $\rho \to \pi_\rho$ from $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$ onto the set of irreducible constituents of $\mathcal{I}_P(\sigma)$, with the properties that

$$\text{(1)} \qquad \operatorname{tr}\big(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)\big) = \sum_{\rho\in\Pi(\tilde{R}_\sigma,\chi_\sigma)} \operatorname{tr}\big(\rho^\vee(r)\big)\operatorname{tr}\big(\pi_\rho(f)\big)$$

and

$$\text{(2)} \qquad \operatorname{tr}\big(\pi_\rho(f)\big) = |\widetilde{R}_\sigma|^{-1} \sum_{r\in\tilde{R}_\sigma} \operatorname{tr}\big(\rho(r)\big)\operatorname{tr}\big(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)\big),$$

for any function $f \in \mathcal{C}(G(F))$. (We are writing $\rho^\vee$ for the contragredient of ρ.)

Consider the set $\widetilde{T}(G)$ of triplets

$$\tau = (M,\sigma,r), \qquad M \in \mathcal{L},\ \sigma \in \Pi_2(M(F)),\ r \in \widetilde{R}_\sigma,$$

which are essential in the sense of [1, §3]. (This means that the subgroup of elements $z \in Z_\sigma$ for which zr is conjugate to r lies in the kernel of χ_σ.) The restricted Weyl group W_0^G of G acts on $\widetilde{T}(G)$, and we write $T(G)$ for the set W_0^G-orbits in $\widetilde{T}(G)$. Set

$$\text{(3)} \qquad f_G(\tau) = \operatorname{tr}\big(\widetilde{R}(r,\sigma)\mathcal{I}_P(\sigma,f)\big), \qquad f \in \mathcal{C}(G(F)).$$

Then $f_G(\tau)$ depends only on the W_0^G-orbit of τ, and is therefore a function on $T(G)$. There is also a symmetry condition

$$f_G(z\tau) = \chi_\tau(z)^{-1} f_G(\tau), \qquad z \in Z_\tau,$$

in which we have written $Z_\tau = Z_\sigma$, $\chi_\tau = \chi_\sigma$ and $z\tau = (M,\sigma,zr)$. Observe that (1) and (2) represent isomorphisms between the two maps $f \to f_G(\pi)$ and $f \to f_G(\tau)$. It will therefore be enough for us to characterize the image of the second map.

Before discussing the image, we should recall [1, §3] that $\widetilde{T}(G)$ is a disjoint union over all $L \in \mathcal{L}$ of spaces $\widetilde{T}_{\mathrm{ell}}(L)$. By definition, $\widetilde{T}_{\mathrm{ell}}(L)$ is the set of triplets

$$(M, \sigma, r), \qquad M \subset L, \ r \in \widetilde{R}^L_{\sigma,\mathrm{reg}},$$

in which M is contained in L and where the null space of r, as a linear transformation on the real vector space

$$\mathfrak{a}_M = X(M)_F \otimes \mathbf{R},$$

is the subspace $\mathfrak{a}_L$. There is an action

$$\tau \longrightarrow \tau_\lambda = (M, \sigma_\lambda, r), \qquad \lambda \in i\mathfrak{a}^*_L,$$

of the real vector space $i\mathfrak{a}^*_L$ on $\widetilde{T}_{\mathrm{ell}}(L)$. This makes $\widetilde{T}(G)$ into a disjoint union of compact tori if F is p-adic, and a disjoint union of Euclidean spaces if F is Archimedean.

We can now define $\mathcal{I}(G(F))$ to be the set of complex valued functions ϕ on $T(G)$ which satisfy the symmetry condition

$$\phi(z\tau) = \chi_\tau(z)^{-1}\phi(\tau), \qquad z \in Z_\tau, \ \tau \in T(G),$$

and which lie in the appropriate space of W_0^G-invariant functions on $\widetilde{T}(G)$. That is, ϕ must be in $C_c^\infty(\widetilde{T}(G))$ if F is p-adic, and in $\mathcal{S}(\widetilde{T}(G))$ if F is Archimedean. In the Archimedean case, we can assume that $F = \mathbf{R}$. Then any representation in $\Pi_2(M(\mathbf{R}))$ can be written uniquely in the form σ_λ, where λ lies in $i\mathfrak{a}^*_M$ and $\sigma \in \Pi_2(M(\mathbf{R}))$ is invariant under the split component $A_M(\mathbf{R})^0$ of the center of $M(\mathbf{R})$. In this case we write μ_{σ_λ} for the linear form that determines the infinitesimal character of σ_λ. Thus, μ_{σ_λ} is a Weyl orbit of elements in the dual of a complex Cartan subalgebra, which we assume is equipped with suitable Hermitian norm $\|\cdot\|$, such that

$$\|\mu_{\sigma_\lambda}\| = \|\mu_\sigma + \lambda\| = \|\mu_\sigma\| + \|\lambda\|.$$

By definition, $\mathcal{S}(\widetilde{T}(G))$ is the space of smooth functions ϕ on $\widetilde{T}(G)$ such that for each $L \in \mathcal{L}$, each integer n, and each invariant differential operator $D = D_\lambda$ on $i\mathfrak{a}^*_L$, transferred in the obvious way

$$D_\tau\phi(\tau) = \lim_{\lambda \to 0} D_\lambda \phi(\tau_\lambda), \qquad \tau \in \widetilde{T}_{\mathrm{ell}}(L),$$

to $\widetilde{T}_{\mathrm{ell}}(L)$, the semi-norm

$$\|\phi\|_{L,D,n} = \sup_{\tau \in \widetilde{T}_{\mathrm{ell}}(L)} \left(|D_\tau \sigma(\tau)| (1 + \|\mu_\tau\|)^n \right)$$

is finite. (We are writing $\mu_\tau = \mu_\sigma$ for $\tau = (M, \sigma, r)$.) In both the real and p-adic cases, $\mathcal{I}(G(F))$ has a natural topology.

THEOREM. *The map*

$$\mathcal{T}_G:\ f \longrightarrow f_G\ ,$$

defined by (3), is an open, continuous and surjective linear transformation from $\mathcal{C}(G(F))$ *onto* $\mathcal{I}(G(F))$.

PROOF. As we have already noted, we shall use the characterization [2], [9, Part B] of the operator valued Fourier transform

$$\mathcal{F}_G:\ \mathcal{C}(G(F)) \longrightarrow \widehat{\mathcal{C}}(G(F))$$

on the Schwartz space. (The discussion of the map $\mathcal{T}_G$ for p-adic groups in [9, Part C, §VI] is incomplete.) The space $\widehat{\mathcal{C}}(G(F))$ consists of smooth operator valued functions

$$\Phi:\ (P,\sigma) \longrightarrow \Phi_P(\sigma)\ \in\ \mathrm{End}(\mathcal{H}_P(\sigma))\,, \qquad P\in\mathcal{P}(M)\,,\ \sigma\in\Pi_2(M(F))\,,\ M\in\mathcal{L}\,,$$

which satisfy a symmetry condition and a growth condition. To describe the symmetry condition, we suppose that $M'\in\mathcal{L}$, $P'\in\mathcal{P}(M')$, and that w belongs to the set $W(\mathfrak{a}_M,\mathfrak{a}_{M'})$ of isomorphisms from $\mathfrak{a}_M$ onto $\mathfrak{a}_{M'}$ obtained by restricting elements in W_0^G to $\mathfrak{a}_M$. Set

$$R_{P'|P}(\widetilde{w},\sigma)\ =\ A(\widetilde{w})R_{w^{-1}P'|P}(\sigma),$$

where $R_{w^{-1}P'|P}(\sigma)$ is the normalized intertwining operator from $\mathcal{H}_P(\sigma)$ to $\mathcal{H}_{w^{-1}P'}(\sigma)$, $\widetilde{w}$ is a representative of w in a fixed maximal compact subgroup K, and $A(\widetilde{w})$ is the canonical map from $\mathcal{H}_{w^{-1}P'}(\sigma)$ to $\mathcal{H}_{P'}(\widetilde{w}\sigma)$. Then Φ must satisfy

$$\Phi_{P'}(\widetilde{w}\sigma)\ =\ R_{P'|P}(\widetilde{w},\sigma)\Phi_P(\sigma)R_{P'|P}(\widetilde{w},\sigma)^{-1} \tag{4}$$

for all such (M',P',w). As for the growth condition, observe that the domain of Φ can be identified with a disjoint union of compact tori if F is p-adic. In this case we require simply that Φ have compact support. If F is Archimedean, we require that the semi-norms

$$\sup_{P,\sigma,\delta_1,\delta_2}\ \|D_\sigma(\Gamma_{\delta_2}\Phi_P(\sigma)\Gamma_{\delta_1})\|(1+\|\mu_\sigma\|)^n(1+\|\mu_{\delta_1}\|)^{m_1}(1+\|\mu_{\delta_2}\|)^{m_2}\ ,$$

determined by integers n, m_1, m_2, and differential operators D_σ, be finite. The elements δ_1 and δ_2 here range over irreducible K-types, and Γ_δ stands for the K-invariant projection of $\mathcal{H}_P(\sigma)$ onto the δ isotypical subspace $\mathcal{H}_P(\sigma)_\delta$. As above, D_σ is assumed to come from an invariant differential operator on $i\mathfrak{a}_M^*$. In each case, $\widehat{\mathcal{C}}(G(F))$ becomes a topological vector space, and the Fourier transform

$$\mathcal{F}_G:\ f \longrightarrow (\mathcal{F}_Gf)_P(\sigma)\ =\ \mathcal{I}_P(\sigma,f)$$

is a topological isomorphism from $\mathcal{C}(G(F))$ onto $\widehat{\mathcal{C}}(G(F))$.

We define a trace map

$$\widehat{\mathcal{T}}_G:\ \widehat{\mathcal{C}}(G(F)) \longrightarrow \mathcal{I}(G(F))$$

by setting

$$(\widehat{\mathcal{T}}_G\Phi)(\tau) = \mathrm{tr}\big(\widetilde{R}(r,\sigma)\Phi_P(\sigma)\big)$$

for any triplet $\tau = (M, \sigma, r)$ in $T(G)$. Our original map $\mathcal{T}_G$ is then the composition of $\mathcal{F}_G$ with $\widehat{\mathcal{T}}_G$. We observe directly from this construction that $\mathcal{T}_G$ maps $\mathcal{C}\big(G(F)\big)$ continuously into $\mathcal{I}\big(G(F)\big)$. Moreover, to prove the remaining assertions that $\mathcal{T}_G$ is open and surjective, it suffices to construct a continuous section

$$h_G : \mathcal{I}\big(G(F)\big) \longrightarrow \widehat{\mathcal{C}}\big(G(F)\big)$$

for $\widehat{\mathcal{T}}_G$.

Suppose first that $F = \mathbb{R}$. Then we shall write the representations in $\Pi_2\big(M(\mathbb{R})\big)$ in the form

$$\sigma_\lambda, \qquad\qquad \sigma \in \Pi_2\big(M(\mathbb{R})/A_M(\mathbb{R})^0\big),\ \lambda \in i\mathfrak{a}_M^*,$$

as above. In this case, R_σ is a product of groups $\mathbb{Z}/2\mathbb{Z}$ [8]. Moreover, the cocycle which defines $\widetilde{R}_\sigma$ splits [8, Theorem 7.1], so we may take $\widetilde{R}_\sigma = R_\sigma$. For any λ, R_{σ_λ} is the subgroup of elements in R_σ which fix λ.

We shall use Vogan's theory of minimal K-types [10], [11], [5, §2.3]. Given $\sigma \in \Pi_2\big(M(\mathbb{R})/A_M(\mathbb{R})^0\big)$, let $A(\sigma)$ denote the set of minimal K-types for the representation $\mathcal{I}_P(\sigma)$. If r belongs to R_σ, we shall write

$$R(r,\sigma)_{\min} = \sum_{\delta\in A(\sigma)} R(r,\sigma)_\delta,$$

where $R(r,\sigma)_\delta$ denotes the restriction of $R(r,\sigma)$ to the δ-isotypical subspace $\mathcal{H}_P(\sigma)_\delta$. Then there is a bijection $\rho \to \delta_\rho$ from the set $\Pi(R_\sigma)$ of (abelian) characters of R_σ onto $A(\sigma)$ with the property that

$$\mathrm{tr}\big(R(r,\sigma)_{\min}\mathcal{I}_P(\sigma,k)\big) = \sum_{\rho\in\Pi(R_\sigma)} \rho^\vee(r)\mathrm{tr}\big(\delta_\rho(k)\big)$$

for each $r \in R_\sigma$ and $k \in K$. This follows from the fact that each δ occurs in $\mathcal{I}_P(\sigma)$ with multiplicity 1, and that moreover any irreducible constituent of $\mathcal{I}_P(\sigma)$ contains exactly one element in $A(\sigma)$. In particular, if $\delta = \delta_\rho$, the operator $R(r,\sigma)_\delta$ is simply equal to the scalar $\rho^\vee(r)$ on $\mathcal{H}_P(\sigma)_\delta$. This suggests that we define operators

$$S_P(r,\sigma) = \sum_{\delta\in A(\sigma)} \deg(\delta)^{-1}R(r,\sigma)_\delta, \qquad\qquad r \in R_\sigma,$$

on

$$\mathcal{H}_P(\sigma)_{\min} = \bigoplus_{\delta\in A(\sigma)} \mathcal{H}_P(\sigma)_\delta.$$

Then

$$\mathrm{tr}\big(R(r_1,\sigma)S_P(r^{-1},\sigma)\big) = \prod_{\rho\in\Pi(R_\sigma)} \rho^\vee(r_1r^{-1}) = \begin{cases} |R_\sigma|, & \text{if } r_1 = r, \\ 0, & \text{otherwise,}\end{cases}$$

for any elements $r_1, r \in R_\sigma$. Suppose that w belongs to $W(\mathfrak{a}_M, \mathfrak{a}_{M'})$, and that λ lies in $i\mathfrak{a}_M^*$. Then the linear transformation $R_{P'|P}(\widetilde{w}, \sigma_\lambda)$ in (4) intertwines the action of K on the various spaces $\mathcal{H}_P(\sigma)_\delta$ and $\mathcal{H}_{P'}(\widetilde{w}\sigma)_\delta$. Consequently, the operator

$$R_{P'|P}(\widetilde{w}, \sigma)^{-1} R_{P'|P}(\widetilde{w}, \sigma_\lambda) \tag{5}$$

acts as a scalar on each of the spaces $\mathcal{H}_P(\sigma)_\delta$, and therefore commutes with $S_P(r, \sigma)$. It follows easily that

$$\begin{aligned} &R_{P'|P}(\widetilde{w}, \sigma_\lambda) S_P(r, \sigma) R_{P'|P}(\widetilde{w}, \sigma_\lambda)^{-1} \\ &= R_{P'|P}(\widetilde{w}, \sigma) S_P(r, \sigma) R_{P'|P}(\widetilde{w}, \sigma)^{-1} \\ &= S_{P'}(wrw^{-1}, \widetilde{w}\sigma) , \end{aligned}$$

for any $r \in R_\sigma$.

In order to construct the section h_G, we choose a function $\beta_M^L \in C_c^\infty(i\mathfrak{a}_M^*/i\mathfrak{a}_L^*)$ for each pair of Levi subgroups $M \subset L$ in $\mathcal{L}$, such that $\beta_M^L(0) = 1$, and such that

$$\beta_{wM}^{wL}(w\lambda) = \beta_M^L(\lambda), \qquad \lambda \in i\mathfrak{a}_M^* ,$$

for any $w \in W_0^G$. Suppose that ϕ belongs to $\mathcal{I}(G(F))$. The domain of ϕ can be represented as the set of W_0^G-orbits of triplets

$$\{\tau = (M, \sigma_{\lambda_L}, r) : \ M \subset L, \ \sigma \in \Pi_2(M(\mathbf{R})/A_M(\mathbf{R})^0), \ \lambda_L \in i\mathfrak{a}_L^*, \ r \in R_{\sigma,\mathrm{reg}}^L\} .$$

We define $h_G(\phi)$ to be the operator valued function

$$\Phi_P(\sigma_\lambda) = |R_\sigma|^{-1} \sum_{L \in \mathcal{L}(M)} \sum_{r \in R_{\sigma,\mathrm{reg}}^L} \beta_M^L(\lambda) \phi(M, \sigma_{\lambda_L}, r) S_P(r^{-1}, \sigma) ,$$

where λ_L denotes the projection onto $i\mathfrak{a}_L^*$ of the variable $\lambda \in i\mathfrak{a}_M^*$. We shall show that this function lies in $\widehat{\mathcal{C}}(G(\mathbf{R}))$, and that its image under $\widehat{\mathcal{T}}_G$ is ϕ.

Take any $w \in W(\mathfrak{a}_M, \mathfrak{a}_{M'})$ as in the symmetry condition (4). Since β_M^L, ϕ and S_P each satisfy their own symmetry conditions, we see that

$$\begin{aligned} &R_{P'|P}(\widetilde{w}, \sigma_\lambda) \Phi_P(\sigma_\lambda) R_{P'|P}(\widetilde{w}, \sigma_\lambda)^{-1} \\ &= |R_\sigma|^{-1} \sum_{L \in \mathcal{L}(M)} \sum_{r \in R_{\sigma,\mathrm{reg}}^L} \beta_{wM}^{wL}(w\lambda) \phi\big(wM, w(\sigma_{\lambda_L}), wrw^{-1}\big) S_{P'}(wr^{-1}w^{-1}, \widetilde{w}\sigma) \\ &= \Phi_{P'}(\widetilde{w}\sigma_\lambda) . \end{aligned}$$

Therefore the symmetry condition (4) holds. To establish the required growth condition, we use the fact that the infinitesimal character of any of the K-types in $A(\sigma)$ may be bounded linearly in terms of the infinitesimal character of σ; there is a constant c such that

$$\|\mu_\delta\| \le c(1 + \|\mu_\sigma\|)$$

for every $\sigma \in \Pi_2(M(\mathbf{R})/A_M(\mathbf{R})^0)$ and $\delta \in A(\sigma)$. The growth condition for ϕ as an element of $\mathcal{I}(G(F))$ then implies the growth condition of $\widehat{\mathcal{C}}(G(F))$ for $\Phi_P(\sigma_\lambda)$. It follows that the function

$$\Phi : \ (P, \sigma_\lambda) \to \Phi_P(\sigma_\lambda)$$

belongs to $\widehat{\mathcal{C}}(G(F))$. In fact the estimates imply that $h_G: \phi \to \Phi$ is a continuous linear map from $\mathcal{I}(G(F))$ into $\widehat{\mathcal{C}}(G(F))$.

Finally, to evaluate $\widehat{\mathcal{T}}_G\Phi$, choose any triplet

$$\tau = (M, \sigma_\lambda, r_1), \qquad r_1 \in R_\sigma, \ r_1\lambda = \lambda,$$

in $T(G)$. Then

$$\begin{aligned}(\widehat{\mathcal{T}}_G\Phi)(\tau) &= \operatorname{tr}\bigl(R(r_1, \sigma_\lambda)\Phi_P(\sigma_\lambda)\bigr) \\ &= |R_\sigma|^{-1} \sum_{L\in\mathcal{L}(M)} \sum_{r\in R^L_{\sigma,\mathrm{reg}}} \beta^L_M(\lambda)\phi(M, \sigma_{\lambda_L}, r)\operatorname{tr}\bigl(R(r_1, \sigma_\lambda)S_P(r^{-1}, \sigma)\bigr) .\end{aligned}$$

The operator $R(r_1, \sigma_\lambda) = R_{P|P}(r_1, \sigma_\lambda)$ here depends implicitly on the group $P \in \mathcal{P}(M)$. However, the trace inside the sum does not, so we are free to choose P so that $R(r_1, \sigma_\lambda)$ equals $R(r_1, \sigma)$. Then as we have seen above, a summand will vanish unless r equals r_1 , in which case it equals

$$\beta^L_M(\lambda)\phi(M, \sigma_{\lambda_L}, r_1)|R_\sigma| .$$

If $L_1 \in \mathcal{L}(M)$ is the group for which r_1 lies in $R^{L_1}_{\sigma,\mathrm{reg}}$, then λ lies in $i\mathfrak{a}^*_{L_1}$. Therefore $\lambda_{L_1} = \lambda$ and $\beta^{L_1}_M(\lambda) = \beta^{L_1}_M(0) = 1$. We obtain

$$(\widehat{\mathcal{T}}_G\Phi)(\tau) = \phi(M, \sigma_\lambda, r_1) = \phi(\tau) .$$

We have verified that $h_G: \phi \to \Phi$ is the required section for $\widehat{\mathcal{T}}_G$, thereby establishing the theorem in the case that $F = \mathbb{R}$.

Now suppose that F is a p-adic field. One could use Schwartz-multipliers to establish the theorem, in the spirit of the corresponding result [3] for Hecke algebras. We shall instead follow an argument which is closer to the discussion above. An element $\phi \in \mathcal{I}(G(F))$ is supported on finitely many connected components in $T(G)$. Using an W^G_0-invariant partition of unity, we can assume that ϕ is supported on a small neighbourhood of some fixed point in $T(G)$. More precisely, we assume that $\phi(\tau)$ vanishes unless τ is of the form (M, σ_λ, r) , where (M, σ) belongs to a fixed orbit of W^G_0 and λ lies in a small neighbourhood $\mathcal{N}_M$ of 0 in $i\mathfrak{a}^*_M$. One reason for localizing around σ is to ensure that for any $\lambda \in \mathcal{N}_M$, R_{σ_λ} is the subgroup of elements in R_σ which fix λ . We shall impose a second condition on the size of $\mathcal{N}_M$ presently.

Choose an open compact subgroup K_0 of $G(F)$, and let $\widetilde{R}(r, \sigma)_{K_0}$ denote the restriction of $\widetilde{R}(r, \sigma)$ to the subspace $\mathcal{H}_P(\sigma)_{K_0}$ of K_0-fixed vectors in $\mathcal{H}_P(\sigma)$. The representation

$$r \longrightarrow \widetilde{R}(r, \sigma)_{K_0} , \qquad r \in \widetilde{R}_\sigma ,$$

of $\widetilde{R}_\sigma$ on $\mathcal{H}_P(\sigma)_{K_0}$ is equivalent to a direct sum

$$\bigoplus_{\rho\in\Pi(\widetilde{R}_\sigma, \chi_\sigma)} \dim(\pi_{\rho,K_0})\rho^\vee ,$$

where π_{ρ,K_0} denotes the $\mathcal{C}(G(F)/\!\!/K_0)$-module of K_0-fixed vectors in the representation π_ρ. We take K_0 to be so small that π_{ρ,K_0} is nonzero for each ρ. Writing $\widetilde{R}(r,\sigma)_{\rho,K_0}$ for the restriction of $\widetilde{R}(r,\sigma)$ to the subspace of $\mathcal{H}_P(\sigma)_{K_0}$ corresponding to ρ, we define operators

$$S_P(r,\sigma) = \sum_{\rho\in\Pi(\tilde{R}_\sigma,\chi_\sigma)} \deg(\rho)\dim(\pi_{\rho,K_0})^{-1}\widetilde{R}(r,\sigma)_{\rho,K_0}, \qquad r\in\widetilde{R}_\sigma,$$

on $\mathcal{H}_P(\sigma)_{K_0}$. For any pair of elements $r_1, r\in\widetilde{R}_\sigma$, we have

$$\begin{aligned}&\operatorname{tr}\big(\widetilde{R}(r_1,\sigma)S_P(r^{-1},\sigma)\big)\\ &= \sum_{\rho\in\Pi(\tilde{R}_\sigma,\chi_\sigma)} \deg(\rho)\operatorname{tr}\big(\rho^\vee(r_1r^{-1})\big) = \begin{cases}|R_\sigma|\chi_\sigma(z), & \text{if } r=r_1z,\ z\in Z_\sigma,\\ 0, & \text{if } r\notin r_1Z_\sigma.\end{cases}\end{aligned}$$

Unlike in the Archimedean case, however, the operator (5) does not generally commute with $S_P(r,\sigma)$. To deal with this complication, we define a function

$$Q_P(\sigma,\lambda) = \sum_{M_1\in\mathcal{L}}\ \sum_{w_1\in W(\mathfrak{a}_M,\mathfrak{a}_{M_1})}\ \sum_{P_1\in\mathcal{P}(M_1)} R_{P_1|P}(\widetilde{w}_1,\sigma_\lambda)^{-1}R_{P_1|P}(\widetilde{w}_1,\sigma),$$

of $\lambda\in i\mathfrak{a}_M^*$. If w and P' are as in (4), we have

$$\begin{aligned}&R_{P'|P}(\widetilde{w},\sigma_\lambda)Q_P(\sigma,\lambda)R_{P'|P}(\widetilde{w},\sigma)^{-1}\\ &= \sum_{M_1,w_1,P_1} R_{P'|P}(\widetilde{w},\sigma_\lambda)R_{P_1|P}(\widetilde{w}_1,\sigma_\lambda)^{-1}R_{P_1|P}(\widetilde{w}_1,\sigma)R_{P'|P}(\widetilde{w},\sigma)^{-1}\\ &= \sum_{M_1,w_1,P_1} R_{P_1|P'}(\widetilde{w}_1\widetilde{w}^{-1},\widetilde{w}\sigma_\lambda)^{-1}R_{P_1|P'}(\widetilde{w}_1\widetilde{w}^{-1},\widetilde{w}\sigma),\end{aligned}$$

by the multiplicative properties of the intertwining operators. Changing variables in the sum over w_1, we see that

$$R_{P'|P}(\widetilde{w},\sigma_\lambda)Q_P(\sigma,\lambda) = Q_{P'}(\widetilde{w}\sigma,w\lambda)R_{P'|P}(\widetilde{w},\sigma).$$

Observe that if $\lambda=0$, $Q_P(\sigma,\lambda)$ is a positive multiple of the identity operator. We assume that the neighbourhood $\mathcal{N}_M$ is so small that the restriction of $Q_P(\sigma,\lambda)$ to $\mathcal{H}_P(\sigma)_{K_0}$ is invertible for every $\lambda\in\mathcal{N}_M$. We can then define

$$S_P(r,\sigma,\lambda) = Q_P(\sigma,\lambda)S_P(r,\sigma)Q_P(\sigma,\lambda)^{-1}, \qquad r\in\widetilde{R}_\sigma.$$

It follows easily that

$$\begin{aligned}&R_{P'|P}(\widetilde{w},\sigma_\lambda)S_P(r,\sigma,\lambda)R_{P'|P}(\widetilde{w},\sigma_\lambda)^{-1}\\ &= Q_{P'}(\widetilde{w}\sigma,w\lambda)R_{P'|P}(\widetilde{w},\sigma)S_P(r,\sigma)R_{P'|P}(\widetilde{w},\sigma)^{-1}Q_{P'}(\widetilde{w}\sigma,w\lambda)^{-1}\\ &= Q_{P'}(\widetilde{w}\sigma,w\lambda)S_{P'}(wrw^{-1},\widetilde{w}\sigma)Q_{P'}(\widetilde{w}\sigma,w\lambda)^{-1}\\ &= S_{P'}(wrw^{-1},\widetilde{w}\sigma,w\lambda).\end{aligned}$$

Notice also that if $r_1\lambda = \lambda$ for an element $r_1 \in \widetilde{R}_\sigma$, and if P is chosen so that $\widetilde{R}(r_1,\sigma_\lambda)$ equals $\widetilde{R}(r_1,\sigma)$, the operators $\widetilde{R}(r_1,\sigma)$ and $Q_P(\sigma,\lambda)$ commute. Therefore

$$\operatorname{tr}\big(\widetilde{R}(r_1,\sigma)S_P(r^{-1},\sigma,\lambda)\big) = \begin{cases} |R_\sigma|\chi_\sigma(z), & \text{if } r = r_1 z,\ z \in Z_\sigma, \\ 0, & \text{if } r \notin r_1 Z_\sigma. \end{cases}$$

This modification allows us to construct the section h_G as we did in the case $F = \mathbb{R}$. We choose the functions $\beta_M^L \in C_c^\infty(i\mathfrak{a}_M^*/i\mathfrak{a}_L^*)$ as above, with the further stipulation that they each be supported on a small neighbourhood of 0. Given ϕ, we define $h_G(\phi)$ to be the operator valued function

$$\Phi_P(\sigma_\lambda) = |R_\sigma|^{-1} \sum_{L\in\mathcal{L}(M)} \sum_{r\in\widetilde{R}^L_{\sigma,\mathrm{reg}}} \beta_M^L(\lambda)\phi(M,\sigma_{\lambda_L},r)S_P(r^{-1},\sigma,\lambda)\ .$$

Our support conditions on β_M^L and ϕ imply that the right hand side vanishes unless λ belongs to $\mathcal{N}_M$, and therefore that $S_P(r^{-1},\sigma,\lambda)$ is well defined. The symmetry condition (4) follows from the remarks above, as in the Archimedean case. The required growth condition is trivial. Consequently, the function

$$\Phi : (P,\sigma_\lambda) \to \Phi_P(\sigma_\lambda)$$

belongs to $\widehat{\mathcal{C}}(G(F))$, and $\phi \to h_G(\phi) = \Phi$ is a continuous linear map from $\mathcal{I}(G(F))$ into $\widehat{\mathcal{C}}(G(F))$. Finally, suppose that τ is any element in $T(G)$. Then $(\widehat{\mathcal{T}}_G\Phi)(\tau)$ vanishes unless τ is of the form

$$(M,\sigma_\lambda,r_1), \qquad \lambda \in \mathcal{N}_M\ ,\ r_1 \in \widetilde{R}_\sigma\ ,\ r_1\lambda = \lambda\ ,$$

in which case we deduce that

$$(\widehat{\mathcal{T}}_G\Phi)(\tau) = \operatorname{tr}\big(\widetilde{R}(r_1,\sigma_\lambda)\Phi_P(\sigma_\lambda)\big) = \phi(\tau)\ ,$$

again as in the Archimedean case. Therefore, the map $h_G : \phi \to \Phi$ is the required section for $\widehat{\mathcal{T}}_G$. We have established the theorem for arbitrary F. □

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