STABILITY AND ENDOSCOPY: INFORMAL MOTIVATION

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The purpose of this note is described in the title. It is an elementary introduction to some of the basic ideas of stability and endoscopy. We shall not discuss the techniques of the theory, which among other things entail a sophisticated use of Galois cohomology. Our aim is rather to persuade a reader that the theory was created in response to some very natural problems in harmonic analysis. The article is intended for people who are starting (or even just thinking of starting) to learn the subject.

Langlands was actually lead to the theory of endoscopy by questions in algebraic geometry, particularly Shimura varieties [17, §1]. However, he quickly realized that the questions had remarkable implications for harmonic analysis. It is in this context that we will discuss the basic ideas.

We begin with a simple form of the trace formula. Suppose that G is a reductive algebraic group defined over a number field F. The adèles $\mathbb{A}$ of F are a locally compact ring in which F embeds as a discrete subring, and the group of F-rational points $G(F)$ embeds as a discrete subgroup of the locally compact group $G(\mathbb{A})$ of adèlic points. We shall be concerned with the case that G is anisotropic, or equivalently, that the quotient space $G(F)\backslash G(\mathbb{A})$ is compact. It is then known that the regular representation

$$\big(R(y)\phi\big)(x) = \phi(xy), \qquad \phi \in L^2\big(G(F)\backslash G(\mathbb{A})\big),\ x, y \in G(\mathbb{A}),$$

of $G(\mathbb{A})$ on the Hilbert space $L^2\big(G(F)\backslash G(\mathbb{A})\big)$ (with the right $G(\mathbb{A})$-invariant measure on $G(F)\backslash G(\mathbb{A})$) decomposes discretely. More precisely, we can write

$$R = \bigoplus_{\pi} m_\pi \pi(f),$$

a direct sum over π in the set $\Pi\big(G(\mathbb{A})\big)$ of irreducible representations of $G\big((\mathbb{A})\big)$, with finite multiplicities $m_\pi \in \mathbb{N} \cup \{0\}$. (If G is not anisotropic, there is a subrepresentation R_{disc} of R which decomposes in this way, at least modulo the split component of the center of G.)

Selberg's original formula gives the trace of the convolution operator

$$R(f) = \bigoplus_{\pi} m_\pi \pi(f)$$

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obtained by integrating R against a test function f in $C_c^\infty\big(G(\mathbb{A})\big)$. (See [21], [2], [8].) On the one hand, the trace of $R(f)$ is a discrete sum

$$I_{\text{disc}}(f) = \sum_{\pi \in \Pi(G(\mathbb{A}))} m_\pi \text{tr}\big(\pi(f)\big) \tag{1}$$

of irreducible characters. The trace formula asserts that the trace of $R(f)$ can also be written as a linear combination

$$I_{\text{ell}}(f) = \sum_{\gamma \in \Gamma(G(F))} a^G(\gamma) I_G(\gamma, f) \tag{2}$$

of orbital integrals

$$I_G(\gamma, f) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)dx$$

of f. Here, $\Gamma\big(G(F)\big)$ stands for the set of conjugacy classes in $G(F)$, $G_\gamma(\cdot)$ denotes the centralizer of γ in $G(\cdot)$, and the coefficients are given by

$$a^G(\gamma) = \text{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\big).$$

The trace formula for compact quotient is thus the identity of the two expansions $I_{\text{ell}}(f)$ and $I_{\text{disc}}(f)$. (The general trace formula ([1], [2]) is considerably more complicated. If G is not anisotropic, $I_{\text{ell}}(f)$ and $I_{\text{disc}}(f)$ are merely the simplest of a number of such expansions on each side, parametrized by conjugacy classes of Levi subgroups of G.)

We should recall that $C_c^\infty\big(G(\mathbb{A})\big)$ is the vector space spanned by complex-valued functions

$$f = f_\infty \cdot f_{\text{fin}} = \prod_{v \in \infty} f_v \cdot \prod_{v \text{ finite}} f_v,$$

in which the Archimedean component f_∞ lies in the usual space of smooth functions of compact support. The nonArchimedean component f_{fin} is required to be a locally constant function of compact support on the group $G(\mathbb{A}_{\text{fin}})$ of finite adèlic points. This second condition implies in particular that for almost all v, f_v is the characteristic function of a hyperspecial maximal compact subgroup of $G(F_v)$ [26, §1.10, §3.1, §3.10].

If f equals $\prod_v f_v$, the global orbital integral is automatically a product

$$I_G(\gamma, f) = \prod_v I_G(\gamma, f_v) \tag{3}$$

of local orbital integrals

$$I_G(\gamma, f_v) = |D^G(\gamma)|_v^{\frac{1}{2}} \int_{G_\gamma(F_v)\backslash G(F_v)} f_v(x_v^{-1}\gamma x_v)dx_v.$$

(The Weyl discriminant

$$D^G(\gamma) = \det\big(1 - \text{Ad}(\gamma)\big)_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

is inserted only for general convenience. It does not appear globally because of the product formula on F^*.) On the spectral side, any irreducible representation is a restricted tensor product

$$\pi = \bigotimes_v \pi_v, \qquad \pi_v \in \Pi\big(G(F_v)\big),$$

of irreducible representations of the local groups [3], and

$$\mathrm{tr}\big(\pi(f)\big) = \prod_v \mathrm{tr}\big(\pi_v(f_v)\big).$$

Automorphic representations are interesting because the components π_v are believed to carry fundamental arithmetic information. The data which parametrize the local sets $\Pi\big(G(F_v)\big)$ are very interesting in themselves, but what is especially important is the global information that implicitly relates the local data for the different components of any π with $m(\pi)$ positive. One hopes to study such information through the trace formula.

A major goal is to prove precise reciprocity laws relating $m(\pi)$ and $m(\pi')$, for representations π and π' of different groups G and G'. The most general pairs (G, G') for which such reciprocity laws should exist are given by Langlands' functoriality conjecture [14], [19]. The general functoriality conjecture is extremely deep, and will undoubtedly need more than just the trace formula for its ultimate resolution. However, there are a significant number of cases for which the trace formula seems ideally suited. It is for these cases that the theory of endoscopy has been designed.

Any discussion of these matters has to begin with the basic case solved by Jacquet and Langlands in 1968 [6, §17]. (See also [4], [8].) In this case, G is the multiplicative group of a quaternion algebra over F (which is actually only anisotropic modulo the center), and G' equals $GL(2)$. The idea is quite simple. The characteristic polynomial for G' and its analogue for G determine a canonical bijection from $\Gamma\big(G(F)\big)$ to a subset of $\Gamma\big(G'(F)\big)$. Indeed, the center of $G'(F)$ is bijective with F^*, while the conjugacy classes of noncentral elements in $G'(F)$ lie in disjoint subsets parametrized naturally by certain quadratic extensions of F. The characteristic polynomial gives an identical parametrization for a subset of the conjugacy classes in $G'(F) = GL(2, F)$. Thus, there is a canonical injection from the set of terms on the geometric side of the trace formula for G to a subset of the terms for G'. Jacquet and Langlands define a correspondence

$$f = \prod_v f_v \;\longrightarrow\; f' = \prod_v f'_v$$

from $C_c^\infty\big(G(\mathbb{A})\big)$ to $C_c^\infty\big(G'(\mathbb{A})\big)$ such that

$$I_G(\gamma, f) = I_{G'}(\gamma', f')$$

if γ' is the image of γ, and such that $I_{G'}(\gamma', f') = 0$ if γ' is not the image of any γ. It is known that

$$a^G(\gamma) = \mathrm{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\big) = \mathrm{vol}\big(G'_{\gamma'}(F)\backslash G'(\mathbb{A})\big) = a^{G'}(\gamma'),$$

and also that the supplementary parabolic terms in the trace formula of G' vanish for the function f'. The geometric sides of the two trace formulas are therefore equal.

Once the two geometric sides have been cancelled, one can easily imagine being able to exploit the resulting equality of spectral sides. The correspondence of functions $f_v \to f'_v$ is defined locally. Moreover, at all places outside a finite set S, $G(F_v)$ is isomorphic with $GL(2, F_v) = G'(F_v)$. At these places, f'_v can simply be taken to be f_v. One can then fix the function $f_S = \prod_{v \in S} f_v$, and regard the difference of the two spectral sides as a linear form in the function $f^S = \prod_{v \notin S} f_v$. In particular, if $\pi = \pi_S \pi^S \in \Pi\big(G(\mathbb{A})\big)$ is a representation with $m(\pi) \neq 0$, there will have to be a term for $G' = GL(2)$ to match the functional

$$f^S \longrightarrow m(\pi)\mathrm{tr}\big(\pi_S(f_S)\big)\,\mathrm{tr}\big(\pi^S(f^S)\big).$$

Combining this argument with the theorem of strong multiplicity one for $GL(2)$ (a general form of which is [7, Theorem 4.4]), one obtains a correspondence $\pi \to \pi'$ such that $\pi_v = \pi'_v$ for every $v \notin S$, and such that $m(\pi)$ equals $m(\pi')$.

The indirectness of the basic argument is part of its charm. The multiplicities $m(\pi)$ and $m(\pi')$ on the two groups are defined quite abstractly, in terms of the traces of two operators. They cannot be compared directly. The trace formulas convert information wrapped up in the multiplicities into concrete linear combinations of orbital integrals. However, these geometric terms become too complicated as f varies (with increasing support, for example) to be of great use for any isolated group. What really drives the argument is local harmonic analysis. It establishes that the geometric terms for G and G', complicated though each may be in isolation, match each other and cancel.

Langlands realized about 1970 that there would be a serious obstruction to extending the argument to other groups. The characteristic polynomial is behind the transfer of conjugacy classes from G to G', and the coefficients of the characteristic polynomial do have analogues for general G. For example, one can take any set of generators for the algebra of G-invariant polynomials on G. These objects can certainly be used to transfer conjugacy classes in G to classes in suitably related groups G'. However, invariant polynomials measure only *geometric* conjugacy classes, that is, conjugacy classes in a group of points over an algebraically closed field. For most G other than a general linear group, there exist nonconjugate elements in $G(F)$ which are conjugate over an algebraic closure $G(\overline{F})$. A similar phenomenon holds for the local groups $G(F_v)$. For example, in the case of $G = SL(2)$ and $F_v = \mathbb{R}$, the elements $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ are conjugate over $G(\mathbb{C})$, but not over $G(\mathbb{R})$. This phenomenon clearly complicates the problem of transferring conjugacy classes.

We are really thinking only of semisimple conjugacy classes here, since we do not want to deal with subtleties of geometric invariant theory. In fact, to focus on the essential problem, it is best to consider only elements γ that are *strongly regular*, which is to say that G_γ is a torus. The strongly regular elements form an open dense subset in any of the local groups $G(F_v)$. Langlands called two strongly regular elements in $G(F_v)$ *stably conjugate* if they were conjugate over $G(\overline{F}_v)$. Stable conjugacy is then an equivalence relation that is weaker than conjugacy. Any stable conjugacy class is a finite union of ordinary conjugacy classes. Now, we are accustomed to thinking of conjugacy classes as being dual to irreducible characters. In the present context, one can argue plausibly that the strongly regular conjugacy classes in $G(F_v)$ are the dual analogues of irreducible *tempered* characters on $G(F_v)$. The relation of stable conjugacy ought then to determine a parallel relation on the set of tempered characters. Langlands quickly realized that in the case $F_v = \mathbb{R}$, there was already a good candidate for such a relation in the work of Harish-Chandra.

One of Harish-Chandra's great achievements was the classification of the discrete series for real groups [5], [22]. Discrete series are of course the basic building blocks of arbitrary tempered representations. We recall that Harish-Chandra's classification consists of a parametrization and character formula that are remarkably similar to those established by Weyl in the special case of compact groups. However, there were two new aspects to Harish-Chandra's generalization. First of all, $G(\mathbb{R})$ can have several conjugacy classes of maximal tori; the basic character formula applies only to a maximal torus $T(\mathbb{R})$ that is compact. Secondly, the real Weyl group $W_{\mathbb{R}}(G,T)$ induced by elements of $G(\mathbb{R})$ is generally smaller than the complex Weyl group $W_{\mathbb{C}}(G,T)$ induced by elements in $G(\mathbb{C})$. For example, if $G = \mathrm{Sp}(2n)$, then $W_{\mathbb{C}}(G,T)$ is isomorphic to a semi-direct product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, while $W_{\mathbb{R}}(G,T)$ corresponds to the subgroup S_n. The discrete series are parametrized by $W_{\mathbb{R}}(G,T)$-orbits of regular characters on $T(\mathbb{R})$, and not the $W_{\mathbb{C}}(G,T)$-orbits that determine the finite dimensional representations of Weyl. In particular, the discrete series occur naturally in finite packets, each of which is bijective with the set $W_{\mathbb{R}}(G,T)\backslash W_{\mathbb{C}}(G,T)$ of cosets. Thinking of the L-functions he had defined earlier [14], Langlands called the relationship defined by this packet structure *L-equivalence*, and he used it as the foundation for a classification of all the irreducible representations of $G(\mathbb{R})$ [15]. (Knapp and Zuckerman [10] later determined the precise structure of the packets for representations outside the discrete series.) Shelstad then completed the theory for real groups [23], [24], [25], by showing among other things that the relationship of L-equivalence on the irreducible tempered characters was indeed dual, in a very precise sense, to the relationship of stable conjugacy on the strongly regular conjugacy classes.

Returning to the trace formula, we could formulate the first question that might come to mind as follows. Is the distribution

$$f \longrightarrow I_{\mathrm{ell}}(f), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big),$$

defined by the geometric side stable? In other words, does it depend only on the stable orbital integrals

$$S_G(\sigma_v, f_v) = \sum_{\gamma_v \in \sigma_v} I_G(\gamma_v, f_v) \tag{4}$$

of the constituents f_v of f? The elements σ_v stand for strongly regular stable conjugacy classes in $G(F_v)$, and γ_v is summed over the conjugacy classes in a stable conjugacy class. At first glance, the answer might seem to be yes. Stable conjugacy can be defined for rational elements $\gamma \in G(F)$, and the volume $a^G(\gamma)$ ought to depend only on the stable class of γ. This would allow us to group the terms in $I_{\mathrm{ell}}(f)$ as sums

$$\sum_{\gamma \in \sigma} I_G(\gamma, f)$$

of global orbital integrals, over the rational conjugacy classes γ in a rational stable class σ. (There will be some elements γ here that are not strongly regular, but this is really a side issue. Our assumption that G is anisotropic insures that the elements γ are at least semisimple.) If we look more closely, however, we find that the answer to the question is no. We have asked that the distribution be stable in *each* function f_v. In particular, if $\sigma_S = \prod_{v\in S} \sigma_v$ is any finite product of local (strongly regular) stable conjugacy classes, with a rational representative σ, then each ordinary conjugacy class $\gamma_S = \prod_{v\in S} \gamma_v$ in σ_S would also have to have a rational representative γ. There are simply not enough rational conjugacy classes in general for this to happen. Contrary to our first impression, then, the distribution $I_{\mathrm{ell}}(f)$ is not generally stable in f.

Thus, the initial observations of Langlands about stable conjugacy had immediate implications for two of the pillars of representation theory: Harish-Chandra's classification of discrete series and Selberg's trace formula. In the first case, there was the problem of constructing a relation on the irreducible tempered representations dual to stable conjugacy. In the case of the trace formula, the problem could be formulated as follows. Express $I_{\mathrm{ell}}(f)$ as the sum of a canonical stable distribution $S^G_{\mathrm{ell}}(f)$ and an explicit error term. The first group to be investigated was $SL(2)$. Labesse and Langlands [13] solved the problem for the anisotropic inner forms of this group (as well as for $SL(2)$ itself), and showed that the solution had remarkable implications for the spectral decomposition. In the general case, Langlands [18] was also able to solve the problem, under the assumption of two conjectures in local harmonic analysis.

Let us describe the main features of Langlands' general solution. The stable part was constructed first, and the error term was then expressed explicitly in terms of the stable parts $S^{G'}_{\mathrm{ell}}$ of trace formulas for groups G' of dimension smaller than G. The groups G', together with the quasi-split inner form G^* of G, are now known as the *elliptic endoscopic groups* for G. They are a family of quasi-split groups whose dual groups ([15, §2], [11, §1]) are of the form

$$\hat{G}' = \hat{G}_s = \mathrm{Cent}(\hat{G}, s)^0.$$

The elements s range over semisimple points in the dual group $\hat{G}$ of G, and are taken up to translation by the center of $\hat{G}$ and up to conjugation by $\hat{G}$. (See [16], [11, §7] and [20, §1.2].) Suppose for example that G is an inner form of a split adjoint group. Then $\hat{G}$ is simply connected, and the centralizer of s in $\hat{G}$ is already connected. The elliptic endoscopic groups in this case are the ones for which $\hat{G}'$ is contained in no proper Levi subgroup of $\hat{G}$. Thus, if G is an orthogonal group $SO(2n+1)$, $\hat{G}$ equals $\mathrm{Sp}(2n,\mathbb{C})$, and s can be taken from among the elements

$$\begin{pmatrix} I_r & 0 & 0 \\ 0 & -I_{2n-2r} & 0 \\ 0 & 0 & I_r \end{pmatrix}, 0 \le r \le \left[\frac{n}{2}\right].$$

The corresponding group $\hat{G}' = \hat{G}_s$ is $\mathrm{Sp}(2r,\mathbb{C}) \times \mathrm{Sp}(2n-2r,\mathbb{C})$, and G' is the split group $SO(2r+1) \times SO(2n-2r+1)$. If G is more general, it is necessary to work with the full L-group

$${}^LG = \hat{G} \rtimes \mathrm{Gal}(\overline{F}/F).$$

In this case there are further groups G' which are constructed by letting $\mathrm{Gal}(\overline{F}/F)$ act by outer automorphisms on $\hat{G}' = \hat{G}_s$ through the nonconnected components of the centralizer of s in $\hat{G}$.

Langlands' stabilization of $I_{\mathrm{ell}}(f)$ was based on a hypothetical transfer

$$f = \prod_v f_v \;\longrightarrow\; f' = \prod_v f'_v \tag{5}$$

of functions on $G(\mathbb{A})$ to functions on any endoscopic group $G'(\mathbb{A})$. Later refinement has given a very precise form to the conjecture. In [20], Langlands and Shelstad constructed local transfer factors, that are explicit complex valued functions

$$\Delta_G(\sigma'_v, \gamma_v)$$

of a stable conjugacy class σ'_v in $G'(F_v)$ and a strongly regular conjugacy class γ_v in $G(F_v)$. They vanish unless σ'_v maps (in a natural sense) to the stable conjugacy class of γ_v. The transfer factors then assume the role of the kernel in a transform

$$f_v \;\longrightarrow\; f'_v(\sigma'_v) = \sum_{\gamma_v} \Delta_G(\sigma'_v, \gamma_v) I_G(\gamma_v, f_v), \qquad f_v \in C_c^\infty\big(G(F_v)\big). \tag{6}$$

The conjecture is that for any $f_v \in C_c^\infty\big(G(F_v)\big)$, there is a function $f'_v \in C_c^\infty\big(G'(F_v)\big)$ whose stable orbital integrals are given by the values of the transform. That is,

$$f'_v(\sigma'_v) = S_{G'}(\sigma'_v, f'_v), \tag{7}$$

for any σ'_v. There is also a supplementary conjecture, known as the fundamental lemma, which applies to the unramified places v of G and G'. The assertion is that if f_v is the characteristic function of a hyperspecial maximal compact subgroup of $G(F_v)$, then f'_v can be taken to be the characteristic function of a hyperspecial maximal compact subgroup of $G'(\mathbb{A})$. Together, the two conjectures imply that there is a transfer correspondence from functions $f = \prod_v f_v$ in $C_c^\infty\big(G(\mathbb{A})\big)$ to functions $f' = \prod_v f'_v$ in $C_c^\infty\big(G'(\mathbb{A})\big)$. (Actually, there is a general problem of embedding ${}^LG'$ into ${}^LG$, which sometimes necessitates replacing G' by a certain central extension $\tilde{G}'$. We shall ignore this complication.)

Given the two local conjectures, Langlands' stabilization of $I_{\mathrm{ell}}(f)$ takes the form of an endoscopic expansion

$$I_{\mathrm{ell}}(f) = \sum_{G'} \iota(G, G') S^{G'}_{\mathrm{ell}}(f'), \tag{8}$$

with explicit coefficients $\iota(G, G')$. The distributions on the right are to be regarded as *stable* trace formulas for the elliptic endoscopic groups G'. They are linear combinations

$$S^{G'}_{\mathrm{ell}}(f') = \sum_{\sigma'} b^{G'}(\sigma') S_{G'}(\sigma', f') \tag{9}$$

over stable conjugacy classes σ' in $G'(F)$, with explicitly defined coefficients $b^{G'}(\sigma')$, of global stable orbital integrals

$$S_{G'}(\sigma', f') = \prod_{v} S_{G'}(\sigma', f'_v). \tag{10}$$

In terms of the original problem, the summand with G' equal to the quasi-split inner form G^* of G (that is, with $s = 1$) is to be regarded as the stable part of $I_{\text{ell}}(f)$, while the rest of the expansion constitutes the error term. Langlands actually dealt only with the strongly regular terms in the original trace formula. To be able to ignore the remaining singular terms, one would have to restrict f by, for example, taking f_v to be supported on the strongly regular set in $G(F_v)$ at some v. Kottwitz [12] was later able to deal with singular terms in $I_{\text{ell}}(f)$.

In the original basic case that G is the multiplicative group of a quaternion algebra, the right hand side of (8) has only one term, which corresponds to the quasi-split inner form $G' = G^* = GL(2)$ of G. The identity then leads to the correspondence $\pi \to \pi'$ of automorphic representations. It is harder to interpret the general case. The original trace formula does tell us that $I_{\text{ell}}(f)$ equals the spectral expansion $I_{\text{disc}}(f)$ defined by the trace of $R(f)$. The identity (8) suggests that $I_{\text{disc}}(f)$ is the sum of a stable part and an error term given precisely in terms of smaller endoscopic groups. That is,

$$I_{\text{disc}}(f) = \sum_{G'} \iota(G, G') S^{G'}_{\text{disc}}(f'). \tag{11}$$

This by itself does not provide a general correspondence of automorphic representations from G to any of the groups G', but it is nonetheless a striking conclusion. Very little is known about the multiplicities $m(\pi)$, especially regarding their stability properties. The identity (11) would give a precise obstruction to the distribution

$$f \longrightarrow I_{\text{disc}}(f) = \operatorname{tr}\big(R(f)\big), \qquad f \in C_c^\infty\big(G(\mathbb{A})\big),$$

being stable, in terms of spectral information on smaller groups. A general distribution on $C_c^\infty\big(G(\mathbb{A})\big)$ could fail to be stable independently at each v in any given finite set S. The general obstruction would have to be measured by many terms, parametrized by products

$$G'_S = \prod_{v \in S} G'_v$$

of *local* endoscopic groups. The products G'_S which are the diagonal image of global endoscopic groups G' are sparse in the set of all products.

We shall conclude with a word on the role of the general trace formula. The formula (8), for anisotropic G, does not immediately imply (11). The problem is that there is no direct formula like (9) (taken in conjunction with (10) and (4)) to define the terms $S^{G'}_{\text{disc}}(f')$. These terms must instead be defined by induction on the dimension of G'. However, this really requires an analogue of (11) for the quasi-split form G^* of G. The inductive definition would take the form

$$S^{G^*}_{\text{disc}}(f) = I^{G^*}_{\text{disc}}(f) - \sum_{G' \neq G^*} \iota(G^*, G') S^{G'}_{\text{disc}}(f'), \qquad f \in C_c^\infty\big(G^*(\mathbb{A})\big),$$

and the conclusion to be drawn from (8) (or rather its analogue for G^*) is simply that $S^{G^*}_{\mathrm{disc}}(f)$ is stable. But G^* is quasi-split, not anisotropic, so have strayed from our original assumption on G. We begin to see that it is rather unnatural to restrict G to being anisotropic, even if we only want to study a simple version of the trace formula. It would be better to keep G arbitrary, and to restrict f so that the geometric side reduces to the form (2). The spectral part $I_{\mathrm{disc}}(f)$ would still have more terms than just the characters $m(\pi)\mathrm{tr}\big(\pi(f)\big)$. (See [1, (4.3) and Theorem 7.1].) However the extra terms are very interesting, and are in any case part of the story. In fact, there are compelling reasons to want to stabilize the full trace formula, with functions f that are unrestricted, even though there are many more terms on each side, and more problems to be solved.

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