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EISENSTEIN SERIES AND THE TRACE FORMULA

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PART I. EISENSTEIN SERIES

The spectral theory of Eisenstein series was begun by Selberg. It was completed by Langlands in a manuscript which was for a long time unpublished but which recently has appeared [1]. The main references are

1. Langlands, *On the functional equations satisfied by Eisenstein series*, Springer-Verlag, Berlin, 1976.
2. ———, *Eisenstein series*, Algebraic Groups and Discontinuous Subgroups, Summer Research Institute (Univ. Colorado, 1965), Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R. I. 1966.
3. Harish-Chandra, *Automorphic forms on semi-simple Lie groups*, Springer-Verlag, Berlin, 1968.

In the first part of these notes we shall try to describe the main ideas in the theory.

Let G be a reductive algebraic matrix group over $\boldsymbol{Q}$. Then $G(\boldsymbol{A})$ is the restricted direct product over all valuations v of the groups $G(\boldsymbol{Q}_v)$. If v is finite, define K_v to be $G(\mathfrak{o}_v)$ if this latter group is a special maximal compact subgroup of $G(\boldsymbol{Q}_v)$. This takes care of almost all v. For the remaining finite v, we let K_v be *any* fixed special maximal compact subgroup of $G(\boldsymbol{Q}_v)$. We also fix a minimal parabolic subgroup P_0, defined over $\boldsymbol{Q}$, and a Levi component M_0 of P_0. Let A_0 be the maximal split torus in the center of M_0. Let $K_{\boldsymbol{R}}$ be a fixed maximal compact subgroup of $G(\boldsymbol{R})$ whose Lie algebra is orthogonal to the Lie algebra of $A_0(\boldsymbol{R})$ under the Killing form. Then $K = \prod_v K_v$ is a maximal compact subgroup of $G(\boldsymbol{A})$.

For most of these notes we shall deal only with standard parabolic subgroups; that is, parabolic subgroups P, defined over $\boldsymbol{Q}$, which contain P_0. Fix such a P. Let N_P be the unipotent radical of P, and let M_P be the unique Levi component of P which contains M_0. Then the split component, A_P, of the center of M_P is contained in A_0. If $X(M_P)_{\boldsymbol{Q}}$ is the group of characters of M_P defined over $\boldsymbol{Q}$, define $\mathfrak{a}_P = \mathrm{Hom}((X_P)_{\boldsymbol{Q}}, \boldsymbol{R})$. Then if $m = \prod_v m_v$ lies in $M(\boldsymbol{A})$, we define a vector $H_M(m)$ in $\mathfrak{a}_P$ by

$$\exp(\langle H_M(m), \chi\rangle) = |\chi(m)| = \prod_v |\chi(m_v)|_v.$$

Let $M_P(\boldsymbol{A})^1$ be the kernel of the homomorphism H_M. Then $M_P(\boldsymbol{A})$ is the direct

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product of $M_P(\boldsymbol{A})^1$ and $A(\boldsymbol{R})^0$, the connected component of 1 in $A(\boldsymbol{R})$. Since $G(\boldsymbol{A})$ equals $N_P(\boldsymbol{A})M_P(\boldsymbol{A})K$, we can write any $x \in G(\boldsymbol{A})$ as nmk, $n \in N_P(\boldsymbol{A})$, $m \in M_P(\boldsymbol{A})$, $k \in K$. We define $H_P(x)$ to be the vector $H_M(m)$ in $\mathfrak{a}_P$.

Let Ω be the restricted Weyl group of (G, A_0). Ω acts on the dual space of $\mathfrak{a}_0$. We identify $\mathfrak{a}_0$ with its dual by fixing a positive definite Ω-invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{a}_0$. This allows us to embed each $\mathfrak{a}_P$ in $\mathfrak{a}_0$. Let Σ_P be the set of roots of (P, A). These are the elements in $X(A)_{\boldsymbol{Q}}$ obtained by decomposing the Lie algebra of N_P under the adjoint action of A_P. They can be regarded as vectors in $\mathfrak{a}_P$. Let Φ_P be the set of simple roots of (P, A). G itself is a parabolic subgroup. We write Z and $\mathfrak{z}$ for A_G and $\mathfrak{a}_G$ respectively. Then Φ_P is a basis of the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{a}_P$.

Suppose that Q is another (standard) parabolic subgroup, with $Q \subset P$. Then $Q^P = Q \cap M_P$ is a parabolic subgroup of M_P. Its unipotent radical is $N_Q^P = N_Q \cap M_P$. We write $\mathfrak{a}_Q^P$ for the orthogonal complement of $\mathfrak{a}_P$ in $\mathfrak{a}_Q$. In general, we shall index the various objects associated with Q^P by the subscript Q and the superscript P. For example, Φ_Q^P stands for the set of simple roots of (Q^P, A_Q). It is the projection onto $\mathfrak{a}_Q^P$ of $\Phi_{P_0}^P \backslash \Phi_{P_0}^Q$. We shall write $\hat{\Phi}_Q^P$ for the basis of the Euclidean space $\mathfrak{a}_Q^P$ which is dual to Φ_Q^P. If P_i is a parabolic subgroup, we shall often use i instead of P_i for a subscript or superscript. If the letter P alone is used, we shall often omit it altogether as a subscript. Finally, we shall always denote the Lie algebras of groups over $\boldsymbol{Q}$ by lower case Gothic letters.

If P and P_1 are parabolic subgroups, let $\Omega(\mathfrak{a}, \mathfrak{a}_1)$ be the set of distinct isomorphisms from $\mathfrak{a}$ onto $\mathfrak{a}_1$ obtained by restricting elements in Ω to $\mathfrak{a}$. P and P_1 are said to be *associated* if $\Omega(\mathfrak{a}, \mathfrak{a}_1)$ is not empty. Suppose that $\mathscr{P}$ is an associated class, and that $P \in \mathscr{P}$.

$$\mathfrak{a}^+ = \mathfrak{a}_P^+ = \{H \in \mathfrak{a} : \langle \alpha, H \rangle > 0, \alpha \in \Phi_P\}$$

is called the *chamber* of P in $\mathfrak{a}$.

LEMMA 1. $\bigcup_{P_1 \subset \mathscr{P}} \bigcup_{s \subset \Omega(\mathfrak{a}, \mathfrak{a}_1)} s^{-1}(\mathfrak{a}_1^+)$ *is a disjoint union which is dense in* $\mathfrak{a}$. $\square$

Before discussing Eisenstein series, we shall define a certain induced representation. Fix P. Let $\mathscr{H}_P^0$ be the space of functions

$$\Phi : N(\boldsymbol{A}) \cdot M(\boldsymbol{Q}) \cdot A(\boldsymbol{R})^0 \backslash G(\boldsymbol{A}) \to \boldsymbol{C}$$

such that

(i) for any $x \in G(\boldsymbol{A})$ the function $m \to \Phi(mx)$, $m \in M(\boldsymbol{A})$, is $\mathscr{Z}_{M(\boldsymbol{R})}$-finite, where $\mathscr{Z}_{M(\boldsymbol{R})}$ is the center of the universal enveloping algebra of $\mathfrak{m}(\boldsymbol{C})$,

(ii) the span of the set of functions $\Phi_k : x \to \Phi(xk)$, $x \in G(\boldsymbol{A})$, indexed by $k \in K$, is finite dimensional.

(iii) $$\|\Phi\|^2 = \int_K \int_{A(\boldsymbol{R})^0 M(\boldsymbol{Q}) \backslash M(\boldsymbol{A})} |\Phi(mk)|^2 dm\, dk < \infty.$$

Let $\mathscr{H}_P$ be the Hilbert space obtained by completing $\mathscr{H}_P^0$. If Λ is in $\mathfrak{a}_C$, the complexification of $\mathfrak{a}$, $\Phi \in \mathscr{H}_P$, and $x, y \in G(\boldsymbol{A})$, put

$$(I_P(\Lambda, y)\Phi)(x) = \Phi(xy) \exp(\langle \Lambda + \rho_P, H_P(xy) \rangle) \exp(-\langle \Lambda + \rho_P, H_P(x) \rangle).$$

Here ρ_P is the vector in $\mathfrak{a}$ such that

$$|\det(\operatorname{Ad} m)_{\mathfrak{n}(\boldsymbol{A})}| = \exp(\langle 2\rho_P, H_M(m) \rangle), \qquad m \in M(\boldsymbol{A}).$$

$I_P(\Lambda) = I_P^G(\Lambda)$ is a representation of $G(\boldsymbol{A})$ induced from a representation of

$P(\boldsymbol{A})$, which in turn is the pull-back of a certain representation, $I_M^M(\Lambda)$ in our notation, of $M(\boldsymbol{A})$. We have

$$I_P(\Lambda, y)^* = I_P(-\bar{\Lambda}, y^{-1}), \qquad y \in G(\boldsymbol{A}),$$

and

$$I_P(\Lambda, f)^* = I_P(-\bar{\Lambda}, f^*), \qquad f \in C_c^\infty(G(\boldsymbol{A})),$$

where $f^*(y) = f(y^{-1})$. In particular, $I_P(\Lambda)$ is unitary if Λ is purely imaginary.

REMARK. It is not difficult to show that $I_M^M(0)$ is the subrepresentation of the regular representation of $M(\boldsymbol{A})$ on $L^2(A(\boldsymbol{R})^0 \cdot M(\boldsymbol{Q})\backslash M(\boldsymbol{A}))$ which decomposes discretely. We can write $I_M^M(0) = \bigoplus_\iota \sigma^\iota$, where $\sigma^\iota = \bigotimes_v \sigma_v^\iota$ is an irreducible representation of $M(\boldsymbol{A})$. If v is any prime and σ_v is an irreducible unitary representation of $M(\boldsymbol{Q}_v)$, define

$$\sigma_{v,\Lambda}(m) = \sigma_v(m)\exp(\langle \Lambda, H_M(m)\rangle), \qquad \Lambda \in \mathfrak{a}_{\boldsymbol{C}}, m \in M(\boldsymbol{Q}_v).$$

If $\sigma_{v,\Lambda}$ is lifted to $P(\boldsymbol{Q}_v)$ and then induced to $G(\boldsymbol{Q}_v)$, the result is a representation $I_P(\sigma_{v,\Lambda})$ of $G(\boldsymbol{Q}_v)$, acting on a Hilbert space $\mathscr{H}_P(\sigma_v)$. In this notation,

$$I_P(\Lambda) = \bigoplus_\iota \bigotimes_v I_P(\sigma_{v,\Lambda}^\iota).$$

Thus $I_P(\Lambda)$ can be completely understood in terms of induced representations and the discrete spectrum of M.

If P and P_1 are fixed, and $s \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$, let w_s be a fixed representative of s in the intersection of $K \cap G(\boldsymbol{Q})$ with the normalizer of A_0. For $\Phi \in \mathscr{H}_P^0$, $\Lambda \in \mathfrak{a}_{\boldsymbol{C}}$, and $x \in G(\boldsymbol{A})$, consider

$$\int_{N_1(\boldsymbol{A})\cap w_sN(\boldsymbol{A})w_s^{-1}\backslash N_1(\boldsymbol{A})} \Phi(w_s^{-1}nx)\exp(\langle \Lambda+\rho_P, H_P(w_s^{-1}nx)\rangle)\, dn \exp(-\langle s\Lambda+\rho_1, H_1(x)\rangle).$$

(We adopt the convention that if H is any closed connected subgroup of N_0, dh is the Haar measure on $H(\boldsymbol{A})$ which makes the volume of $H(\boldsymbol{Q})\backslash H(\boldsymbol{A})$ one. This defines a unique quotient measure dn on $N_1(\boldsymbol{A}) \cap w_sN(\boldsymbol{A})w_s^{-1}\backslash N_1(\boldsymbol{A})$.)

LEMMA 2. *Suppose that* $\langle \alpha, \operatorname{Re} \Lambda - \rho_P\rangle > 0$ *for each* α *in* Σ_P *such that* $s\alpha$ *belongs to* $-\Sigma_{P_1}$. *Then the above integral converges absolutely.* □

The integral, for Λ as in the lemma, defines a linear operator from $\mathscr{H}_P^0$ to $\mathscr{H}_{P_1}^0$, which we denote by $M(s, \Lambda)$. Intertwining integrals play an important role in the harmonic analysis of groups over local fields, so it is not surprising that $M(s, \Lambda)$ arises naturally in the global theory.

LEMMA 3. $M(s, \Lambda)^* = M(s^{-1}, -s\bar{\Lambda})$. *Moreover, if* $f \in C_c^\infty(G(\boldsymbol{A}))^K$, *the* K*-conjugate invariant functions in* $C_c^\infty(G(\boldsymbol{A}))$,

$$M(s, \Lambda)I_P(\Lambda, f) = I_P(s\Lambda, f)\, M(s, \Lambda). \quad \square$$

We now define Eisenstein series:

LEMMA 4. *If* $\Phi \in \mathscr{H}_P^0$, $x \in G(\boldsymbol{A})$ *and* $\Lambda \in \mathfrak{a}_{\boldsymbol{C}}$, *with* $\operatorname{Re} \Lambda \in \rho_P + \mathfrak{a}^+$, *the series*

$$E(x, \Phi, \Lambda) = \sum_{\delta\in P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})} \Phi(\delta x) \exp(\langle \Lambda + \rho_P, H_P(\delta x)\rangle)$$

converges absolutely. □

The principal results on Eisenstein series are contained in the following:

MAIN THEOREM. (a) *Suppose that* $\Phi \in \mathscr{H}_P^0$. $E(x, \Phi, \Lambda)$ *and* $M(s, \Lambda)\,\Phi$ *can be analytically continued as meromorphic functions to* $\mathfrak{a}_C$. *On* $i\mathfrak{a}$, $E(x, \Phi, \Lambda)$ *is regular, and* $M(s, \Lambda)$ *is unitary. For* $f \in C_c^\infty(G(A))^K$ *and* $t \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$, *the following functional equations hold*:

(i) $E(x, I_P(\Lambda, f)\,\Phi, \Lambda) = \int_{G(A)} f(y) E(xy, \Phi, \Lambda)\, dy$,

(ii) $E(x, M(s, \Lambda)\,\Phi, s\Lambda) = E(x, \Phi, \Lambda)$,

(iii) $M(ts, \Lambda)\,\Phi = M(t, s\Lambda) M(s, \Lambda)\Phi$.

(b) *Let* $\mathscr{P}$ *be an associated class of parabolic subgroups. Let* $\hat{L}_{\mathscr{P}}$ *be the set of collections* $F = \{F_P : P \in \mathscr{P}\}$ *of measurable functions* F_P: $i\mathfrak{a} \to \mathscr{H}_P$ *such that*

(i) *If* $s \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$,

$$F_{P_1}(s\Lambda) = M(s, \Lambda) F_P(\Lambda),$$

(ii)
$$\|F\|^2 = \sum_{P \in \mathscr{P}} n(A)^{-1} \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}_P} \|F_P(\Lambda)\|^2 d\Lambda < \infty,$$

where $n(A)$ *is the number of chambers in* $\mathfrak{a}$. *Then the map which sends* F *to the function*

$$\sum_{P \in \mathscr{P}} n(A)^{-1} \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}} E(x, F_P(\Lambda), \Lambda)\, d\Lambda,$$

defined for F *in a dense subspace of* $\hat{L}_{\mathscr{P}}$, *extends to a unitary map from* $\hat{L}_{\mathscr{P}}$ *onto a closed* $G(A)$-*invariant subspace* $L^2_{\mathscr{P}}(G(Q)\backslash G(A))$ *of* $L^2(G(Q)\backslash G(A))$. *Moreover, we have an orthogonal decomposition*

$$L^2(G(Q)\backslash G(A)) = \bigoplus_{\mathscr{P}} L^2_{\mathscr{P}}(G(Q)\backslash G(A)). \quad \square$$

The theorem states that the regular representation of $G(A)$ on $L^2(G(Q)\backslash G(A))$ is the direct sum over a set of representatives $\{P\}$ of associated classes of parabolic subgroups, of the direct integrals $\int_{i\mathfrak{a}_P^+}^{\oplus} I_P(\Lambda)\, d\Lambda$.

The theorem looks relatively straightforward, but the proof is decidedly round about. The natural inclination might be to start with a general Φ in $\mathscr{H}_P^0$ and try to prove directly the analytic continuation of $M(s, \Lambda)\Phi$ and $E(x, \Phi, \Lambda)$. This does not seem possible. One does not get any idea how the proof will go, for general Φ, until p. 231 of [**1**], the second last page of Langlands' original manuscript. Rather Langlands' strategy was to prove all the relevant statements of the theorem for Φ in a certain subspace $\mathscr{H}^0_{P,\mathrm{cusp}}$ of $\mathscr{H}_P^0$. He was then able to finesse the theorem from this special case.

Let $\mathscr{H}_{P,\mathrm{cusp}}$ be the space of measurable functions Φ on $N(A)M(Q)A(R)^0\backslash G(A)$ such that

(i) $\|\Phi\|^2 = \int_K \int_{M(Q)A(R)^0\backslash M(A)} |\Phi(mk)|^2\, dm\, dk < \infty$,

(ii) for any $Q \supsetneqq P$, and $x \in G(A)$, $\int_{N_Q(Q)\backslash N_Q(A)} \Phi(nx)\, dn = 0$.

It is a right $G(A)$-invariant Hilbert space.

LEMMA 5. *If* $f \in C_c^N(G(A))$ *for some large* N, *the map* $\Phi \to \Phi * f$, $\Phi \in \mathscr{H}_{G,\mathrm{cusp}}$, *is a Hilbert-Schmidt operator on* $\mathscr{H}_{G,\mathrm{cusp}}$. $\square$

This lemma, combined with the spectral theorem for compact operators, leads to

COROLLARY. *$\mathscr{H}_{G,\mathrm{cusp}}$ decomposes into a direct sum of irreducible representations of $G(A)$, each occurring with finite multiplicity.* $\square$

$\mathscr{H}_{G,\mathrm{cusp}}$ is called the space of *cusp forms* on $G(A)$. It follows from the corollary, applied to M, that any function in $\mathscr{H}_{P,\mathrm{cusp}}$ is a limit of functions in $\mathscr{H}_P^0$. Therefore $\mathscr{H}_{P,\mathrm{cusp}}$ is a subspace of $\mathscr{H}_P$. It is closed and invariant under $I_P(\Lambda)$.

Let $\mathscr{S}(G)$ be the collection of triplets $\chi = (\mathscr{P}, \mathscr{V}, W)$, where W is an irreducible representation of K, $\mathscr{P}$ is an associated class of parabolic subgroups, and $\mathscr{V}$ is a collection of subspaces

$$\{V_P \subset \mathscr{H}^M_{M,\mathrm{cusp}}, \text{ the space of cusp forms on } M(A)\}_{P\in\mathscr{P}},$$

such that

(i) if $P \in \mathscr{P}$, V_P is the eigenspace of $\mathscr{H}^M_{M,\mathrm{cusp}}$ associated to a complex valued homomorphism of $\mathscr{Z}_{M(\boldsymbol{R})}$, and

(ii) if $P, P' \in \mathscr{P}$, and $s \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$, V_{P_1} is the space of functions obtained by conjugating functions in V_P by w_s.

If $P \in \mathscr{P}$, define $\mathscr{H}_{P,\chi}$ to be the space of functions Φ in $\mathscr{H}^0_{P,\mathrm{cusp}}$ such that for each $x \in G(A)$,

(i) the function $k \to \Phi(xk)$, $k \in K$, is a matrix coefficient of W, and

(ii) the function $m \to \Phi(mx)$, $m \in M(A)$, belongs to V_P. $\mathscr{H}_{P,\chi}$ is a finite dimensional space which is invariant under $I_P(\Lambda, f)$ for any $f \in C_c^\infty(G(A))^K$. $\mathscr{H}_{P,\mathrm{cusp}}$ is the orthogonal direct sum over all $\chi = (\mathscr{P}, \mathscr{V}, W)$, for which $P \in \mathscr{P}$, of the spaces $\mathscr{H}_{P,\chi}$.

Fix $\chi = (\mathscr{P}, \mathscr{V}, W)$ and fix $P \in \mathscr{P}$. Suppose that we have an analytic function

$$\Lambda \to \Phi(\Lambda) = \Phi(\Lambda, x), \qquad \Lambda \in \mathfrak{a}_C,\ x \in N(A)M(\boldsymbol{Q})A(\boldsymbol{R})^0\backslash G(A),$$

of Paley-Wiener type, from $\mathfrak{a}_C$ to the finite dimensional space $\mathscr{H}_{P,\chi}$. Then

$$\phi(x) = \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\mathrm{Re}\Lambda=\Lambda_0} \exp(\langle \Lambda + \rho_P, H_P(x)\rangle)\Phi(\Lambda, x)\, d\Lambda$$

is a function on $N(A)M(\boldsymbol{Q})\backslash G(A)$ which is independent of the point $\Lambda_0 \in \mathfrak{a}$. It is compactly supported in the $A(\boldsymbol{R})^0$-component of x.

LEMMA 6. *For Φ as above, the function*

$$\hat{\phi}(x) = \sum_{\delta\in P(\boldsymbol{Q})\backslash G(A)} \phi(\delta x)$$

converges absolutely and belongs to $L^2(G(\boldsymbol{Q})\backslash G(A))$. Let $L^2_\chi(G(\boldsymbol{Q})\backslash G(A))$ be the closed subspace generated by all such $\hat{\phi}$. Then there is an orthogonal decomposition

$$L^2(G(\boldsymbol{Q})\backslash GA)) = \bigoplus_{\chi\in\mathscr{S}(G)} L^2_\chi(G(\boldsymbol{Q})\backslash G(A)). \quad \square$$

Suppose that $\Lambda_0 \in \rho_P + \mathfrak{a}^+$. Then

$$\begin{aligned}\hat{\phi}(x) &= \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\mathrm{Re}\,\Lambda=\Lambda_0} \sum_{\delta\in P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})} \exp(\langle \Lambda + \rho_P, H_P(\delta x)\rangle)\, \Phi(\Lambda, \delta x)\, d\Lambda \\ &= \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\mathrm{Re}\,\Lambda=\Lambda_0} E(x, \Phi(\Lambda), \Lambda)\, d\Lambda.\end{aligned}$$

Suppose that $\Phi_1(\Lambda_1, x)$ is another function, associated to P_1. We want an inner product formula for

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\hat{\phi}_1(x)}\, dx$$

in terms of Φ and Φ_1. The inner product is

$$\begin{aligned}\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \sum_{\delta\in P_1(\mathbf{Q})\backslash G(\mathbf{Q})} & \hat{\phi}(x)\overline{\phi_1(\delta x)}\, dx \\ &= \int_{P_1(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\phi_1(x)}\, dx \\ &= \left(\frac{1}{2\pi i}\right)^{\dim A_1} \int_{\mathrm{Re}\Lambda=\Lambda_0} \int_K \int_{A_1(\mathbf{R})^0 M_1(\mathbf{Q})\backslash M_1(\mathbf{A})} \int_{A_1(\mathbf{R})^0} E(namk, \Phi(\Lambda), \Lambda) \\ &\quad \cdot \exp(-2\langle \rho_1, H_1(x)\rangle)\, \overline{\phi_1(amk)}\, da\, dm\, dk.\end{aligned}$$

LEMMA 7. *Suppose that P and P_1 are of the same rank. If $\Phi \in \mathscr{H}_{P,\chi}$ and* $\mathrm{Re}\,\Lambda \in \rho_P + \mathfrak{a}^+$, *then*

$$\int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} E(nx, \Phi, \Lambda)\, dn = \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} (M(s,\Lambda)\Phi)(x) \exp(\langle s\Lambda + \rho_1, H_1(x)\rangle).$$

(*Of course, the right-hand side is* 0 *if $\Omega(\mathfrak{a}, \mathfrak{a}_1)$ is empty; that is, if P and P_1 are not associated.*)

PROOF. Let $\{\Omega\}$ be the set of $s \in \Omega$ such that $s^{-1}\alpha > 0$ for every $\alpha \in \Phi_0^P$. Then $\{\Omega\}$ is a set of representatives in Ω of the left cosets of Ω modulo the Weyl group of M. By the Bruhat decomposition,

$$\begin{aligned}\int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} & E(nx, \Phi, \Lambda)\, dn \\ &= \sum_{t\in\{\Omega\}} \int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} \sum_{\{\nu\in w_t^{-1}P(\mathbf{Q})w_t\cap N_0(\mathbf{Q})\backslash N_0(\mathbf{Q})\}} \Phi(w_t\,\nu n x) \\ &\quad \cdot \exp(\langle \Lambda + \rho_P, H_P(w_t\nu n x)\rangle)\, dn,\end{aligned}$$

$w_t N_1 w_t^{-1} \cap M$ is the unipotent radical of a standard parabolic subgroup of M. If the group is not M itself the term corresponding to s above is 0, since Φ is cuspidal. The group is M itself if and only if $s = t^{-1}$ maps $\mathfrak{a}$ onto $\mathfrak{a}_1$. In this case $w_t^{-1}Pw_t \cap N_0\backslash N_0$ is isomorphic to $w_t^{-1}Nw_t \cap N_1\backslash N_1$. Therefore the above formula equals

$$\begin{aligned}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} & \mathrm{vol}(w_s N(\mathbf{Q}) w_s^{-1} \cap N(\mathbf{Q})\backslash w_s N(\mathbf{A}) w_s^{-1}\backslash N(\mathbf{A})) \\ &\cdot \int_{w_s N(\mathbf{A}) w_s^{-1}\cap N(\mathbf{A})\backslash N(\mathbf{A})} \Phi(w_s^{-1} nx) \exp(\langle \Lambda + \rho_P, H_P(w_s^{-1}nx)\rangle)\, dn.\end{aligned}$$

The volume is one by our choice of measure. The lemma therefore follows. □

Combining the lemma with the Fourier inversion formula, one obtains

COROLLARY. *Suppose that P and P_1 are associated, and that ϕ and ϕ_1 are as in the discussion preceding Lemma* 7. *Then*

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\hat{\phi}_1(x)}\,dx$$
$$= \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\Lambda_0 + i\mathfrak{a}} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} (M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\,d\Lambda,$$

where Λ_0 *is any point in* $\rho_P + \mathfrak{a}^+$. □

The proofs of Lemmas 1–6 are based on rather routine and familiar estimates. This is the point at which the serious portion of the proof of the Main Theorem should begin. There are two stages. The first stage is to complete the analytic continuation and functional equations for Φ a vector in $\mathscr{H}^0_{P,\mathrm{cusp}}$. This is nicely described in **[2]**, so we shall skip it altogether. The second stage, done in Chapter 7 of **[1]**, is the spectral decomposition of $L^2_\chi(G(\mathbf{Q})\backslash G(\mathbf{A}))$. From this all the remaining assertions of the Main Theorem follow. We shall try to give an intuitive description of the argument.

Fix $\chi = (\mathscr{P}_\chi, \mathscr{V}_\chi, W)$. The argument in Chapter 7 of **[1]** is based on an intricate induction on the dimension of (A/Z), for $P \in \mathscr{P}_\chi$. For $\Phi \in \mathscr{H}_{P,\chi}$, $P \in \mathscr{P}_\chi$, we are assuming that $E(x, \Phi, \Lambda)\Phi$ and $M(s, \Lambda)\Phi$ are meromorphic on $\mathfrak{a}_C$. In the process of proving this, one shows that the singularities of these two functions are hyperplanes of the form $\mathfrak{r} = \{\Lambda \in \mathfrak{a}_C : \langle \alpha, \Lambda\rangle = \mu,\ \mu \in \mathbf{C},\ \alpha \in \Sigma_P\}$, and that only finitely many $\mathfrak{r}$ meet $\mathfrak{a}^+ + i\mathfrak{a} = \{\Lambda \ni \mathfrak{a}_C : \langle \alpha, \operatorname{Re}\Lambda\rangle > 0,\ \alpha \in \Phi_P\}$.

Let $L^2_{\mathscr{P}_\chi,\chi}(G(\mathbf{Q})\backslash G(\mathbf{A}))$ be the closed subspace of $L^2_\chi(G(\mathbf{Q})\backslash G(\mathbf{A}))$ generated by functions $\hat{\phi}(x)$, where $\hat{\phi}$ comes from a function $\Phi(\Lambda)$, as above, which vanishes on the finite set of singular hyperplanes which meet $\mathfrak{a}^+ + i\mathfrak{a}$. If $\phi_1(x)$ comes from $\Phi_1(\Lambda_1)$, for $P_1 \in \mathscr{P}_\chi$, we can apply Cauchy's theorem to the formula for

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\hat{\phi}_1(x)}\,dx.$$

The result is

$$\left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} (M(s,\Lambda)\Phi(\Lambda), \Phi_1(s\Lambda))\,d\Lambda,$$

since $-s\bar{\Lambda} = s\Lambda$ on $i\mathfrak{a}$. Changing variables in the integral and sum, we obtain

$$\left(\frac{1}{2\pi i}\right)^{\dim A} n(A)^{-1} \sum_{P_2\in\mathscr{P}_\chi} \sum_{t\in\Omega(\mathfrak{a}_2,\mathfrak{a})} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} \cdot \int_{i\mathfrak{a}_2} (M(s, t\Lambda)\Phi(t\Lambda), \Phi_1(t\Lambda))\,d\Lambda$$
$$= \left(\frac{1}{2\pi i}\right)^{\dim A} n(A)^{-1} \sum_{P_2} \sum_{r\in\Omega(\mathfrak{a}_2,\mathfrak{a}_1)} \sum_{t\in\Omega(\mathfrak{a}_2,\mathfrak{a})} \cdot \int_{i\mathfrak{a}_2} (M(rt^{-1}, t\Lambda)\Phi(t\Lambda), \Phi_1(r\Lambda))\,d\Lambda$$
$$= \left(\frac{1}{2\pi i}\right)^{\dim A} n(A)^{-1} \sum_{P_2}\sum_{r}\sum_{t} \int_{i\mathfrak{a}_2} (M(t,\Lambda)^{-1}\Phi(t\Lambda), M(r,\Lambda)^{-1}\Phi_1(r\Lambda))\,d\Lambda$$
$$= \sum_{P_2\in\mathscr{P}_\chi} n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}_2} (F_{P_2}(\Lambda), F_{1,P_2}(\Lambda))\,d\Lambda,$$

where

$$F_{1,P_2}(\Lambda) = \sum_{r\in\Omega(\mathfrak{a}_2,\mathfrak{a}_1)} M(r,\Lambda)^{-1}\Phi_1(r\Lambda), \tag{1}$$

and F_{P_2} is defined similarly. Define $\hat{L}_{\mathscr{P}_\chi,\chi}$ to be the subspace of the space $\hat{L}_{\mathscr{P}_\chi}$ (defined in the statement of the Main Theorem) consisting of those collections $\{F_{P_2}: P_2 \in \mathscr{P}_\chi\}$ such that F_{P_2} takes values in $\mathscr{H}_{P_2,\chi}$. We have just exhibited an isometric isomorphism from a dense subspace of $\hat{L}_{\mathscr{P}_\chi,\chi}$ to a dense subspace of $L^2_{\mathscr{P}_\chi,\chi}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$. Suppose that $\{F_{P_2}\}$ is a collection of functions in $\hat{L}_{\mathscr{P}_\chi,\chi}$ each of which is smooth and compactly supported. Let $h(x)$ be the corresponding function in $L^2_{\mathscr{P}_\chi,\chi}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$ defined by the above isomorphism. We would like to prove that $h(x)$ equals

$$h'(x) = \sum_{P_2\in\mathscr{P}_\chi} n(A_2)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A}\int_{i\mathfrak{a}_2} E(x, F_{P_2}(\Lambda), \Lambda)\, d\Lambda.$$

If $\hat{\phi}_1(x)$ is as above, the same argument as that of the corollary to Lemma 7 shows that

$$\int_{G(\boldsymbol{Q})\backslash G(\boldsymbol{A})} h'(x)\overline{\hat{\phi}_1(x)}\,dx = \sum_{P_2} n(A_2)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A_2} \cdot\int_{i\mathfrak{a}_2}\sum_{r\in\Omega(\mathfrak{a}_2,\mathfrak{a}_1)}(M(r,\Lambda)F_{P_2}(\Lambda), \Phi_1(r\Lambda))\,d\Lambda.$$

Since the projection of $\hat{\phi}_1(x)$ onto $L^2_{\mathscr{P}_\chi,\chi}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$ corresponds to the collection $\{F_{1,P_2}\}$ defined by (1), this equals

$$\sum_{P_2} n(A_2)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A_2}\int_{i\mathfrak{a}_2}(F_{P_2}(\Lambda), F_{1,P_2}(\Lambda))\,d\Lambda = \int_{G(\boldsymbol{Q})\backslash G(\boldsymbol{A})} h(x)\overline{\hat{\phi}_1(x)}\,dx.$$

We have shown that $h(x) = h'(x)$. This completes the first stage of Langlands' induction.

To begin the second stage, Langlands lets Q be the projection of $L^2_\chi(G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$ onto the orthogonal complement of $L^2_{\mathscr{P}_\chi,\chi}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$. Then for any $\hat{\phi}(x)$ and $\hat{\phi}_1(x)$ corresponding to $\Phi(\Lambda)$ and $\Phi_1(\Lambda_1)$, $(Q\hat{\phi}, \hat{\phi}_1)$ equals

$$\left(\frac{1}{2\pi i}\right)^{\dim A}\left(\int_{\Lambda_0+i\mathfrak{a}}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)}(M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\,d\Lambda - \int_{i\mathfrak{a}}\sum_{s}(M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\,d\Lambda\right).$$

Choose a path in $\mathfrak{a}^+$ from Λ_0 to 0 which meets any singular hyperplane $\mathfrak{r}$ of $\{M(s,\Lambda): s\in\Omega(\mathfrak{a},\mathfrak{a}_1)\}$ in at most one point $Z(\mathfrak{r})$. Any such $\mathfrak{r}$ is of the form $X(\mathfrak{r}) + \mathfrak{r}^\vee_{\mathcal{C}}$, where $\mathfrak{r}^\vee$ is a real vector subspace of $\mathfrak{a}$ of codimension one, and $X(\mathfrak{r})$ is a vector in $\mathfrak{a}$ orthogonal to $\mathfrak{r}^\vee$. The point $Z(\mathfrak{r})$ belongs to $X(\mathfrak{r}) + \mathfrak{r}^\vee$. By the residue theorem $(Q\hat{\phi}, \hat{\phi}_1)$ equals

$$\left(\frac{1}{2\pi i}\right)^{\dim A-1}\sum_{\mathfrak{r}}\int_{Z(\mathfrak{r})+i\mathfrak{r}^\vee}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)}\mathrm{Res}_{\mathfrak{r}}(M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\,d\Lambda.$$

The obvious tactic at this point is to repeat the first stage of the induction with Λ_0 replaced by $Z(\mathfrak{r})$, 0 replaced by $X(\mathfrak{r})$, and $E(x,\Phi,\Lambda)$ by $\mathrm{Res}_{\mathfrak{r}}E(x,\Phi,\Lambda)$.

Suppose that $\mathfrak{r}^\vee = \{H \in \mathfrak{a} : \langle \alpha, H \rangle = 0\}$ for a simple root $\alpha \in \Phi_P$. Then $\mathfrak{r}^\vee = \mathfrak{a}_R$, for R a parabolic subgroup of G containing P. If $\Phi \in \mathscr{H}_{P,\chi}$, define

$$E^R(x, \Phi, \Lambda) = \sum_{P(Q)\backslash R(Q)} \Phi(\delta x) \exp(\langle \Lambda + \rho_P, H(\delta x) \rangle).$$

This is essentially a cuspidal Eisenstein series on the group $M_R(A)$. It converges for suitable $\Lambda \in \mathfrak{a}_C$ and can be meromorphically continued. It is clear that

$$E(x, \Phi, \Lambda) = \sum_{R(Q)\backslash G(Q)} E^R(\delta x, \Phi, \Lambda)$$

whenever the right-hand side converges. Suppose that $\Lambda \in \mathfrak{r}$, and $\Lambda = X(\mathfrak{r}) + \Lambda^\vee$, $\Lambda^\vee \in \mathfrak{r}_C^\vee$, $\operatorname{Re} \Lambda^\vee \in \rho_R + \mathfrak{a}_R^+$. Then for any small positive ε,

$$\begin{aligned}
\operatorname{Res}_{\mathfrak{r}} E(x, \Phi, \Lambda) &= \frac{1}{2\pi i} \int_0^{2\pi} E(x, \Phi, \Lambda + \varepsilon e^{2\pi i\theta} X(\mathfrak{r}))\, d\theta \\
&= \sum_{R(Q)\backslash G(Q)} \left(\frac{1}{2\pi i} \int_0^{2\pi} E^R(\delta x, \Phi, \Lambda + \varepsilon e^{2\pi i\theta} X(\mathfrak{r}))\, d\theta \right) \\
&= \sum_{R(Q)\backslash G(Q)} \Phi^\vee(\delta x) \exp(\langle \Lambda^\vee + \rho_R, H_R(\delta x) \rangle),
\end{aligned}$$

where

$$\Phi^\vee(y) = \frac{1}{2\pi i} \int_0^{2\pi} E^R(y, \Phi, (1 + \varepsilon) X(\mathfrak{r}) e^{2\pi i\theta})\, d\theta,$$

the residue at $X(\mathfrak{r})$ of an Eisenstein series in one variable. One shows that the function $m \to \Phi^\vee(my)$, $m \in M_R(Q)\backslash M_R(A)$, is in the discrete spectrum. Thus

$$\operatorname{Res}_{\mathfrak{r}} E(x, \Phi, \Lambda) = E(x, \Phi^\vee, \Lambda^\vee),$$

the Eisenstein series over $R(Q)\backslash G(Q)$ associated to a vector $\Phi^\vee$ in $\mathscr{H}_R \backslash \mathscr{H}_{R,\text{cusp}}$. Its analytic continuation is immediate. Let $\mathscr{H}_{R,\chi}$ be the finite dimensional subspace of $\mathscr{H}_R$ consisting of all those vectors $\Phi^\vee$. By examining $\operatorname{Res}_{\mathfrak{r}} M(s, \Lambda)$, one obtains the operators

$$M(s, \Lambda^\vee) : \mathscr{H}_{R,\chi} \to \mathscr{H}_{Q,\chi} \qquad \Lambda^\vee \in \mathfrak{a}_{R,C},$$

for $s \in \Omega(\mathfrak{a}_R, \mathfrak{a}_Q)$. Their analytic continuation then comes without much difficulty.

Let $\mathscr{P}'$ be the class of parabolic subgroups associated to R. In carrying out the second stage of the induction one defines subspaces $L^2_{\mathscr{P}',\chi}(G(Q)\backslash G(A)) \subset L^2_\chi(G(Q)\backslash G(A))$ and $\hat{L}^2_{\mathscr{P}',\chi} \subset \hat{L}^2_{\mathscr{P}'}$, and as above, obtains an isomorphism between them. In the process, one proves the functional equations in (a) of the Main Theorem for vectors $\Phi^\vee \in \mathscr{H}_{R,\chi}$.

The pattern is clear. For R now *any* standard parabolic subgroup and $\mathscr{P}$ *any* associated class, one eventually obtains a definition of spaces $\mathscr{H}_{R,\chi}$, $\hat{L}_{\mathscr{P},\chi}$ and $L^2_{\mathscr{P},\chi}(G(Q)\backslash G(A))$. By definition, $\mathscr{H}_{R,\chi}$ is $\{0\}$ unless R contains an element of $\mathscr{P}_\chi$, and the other two spaces are $\{0\}$ unless an element of $\mathscr{P}$ contains an element of $\mathscr{P}_\chi$. If $\mathscr{P}$ is the associated class of R, there corresponds a stage of the induction in which one proves part (a) of the Main Theorem for vectors $\Phi^\vee$ in $\mathscr{H}_{R,\chi}$ and part (b) for the spaces $\hat{L}_{\mathscr{P},\chi}$ and $L^2_{\mathscr{P},\chi}(G(Q)\backslash G(A))$. Finally, one shows that

$$L^2_\chi(G(Q)\backslash G(A)) = \bigoplus_{\mathscr{P}} L^2_{\mathscr{P},\chi}(G(Q)\backslash G(A)).$$

Notice that if R and $\mathscr{P}$ are fixed,

$$\mathscr{H}_R = \bigoplus_\chi \mathscr{H}_{R,\chi}, \quad \hat{L}_{\mathscr{P}} = \bigoplus_\chi \hat{L}_{\mathscr{P},\chi},$$

and

$$L^2_{\mathscr{P}}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})) = \bigoplus_\chi L^2_{\mathscr{P},\chi}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})).$$

The last decomposition together with the Main Theorem yields

$$L^2(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})) = \bigoplus_{\mathscr{P},\chi} L^2_{\mathscr{P},\chi}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})). \tag{2}$$

This completes our description of the proof of the Main Theorem. It is perhaps a little too glib. For one thing, we have not explained why it suffices to consider only those $\mathfrak{r}$ above such that $\mathfrak{r}^\vee = \{\Lambda \in \mathfrak{a} : \langle \alpha, \Lambda \rangle = 0\}$. Moreover, we neglected to mention a number of serious complications that arise in higher stages of the induction. Some of them are described in Appendix III of **[1]**. We shall only remark that most of the complications exist because eventually one has to study points $X(\mathfrak{r})$ and $Z(\mathfrak{r})$ which lie *outside* the chamber $\mathfrak{a}^+$, where the behavior of the functions $M(s, \Lambda)$ is a total mystery.

PART II. THE TRACE FORMULA

In this section we shall describe a trace formula for G. We have not yet been able to prove as explicit a formula as we would like for general G. We shall give a more explicit formula for GL_3 in the next section. In the past most results have been for groups of rank one. The main references are

4. H. Jacquet and R. Langlands, *Automorphic forms on* GL(2), Springer-Verlag, Berlin, 1970.

5. M. Duflo and J. Labesse, *Sur la formule des traces de Selberg*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 193–284.

6. J. Arthur, *The Selberg trace formula for groups of F-rank one*, Ann. of Math. (2) **100** (1974), 326–385.

We shall also quote from

7. J. Arthur, *The characters of discrete series as orbital integrals*, Invent. Math. **32** (1976), 205–261.

Let R be the regular representation of $G(\boldsymbol{A})$ on $L^2(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$. If $\xi \in i\mathfrak{z}$, recall that R_ξ is the twisted representation on $L^2(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$ given by $R_\xi(x) = R(x)\exp(\langle \xi, H_G(x)\rangle)$. We are really interested in the regular representation of $G(\boldsymbol{A})$ on $L^2(G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$; but this representation is a direct integral over $\xi \in i\mathfrak{z}$ of the representations R_ξ, so it is good enough to study these latter ones. The decomposition (2), quoted in Part I, is equivalent to

$$L^2(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})) = \bigoplus_{\mathscr{P},\chi} L^2_{\mathscr{P},\chi}(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})).$$

Suppose that $f \in C_c^\infty(G(\boldsymbol{A}))^K$. Then this last decomposition is invariant under the operator $R_\xi(f)$. $R_\xi(f)$ is an integral operator with kernel

$$K(x, y) = \sum_{\gamma \in G(\boldsymbol{Q})} f_\xi(x^{-1}\gamma y),$$

where

$$f_\xi(u) = \int_{Z(\mathbf{R})^0} f(zu) \exp(\langle \xi, H_G(zu)\rangle)\, dz, \qquad u \in Z(\mathbf{R})^0\backslash G(\mathbf{A}).$$

The following result is essentially due to Duflo and Labesse.

LEMMA 1. *For every $N \geqq 0$ we can express f as a finite sum of functions of the form*

$$f^1 * f^2, \qquad f^i \in C_c^N(G(\mathbf{A}))^K. \quad \square$$

Let $\mathscr{S}_E(G)$ be the set of $\chi = (\mathscr{P}, \mathscr{V}, W)$ in $\mathscr{S}(G)$ such that $\mathscr{P} \neq \{G\}$. Let $R_{E,\xi}(f)$ be the restriction of $R_\xi(f)$ to $\bigoplus_{\mathscr{P}} \bigoplus_{\chi \in \mathscr{S}_E(G)} L^2_{\mathscr{P},\chi}\ (Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(\mathbf{A}))$. Then

$$R_{\mathrm{cusp},\xi}(f) = R_\xi(f) - R_{E,\xi}(f)$$

is the restriction of $R_\xi(f)$ to the space of cusp forms. It is a finite sum of compositions $R_{\mathrm{cusp},\xi}(f^1) R_{\mathrm{cusp},\xi}(f^2)$ of Hilbert-Schmidt operators and so is of trace class. For each P and χ let $\mathscr{B}_{P,\chi}$ be a fixed orthonormal basis of the finite dimensional space $\mathscr{H}_{P,\chi}$. Finally, recall that $\mathfrak{a}^G = \mathfrak{a}_P^G$ is the orthogonal complement of $\mathfrak{z} = \mathfrak{a}_G$ in $\mathfrak{a}$. If Λ is in $i\mathfrak{a}^G$, we shall write Λ_ξ for the vector $\Lambda + \xi$ in $i\mathfrak{a}$.

LEMMA 2. *$R_E(f)$ is an integral operator with kernel*

$$K_E(x, y) = \sum_{\chi \in \mathscr{S}_E(G)} \sum_P n(A)^{-1} \left(\frac{1}{2\pi i}\right)^{\dim(A/Z)} \cdot \int_{i\mathfrak{a}^G} \left\{ \sum_{\Phi \in \mathscr{B}_{P,\chi}} E(x, I_P(\Lambda_\xi, f)\Phi, \Lambda) \overline{E(y, \Phi, \Lambda)} \right\} d\Lambda.$$

The lemma would follow from the spectral decomposition described in the last section if we could show that the integral over Λ and sum over χ converged and was locally bounded. We can assume that $f = f^1 * f^2$. If

$$K_{P,\chi}(f, x, y) = \sum_{\Phi \in \mathscr{B}_{P,\chi}} E(x, I_P(\Lambda, f)\, \Phi, \Lambda) \overline{E(y, \Phi, \Lambda)},$$

a finite sum, then one easily verifies that

$$|K_{P,\chi}(f, x, y)| \leq K_{P,\chi}(f^1 * (f^1)^*, x, x)^{1/2} K_{P,\chi}((f^2)^* * f^2, y, y)^{1/2}.$$

By applying Schwartz' inequality to the sum over χ, P and the integral over Λ, we reduce to the case that $f = f^1 * (f^1)^*$ and $x = y$. But then $R_{E,\xi}(f)$ is the restriction of the positive semidefinite operator $R_\xi(f)$ to an invariant subspace. The integrand in the expression for $K_E(x, x)$ is nonnegative, and the integral itself is bounded by $K(x, x)$. This proves the lemma. $\square$

The proof of the lemma can be modified to show that $K_E(x, y)$ is continuous in each variable. The same is therefore true of

$$K_{\mathrm{cusp}}(x, y) = K(x, y) - K_E(x, y).$$

One proves without difficulty

LEMMA 3. *The trace of $R_{\mathrm{cusp},\xi}(f)$ equals*

$$\int_{Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(\mathbf{A})} K_{\mathrm{cusp}}(x, x)\, dx. \quad \square$$

Of course neither K nor K_E is integrable over the diagonal. It turns out, however, that there is a natural way to modify the kernels so that they are integrable. Given P, let τ_P (resp. $\hat{\tau}_P$) be the characteristic function of $\{H \in \mathfrak{a}_0: \alpha(H) > 0, \alpha \in \Phi_P$ (resp. $\mu(H) > 0, \mu \in \hat{\Phi}_P)\}$. (Recall that $\hat{\Phi}_P$ is the basis of $\mathfrak{a}^G$ which is dual to Φ_P.) Then $\tau_P \leqq \hat{\tau}_P$. Suppose that $T \in \mathfrak{a}_0$.

LEMMA 4. *For any P, $\sum_{\delta \in P(Q)\backslash G(Q)} \hat{\tau}_P(H(\delta x) - T)$ is a locally bounded function of $x \in G(A)$. In particular the sum is finite.* □

We will now take $T \in \mathfrak{a}_0^+$. We shall assume that the distance from T to each of the walls of $\mathfrak{a}_0^+$ is arbitrarily large. We shall modify $K(x, x)$ by regarding it as the term corresponding to $P = G$ in a sum of functions indexed by P. If x remains within a large compact subset of $Z(\boldsymbol{R})^0G(Q)\backslash G(A)$, depending on T, the functions corresponding to $P \neq G$ will vanish. They are defined in terms of

$$K^P(x, y) = \int_{N(A)} \sum_{\mu \subset M(Q)} f_\xi(x^{-1}n\mu y)\, dn,$$

the kernel of $R_\xi^P(f)$, where R^P is the regular representation of $G(A)$ on $L^2\left(Z(\boldsymbol{R})^0N(A)M(Q)\backslash G(A)\right)$. Define the modified function to be

$$k^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \subset P(Q)\backslash G(Q)} K^P(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T).$$

For each x this is a finite sum. The function obtained turns out to be integrable over $Z(\boldsymbol{R})^0G(Q)\backslash G(A)$. We shall give a fairly detailed sketch of the proof of this fact because it is typical of the proofs of later results, which we shall only state.

We begin by partitioning $Z(\boldsymbol{R})^0 \cdot G(Q)\backslash G(A)$ into disjoint sets, indexed by P, which depend on T. Fix a Siegel set $\mathfrak{r}$ in $Z(\boldsymbol{R})^0\backslash G(A)$ such that $Z(\boldsymbol{R})^0\backslash G(A) = G(Q)\mathfrak{r}$. Consider the set of $x \in \mathfrak{r}$ such that $\mu(H_0(x) - T) < 0$ for every $\mu \in \hat{\Phi}_0$. It is a compact subset of $\mathfrak{r}$. The projection, $G(T)$, of this set onto $Z(\boldsymbol{R})^0G(Q)\backslash G(A)$ remains compact. For any P we can repeat this process on M, to obtain a compact subset of $A(\boldsymbol{R})^0M(Q)\backslash M(A)$, which of course depends on T. Let $F^P(m, T)$ be its characteristic function. Extend it to a function on $G(A)$ by

$$F^P(nmk, T) = F^P(m, T), \qquad n \in N(A), m \in M(A), k \in K.$$

This gives a function on $N(A)M(Q)A(\boldsymbol{R})^0\backslash G(A)$. If $Q \subset P$, define τ_Q^P (resp. $\hat{\tau}_Q^P$) to be the characteristic function of

$$\{H \in \mathfrak{a}_0: \alpha(H) > 0, \alpha \in \Phi_Q^P\,(\text{resp. } \mu(H) > 0, \mu \in \hat{\Phi}_Q^P)\}.$$

The following lemma gives our partition of $N(A)M(Q)A(\boldsymbol{R})^0\backslash G(A)$. It is essentially a restatement of standard results from reduction theory.

LEMMA 5. *Given P,*

$$\sum_{\{Q: P_0 \subset Q \subset P\}} \sum_{\delta \in Q(Q)\backslash P(Q)} F^Q(\delta x, T)\tau_Q^P(H(\delta x) - T)$$

equals 1 for almost all $x \in G(A)$. □

We can now study the function $k^T(x)$. It equals

$$\sum_{\{Q,\, P:P_0\subset Q\subset P\}} (-1)^{\dim(A/Z)} \sum_{\delta\in Q(\mathbf{Q})\backslash G(\mathbf{Q})} F^Q(\delta x,\, T)\tau_Q^P(H(\delta x) - T) \cdot \hat{\tau}_P(H(\delta x) - T)K^P(\delta x, \delta x).$$

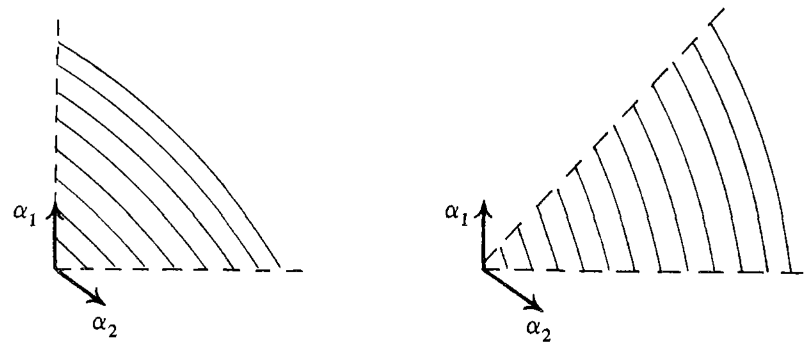
Now for any $H \in \mathfrak{a}_0$ we can surely write

$$\tau_Q^P(H)\hat{\tau}_P(H) = \sum_{\{P_1,\, P_2:P\subset P_1\subset P_2\}} (-1)^{\dim(A_1/A_2)}\tau_Q^2(H)\hat{\tau}_2(H) = \sum_{\{P_1:P_1\supset P\}} \sigma_Q^1(H),$$

where

$$\sigma_Q^1(H) = \sigma_Q^{P_1}(H) = \sum_{\{P_2:P_2\supset P_1\}} (-1)^{\dim(A_1/A_2)}\tau_Q^2(H)\hat{\tau}_2(H).$$

To study σ_Q^1, consider the case that $G = \mathrm{GL}_3$. Then $\mathfrak{a}_0/\mathfrak{z}$ is two dimensional, spanned by simple roots α_1 and α_2. Let $Q = P_0$, and let P_1 be the maximal parabolic subgroup such that $\mathfrak{a}_1 = \{H \in \mathfrak{a}_0\colon \alpha_1(H) = 0\}$. Then σ_Q^1 is the difference of the characteristic functions of the following sets:



$P_2 = P_1$ $\qquad$ $P_2 = G$

The next lemma generalizes what is clear from the diagrams.

LEMMA 6. *Suppose that H is a vector in the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{a}_Q$. If $\sigma_Q^1(H) \neq 0$, and $H = H_* + H^*$, $H_* \in \mathfrak{a}_Q^1$, $H^* \in \mathfrak{a}_1$, then $\alpha(H_*) > 0$ for each $\alpha \in \Phi_Q^1$ and $\|H^*\| < c\|H_*\|$ for a constant c depending only on G. In other words H^* belongs to a compact set, while H_* belongs to the positive chamber in $\mathfrak{a}_Q^1$.* $\square$

We write $k^T(x)$ as

$$\sum_{Q\subset P_1} \sum_{\delta\in Q(\mathbf{Q})\backslash G(\mathbf{Q})} F^Q(\delta x,\, T)\,\sigma_Q^1(H(\delta x) - T) \sum_{\{P:Q\subset P\subset P_1\}} (-1)^{\dim(A/Z)}K^P(\delta x, \delta x).$$

The integral over $Z(\mathbf{R})^0G(\mathbf{Q})\backslash G(\mathbf{A})$ of the absolute value of $k^T(x)$ is bounded by the sum over $Q \subset P_1$ of the integral over $Q(\mathbf{Q})Z(\mathbf{R})^0\backslash G(\mathbf{A})$ of the product of $F^Q(x, T)\sigma_Q^1(H(x) - T)$ and

$$\left| \sum_{\{P:Q\subset P\subset P_1\}} (-1)^{\dim(A/Z)} \int_{N(\mathbf{A})} \sum_{\mu\in M(\mathbf{Q})} f_\xi(x^{-1}\, n\mu x)\, dn \right|. \tag{1}$$

We can assume that for a given x the first function does not vanish. Then by the last lemma, the projection of $H_Q(x)$ onto $\mathfrak{a}_Q^1$ is large. Conjugation by x^{-1} tends to stretch any element $\mu \in M(Q)$ which does not lie in $Q(Q)$. Since f is compactly supported, we can choose T so large that the only μ which contribute nonzero summands in (1) belong to $Q^P(Q) = Q(Q) \cap M(Q) = M_Q(Q)N_Q^P(Q)$. Thus, (1) equals

$$\left| \sum_{\mu \in M_Q(Q)} \sum_{\{P: Q \subset P \subset P_1\}} (-1)^{\dim(A/Z)} \int_{N(A)} \sum_{\nu \in N_Q^P(Q)} f_\xi(x^{-1}\mu\nu n x)\, dn \right|,$$

which is bounded by

$$\sum_{\mu \in M_Q(Q)} \left| \sum_{\{P: Q \subset P \subset P_1\}} (-1)^{\dim(A/Z)} \int_{N(A)} \sum_{\nu \in N_Q^P(Q)} f_\xi(x^{-1}\mu\nu n x)\, dn \right|.$$

Recall that $\mathfrak{n}_0$ is the Lie algebra of N_0. Let $\langle\ ,\ \rangle$ denote the canonical bilinear form on $\mathfrak{n}_0$. If $X \in \mathfrak{n}_0$, let $e(X)$ be the matrix in N_0 obtained by adding X to the identity. Then e is an isomorphism (of varieties over Q) from $\mathfrak{n}_0$ onto N_0. Let ψ be a nontrivial character on A/Q. Applying the Poisson summation formula to $\mathfrak{n}_Q^P$, we see that (1) is bounded by

$$\sum_\mu \left| \sum_P (-1)^{\dim(A/Z)} \sum_{\zeta \in \mathfrak{n}_Q^P(Q)} \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu e(X) x)\psi(\langle X, \zeta\rangle)\, dX \right|.$$

If $\mathfrak{n}_Q^1(Q)'$ is the set of elements in $\mathfrak{n}_Q^1(Q)$, which do not belong to any $\mathfrak{n}_Q^P(Q)$ with $Q \subset P \subsetneqq P_1$, this expression equals

$$\sum_\mu \left| \sum_{\zeta \in \mathfrak{n}_Q^1(Q)'} \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu e(X) x)\psi(\langle X, \zeta\rangle)\, dX \right|.$$

Therefore the integral of $|k^T(x)|$ is bounded by the sum over $Q \subset P_1$ and $\mu \in M_Q(Q)$ of the integral over (n^*, n_*, a, m, k) in

$$N_1(Q)\backslash N_1(A) \times N_Q^1(Q)\backslash N_Q^1(A) \times Z(R)^0\backslash A_Q(R)^0 \times A_Q(R)^0 M_Q(Q)\backslash M_Q(A) \times K$$

of

$$F^Q(m, T)\, \sigma_Q^1(H(a) - T) \exp(-2\langle \rho_Q, H(a)\rangle)$$

$$\cdot \sum_{\zeta \in \mathfrak{n}_Q^1(Q)'} \left| \int_{\mathfrak{n}_Q(A)} f_\xi(k^{-1}m^{-1}a^{-1}n_*^{-1}n^{*-1} \cdot \mu e(x)\cdot n^* n_* amk)\psi(\langle X, \zeta\rangle)\, dX \right|.$$

The integral over n^* goes out. The integrals over k and m can be taken over compact sets. It follows from the last lemma that the set of points $\{a^{-1} n_* a\}$, indexed by those n_* and a for which the integrand is not zero, is relatively compact. Therefore there is a compact set C in $Z(R)^0\backslash G(A)$ such that the integral of $|k^T(x)|$ is bounded by the sum over $Q \subset P_1$, $\mu \in M_Q(Q)$ and the integral over $x \in C$ of

$$\int_{Z(R)^0\backslash A_Q(R)^0} \exp(-2\langle \rho_Q, H(a)\rangle)\, \sigma_Q^1(H(a) - T)$$

$$\cdot \sum_{\zeta \in \mathfrak{n}_Q^1(Q)'} \left| \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu a^{-1} e(X) a x)\psi(\langle X, \zeta\rangle)\, dX \right| da$$

$$= \int_{Z(R)^0\backslash A_Q(R)^0} \sigma_Q^1(H(a) - T) \sum_\zeta \left| \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu e(X) x) \cdot \psi(\langle X, \mathrm{Ad}(a)\zeta\rangle)\, dX \right| da.$$

Since f is compactly supported, the sum over μ is finite. If $Q = P_1$, σ_Q^1 equals 0, unless of course $Q = P_1 = G$, when it equals 1. If $Q \subsetneqq P_1$,

$$Y \to \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu e(X)x)\psi(\langle X, Y\rangle)\,dX, \qquad Y \in \mathfrak{n}_Q^1(A),$$

is the Fourier transform of a Schwartz-Bruhat function on $\mathfrak{n}_Q^1(A)$, and is continuous in x. If $H(a) = H_* + H^*$, $H_* \in \mathfrak{a}_Q^1$, $H^* \in \mathfrak{a}_1/\mathfrak{z}$, H^* must remain in a compact set. Since H_* lies in the positive chamber of $\mathfrak{a}_Q^1$, far from the walls, $\mathrm{Ad}(a)$ stretches any element ζ in $\mathfrak{n}_Q^1(Q)'$. In fact as H_* goes to infinity in any direction, $\mathrm{Ad}(a)\zeta$ goes to infinity. Here it is crucial that ζ not belong to any $\mathfrak{n}_Q^P(Q)$, $Q \subset P \subsetneqq P_1$. It follows that if $Q \subsetneqq P_1$, the corresponding term is finite and goes to 0 exponentially in T. Thus the dominant term is the only one left, that corresponding to $Q = P_1 = G$. It is an integral over the compact set $G(T)$. We have sketched the proof of

THEOREM 1. *We can choose $\varepsilon > 0$ such that for any $T \in \mathfrak{a}_0^+$, sufficiently far from the walls,*

$$\int_{Z(R)^0G(Q)\backslash G(A)} k^T(x)\,dx = \int_{G(T)} K(x, x)\,dx + O(e^{-\varepsilon\|T\|}). \quad \square$$

We would expect the integral of $k^T(x)$ to break up into a sum of terms corresponding to conjugacy classes in $G(Q)$. It seems, however, that a certain equivalence relation in $G(Q)$, weaker than conjugacy, is more appropriate. If $\mu \in G(Q)$, let μ_s be its semisimple component relative to the Jordan decomposition. Call two elements μ and μ' in $G(Q)$ *equivalent* if μ_s and μ'_s are $G(Q)$-conjugate. Let $\mathscr{C}$ be the set of equivalence classes in $G(Q)$. If $\mathfrak{o} \in \mathscr{C}$, define

$$K_{\mathfrak{o}}^P(x, y) = \sum_{\mu \in M(Q)\cap\mathfrak{o}} \int_{N(A)} f_\xi(x^{-1}\mu n y)\,dn,$$

and

$$k_{\mathfrak{o}}^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(Q)\backslash G(Q)} K_{\mathfrak{o}}^P(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T).$$

Then

$$k^T(x) = \sum_{\mathfrak{o} \in \mathscr{C}} k_{\mathfrak{o}}^T(x).$$

If $\mu \in G$ and H is a closed subgroup of G, let H_μ denote the centralizer of μ in H.

LEMMA 7. *Fix P. Then for $\mu \in M(Q)$ and $\phi \in C_c^\infty(N(A))$,*

$$\sum_{\zeta \in N_{\mu_s}(Q)\backslash N(Q)} \sum_{\nu \in N_{\mu_s}(Q)} \phi(\mu^{-1}\zeta^{-1}\mu\nu\zeta) = \sum_{\nu \in N(Q)} \phi(\nu). \quad \square$$

This lemma is easily proved. It implies that for $\mathfrak{o} \in \mathscr{C}$, $P(Q) \cap \mathfrak{o} = (M(Q) \cap \mathfrak{o})N(Q)$. If this fact is combined with the proof of Theorem 1 we obtain a stronger version.

THEOREM 1*. *There are positive constants C and ε such that*

$$\sum_{\mathfrak{o} \in \mathscr{C}} \int_{Z(R)^0G(Q)\backslash G(A)} |k_{\mathfrak{o}}^T(x) - F^G(x, T)K_{\mathfrak{o}}^G(x, x)|\,dx \leqq Ce^{-\varepsilon\|T\|}. \quad \square$$

The integral of $k_{\mathfrak{o}}^T(x)$ cannot be computed yet. What we must do is replace $k_{\mathfrak{o}}^T(x)$ by a different function. Define

$$J_{\mathfrak{o}}^P(x, y) = \sum_{\mu \in M(\mathbf{Q}) \cap \mathfrak{o}} \sum_{\zeta \in N_{\mu_s}(\mathbf{Q}) \backslash N(\mathbf{Q})} \int_{N_{\mu_s}(A)} f_\xi(x^{-1}\zeta^{-1}\mu n \zeta y)\, dn.$$

It is obtained from $K_{\mathfrak{o}}^P(x, y)$ by replacing a part of the integral over $N(A)$ by the corresponding sum over $\mathbf{Q}$-rational points. Define

$$j_{\mathfrak{o}}^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(\mathbf{Q}) \backslash G(\mathbf{Q})} J_{\mathfrak{o}}^P(\delta x, \delta x) \hat{\tau}_P(H(\delta x) - T).$$

The proof of the following is similar to that of Theorem 1*.

THEOREM 2. *There are positive constants C and ε such that*

$$\sum_{\mathfrak{o} \in \mathscr{C}} \int_{Z(\mathbf{R})^0 G(\mathbf{Q}) \backslash G(A)} \left| j_{\mathfrak{o}}^T(x) - F^G(x, T) K_{\mathfrak{o}}^G(x, x) \right| dx \leqq C e^{-\varepsilon \|T\|}. \quad \square$$

Suppose that T_1 is a point in $T + \mathfrak{a}_0^+$. By integrating the difference of $k_{\mathfrak{o}}^T(x)$ and $k_{\mathfrak{o}}^{T_1}(x)$ one proves inductively that the integral of $k_{\mathfrak{o}}^T(x)$ is a polynomial in T. The same goes for the integral of $j_{\mathfrak{o}}^T(x)$. Since the integrals of $k_{\mathfrak{o}}^T(x)$ and $j_{\mathfrak{o}}^T(x)$ differ by an expression which approaches 0 as T approaches ∞, they must be equal. Summarizing what we have said so far, we have

THEOREM 3.

$$\int_{Z(\mathbf{R})^0 G(\mathbf{Q}) \backslash G(A)} k^T(x)\, dx = \sum J_{\mathfrak{o}}^T(f_\xi),$$

where $J_{\mathfrak{o}}^T(f_\xi) = \int_{Z(\mathbf{R})^0 G(\mathbf{Q}) \backslash G(A)} j_{\mathfrak{o}}^T(x)\, dx$. $\square$

Suppose that the class $\mathfrak{o}$ consists entirely of semisimple elements, so that $\mathfrak{o}$ is an actual conjugacy class. The centralizer of any element in $\mathfrak{o}$ is anisotropic modulo its center. The split component of the center is $G(\mathbf{Q})$-conjugate to the split component of a standard parabolic subgroup. Thus, we can find $\mu_0 \in \mathfrak{o}$ and a standard parabolic P_0 such that the identity component of G_{μ_0} is contained in M_0 and is anisotropic modulo A_0. We shall say that the class $\mathfrak{o}$ is *unramified* if G_{μ_0} itself is contained in M_0. Assume that this is the case. We shall show how to express $J_{\mathfrak{o}}^T(f_\xi)$ as a weighted orbital integral of f_ξ.

Given any P, suppose that $\mu \in \mathfrak{o} \cap M(\mathbf{Q})$. Then by the same argument, we can choose an element s in $\bigcup_{P_1} \Omega(\mathfrak{a}_0, \mathfrak{a}_1)$ and an element $\eta \in M(\mathbf{Q})$ such that $s\mathfrak{a}_0$ contains $\mathfrak{a}$, and $\mu = \eta w_s \mu_0 w_s^{-1} \eta^{-1}$. Let $\Omega(\mathfrak{a}_0; P)$ be the set of elements s in $\bigcup_{P_1} \Omega(\mathfrak{a}_0, \mathfrak{a}_1)$ such that if $\mathfrak{a}_1 = s\mathfrak{a}_0$, $\mathfrak{a}_1$ contains $\mathfrak{a}$, and $s^{-1}\alpha$ is positive for every root α in $\Phi_{P_1}^P$. Then if we demand that the element s above lie in $\Omega(\mathfrak{a}_0; P)$, it is uniquely determined. Thus $J_{\mathfrak{o}}^P(\delta x, \delta x)$ equals

$$\sum_{s \in \Omega(\mathfrak{a}_0; P)} \sum_{\eta \in M_{w_s \mu_0 w_s^{-1}}(\mathbf{Q}) \backslash M(\mathbf{Q})} \sum_{\zeta \in N(\mathbf{Q})} f_\xi(x^{-1}\delta^{-1}\zeta^{-1}\eta^{-1} w_s \mu_0 w_s^{-1} \eta \zeta \delta x)$$

$$= \sum_{s \in \Omega(\mathfrak{a}_0; P)} \sum_{\zeta \in M_{w_s \mu_0 w_s}(\mathbf{Q}) \backslash P(\mathbf{Q})} f_\xi(x^{-1}\delta^{-1}\zeta^{-1} w_s \mu_0 w_s^{-1} \zeta \delta x).$$

Therefore $j_{\mathfrak{o}}^T(x)$ equals

$$\sum_P (-1)^{\dim(A/Z)} \sum_{s\in\Omega(\mathfrak{a}_0;P)} \sum_{\delta\in M_{w_s\mu_0 w_s^{-1}}(\mathbf{Q})\backslash G(\mathbf{Q})} f_\xi(x^{-1}\delta^{-1}w_s\mu_0 w_s^{-1}\delta x)\hat{\tau}_P(H(\delta x) - T).$$

Since the centralizer of $w_s\mu_0 w_s^{-1}$ in G is contained in M, this equals

$$\sum_{\delta\in G_{\mu_0}(\mathbf{Q})\backslash G(\mathbf{Q})} f_\xi(x^{-1}\delta^{-1}\mu\delta x) \sum_P (-1)^{\dim(A/Z)} \sum_{s\in\Omega(\mathfrak{a}_0;P)} \hat{\tau}_P(H(w_s\delta x) - T).$$

Then $J_{\mathfrak{o}}^T(f_\xi)$ equals

$$\text{(2)}\qquad \operatorname{vol}(A_0(\mathbf{R})^0 G_{\mu_0}(\mathbf{Q})\backslash G_{\mu_0}(\mathbf{A})) \int_{G_{\mu_0}(\mathbf{A})\backslash G(\mathbf{A})} f_\xi(x^{-1}\mu x)v(x, T)\,dx,$$

where

$$v(x, T) = \int_{Z(\mathbf{R})^0\backslash A_0(\mathbf{R})^0} \left\{ \sum_P (-1)^{\dim(A/Z)} \sum_{s\in\Omega(\mathfrak{a}_0;P)} \hat{\tau}_P(H(w_s a x) - T)\,da \right\}.$$

The expression in the brackets is compactly supported in a. In fact it follows from the results of [7, §§2, 3] that $v(x, T)$ equals the volume in $\mathfrak{a}_0/\mathfrak{z}$ of the convex hull of the projection onto $\mathfrak{a}_0/\mathfrak{z}$ of $\{s^{-1}T - s^{-1}H(w_s x);\ s \in \bigcup_{P_1}\Omega(\mathfrak{a}_0, \mathfrak{a}_1)\}$. It was Langlands who surmised that the volume of a convex hull would play a role in the trace formula.

By studying how far $J_{\mathfrak{o}}^T(f_\xi)$ differs from an invariant distribution, I hope to express $J_{\mathfrak{o}}^T(f_\xi)$, for general $\mathfrak{o}$, as a limit of the distributions for which $\mathfrak{o}$ is as above, at least modulo an invariant distribution that lives on the unipotent set of $G(\mathbf{A})$. However, this has not yet been done.

The study of $K_E(x, x)$ parallels what we have just done. The place of $\{\mathfrak{o}\in\mathscr{C}\}$ is now taken by $\{\chi \in \mathscr{S}_E(G)\}$. Given P, and $\chi \in \mathscr{S}_E(G)$, define

$$K_\chi^P(x, y) = \sum_{Q\subset P} n^P(A_Q)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim(A_Q/Z)} \cdot \int_{i\mathfrak{a}_Q^G} \left\{ \sum_{\Phi\in\mathscr{B}_{Q,\chi}} E^P(x, I_Q(\Lambda_\xi, f)\Phi, \Lambda)\,\overline{E^P(y, \Phi, \Lambda)} \right\} d\Lambda,$$

where $n^P(A_Q)$ is the number of chambers in $\mathfrak{a}_Q/\mathfrak{a}$. Then if $P \neq G$,

$$K^P(x, y) = \sum_{\chi\in\mathscr{S}_E(G)} K_\chi^P(x, y).$$

The convergence of the sum over χ and the above integral over Λ is established by the argument of Lemma 2. Define

$$k_\chi^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta\in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_\chi^P(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T).$$

Then

$$K_{\text{cusp}}(x, x) = \sum_{\mathfrak{o}\in\mathscr{C}} k_{\mathfrak{o}}^T(x) - \sum_{\chi\in\mathscr{S}_E(G)} k_\chi^T(x).$$

We would like to be able to integrate the function $k_\chi^T(x)$. But as before, we will have to replace it with a new function $j_\chi^T(x)$ before this can be done.

To define the new function, we need to introduce a truncation operator. In form it resembles the way we modified the kernel $K(x, x)$, but it applies to any continuous function ϕ on $Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})$. Define a new function on $Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})$ by

$$(\Lambda^T \phi)(x) = \sum_{P \supset P_0} (-1)^{\dim(A/Z)} \sum_{\delta \in P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})} \hat{\tau}_P(H(\delta x) - T) \cdot \int_{N(\boldsymbol{Q})\backslash N(\boldsymbol{A})} \phi(n\delta x)\, dn.$$

Λ^T has some agreeable properties. It leaves any cusp form invariant. It is a self-adjoint operator. These facts are clear. It is less clear, but true, that $\Lambda^T \circ \Lambda^T = \Lambda^T$. If $\chi \in \mathscr{S}_E(G)$, define

$$J_\chi^T(x, y) = \sum_P n(A)^{-1} \left(\frac{1}{2\pi i}\right)^{\dim(A/Z)} \cdot \int_{i\mathfrak{a}^G} \left\{ \sum_{\phi \in \mathscr{B}_{P,\chi}} E(x, I_P(\Lambda_\xi, f)\Phi, \Lambda) \overline{\Lambda^T E(y, \Phi, \Lambda)} \right\} d\Lambda.$$

If we apply Λ^T to the second variable in $K_E(x, y)$ we obtain $\sum_{\chi \in \mathscr{S}_E(G)} J_\chi^T(x, y)$. Define $j_\chi^T(x) = J_\chi^T(x, x)$.

THEOREM 4. *The function* $\sum_{\chi \in \mathscr{S}_E(G)} |j_\chi^T(x)|$ *is integrable over* $Z(\boldsymbol{R})^0 G(\boldsymbol{Q})\backslash G(\boldsymbol{A})$. *For any* χ, *the integral of* $j_\chi^T(x)$ *equals*

$$\sum_P n(A)^{-1} \left(\frac{1}{2\pi i}\right)^{\dim(A/Z)} \cdot \int_{i\mathfrak{a}^G} \sum_{\Phi \in \mathscr{B}_{P,\chi}} \int_{Z(\boldsymbol{R})^0 G(\boldsymbol{Q})\backslash G(\boldsymbol{A})} \Lambda^T E(x, I_P(\Lambda_\xi, f)\Phi, \Lambda) \overline{\Lambda^T E(x, \Phi, \Lambda)}\, dx\, d\Lambda.$$

The first statement of the theorem comes from a property of Λ^T. Namely, if ϕ is a smooth function on $Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})$ any of whose derivatives (with respect to the universal enveloping algebra of $\mathfrak{g}(\boldsymbol{C})$) are slowly increasing in a certain sense, then $\Lambda^T \phi$ is rapidly decreasing. The proof of this property is similar to the proof of Theorem 2. The Poisson formula can no longer be used, but one uses [**3**, Lemma 10] instead. Given the fact that $\Lambda^T \circ \Lambda^T = \Lambda^T$, the other half of the theorem is a statement about the interchange of the integrals over x and Λ. By the proof of Lemma 2 we can essentially assume the integrand is nonnegative. The result follows. $\square$

THEOREM 5. *There are positive constants C and* ε *such that*

$$\sum_{\chi \in \mathscr{S}_E(G)} \int_{Z(\boldsymbol{R})^0 G(\boldsymbol{Q})\backslash G(\boldsymbol{A})} \left| k_\chi^T(x) - j_\chi^T(x) \right| dx \leqq C e^{-\varepsilon \|T\|}.$$

It turns out that this theorem can be proved by studying the function $k^T(x) - \Lambda_y^T K(x, y)$, at $x = y$. This is essentially the sum over all χ of the above integrand (without the absolute value bars). The point is that the new function is easier to study because it has a manageable expression in terms of f. $\square$

Combining Theorems 4 and 5, we see that $\int_{Z(\boldsymbol{R})^0 G(\boldsymbol{Q})\backslash G(\boldsymbol{A})} \sum_{\chi \in \mathscr{S}_E(G)} |k_\chi^T(x)|\, dx$ is finite. In particular, each $k_\chi^T(x)$ is integrable. With a little more effort it can be shown that the integrals of $k_\chi^T(x)$ and $j_\chi^T(x)$ are actually equal. We shall denote the common value by $J_\chi^T(f_\xi)$. It is a polynomial in T. We have shown that

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\mathfrak{o}\in\mathscr{C}} J^T_{\mathfrak{o}}(f_\xi) - \sum_{\chi\in\mathscr{S}_E(G)} J^T_\chi(f_\xi)$$

for any suitably large $T \bullet \mathfrak{a}_0^+$. The right-hand side is a polynomial in T while the left-hand side is independent of T. Letting $J_{\mathfrak{o}}(f_\xi)$ and $J_\chi(f_\xi)$ be the constant terms of the polynomials $J^T_{\mathfrak{o}}(f_\xi)$ and $J^T_\chi(f_\xi)$ we have

THEOREM 6. *For any* $f \in C_c^\infty(G(\boldsymbol{A}))^K$,

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\mathfrak{o}\in\mathscr{C}} J_{\mathfrak{o}}(f_\xi) - \sum_{\chi\in\mathscr{S}_E(G)} J_\chi(f_\xi). \quad \square$$

Suppose that $\chi = (\mathscr{P}, \mathscr{V}, W)$ and that $P \bullet \mathscr{P}$. Then if $\Phi \in \mathscr{H}_{P,\chi}$, $\Lambda^T E(x, \Phi, \Lambda)$ equals the function denoted $E''(x, \Phi, \Lambda)$ by Langlands in [2, §9]. This is, in fact, what led us to the definition of Λ^T in the first place. If Φ' is another vector in $\mathscr{H}_{P,\chi}$, Langlands has proved the elegant formula

$$\int_{Z(\boldsymbol{R})^0\cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})} E''(x, \Phi', \Lambda')\overline{E''(x, \Phi, \Lambda)}\, dx$$
$$= \sum_{P_2} \sum_{t\in\Omega(\mathfrak{a},\mathfrak{a}_2)} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_2)} \frac{e^{\langle t\Lambda' + s\bar{\Lambda}, T\rangle}}{\prod_{\alpha\in\Phi_2}\langle t\Lambda' + s\bar{\Lambda}, \alpha\rangle} (M(t, \Lambda')\Phi', M(s, \Lambda)\Phi)$$

(see [2, §9]. Actually the formula quoted by Langlands is slightly more complicated, but it can be reduced to what we have stated.) In this formula, we can set $\Lambda' = \Lambda$ and $\Phi' = I_P(\Lambda, f)\Phi$. We can then sum over all Φ in $\mathscr{B}_{P,\chi}$ and integrate over $\Lambda \in i\mathfrak{a}^G$. The result is *not* a polynomial in T. To obtain $J^T_\chi(f_\xi)$ we would have to consider all P, not just those in the associated class $\mathscr{P}$. The best hope seems to be to calculate residues in Λ and Λ' separately in the above formula. For GL_3 the result turns out to be relatively simple.

PART III. THE CASE OF GL_3

In this last section we shall give the results of further calculations. They can be stated for general G but at this point they can be proved only for $G = \mathrm{GL}_3$. Our aim is to express the trace of $R_{\mathrm{cusp},\xi}(f)$ in terms of the invariant distributions defined in

8. J. Arthur, *On the invariant distributions associated to weighted orbital integrals*, preprint.

A trace formula for K-bi-invariant functions has also been proved in

9. A. B. Venkov, *On Selberg's trace formula for* $\mathrm{SL}_3(\boldsymbol{Z})$, Soviet Math. Dokl. **17** (1976), 683–687.

First we remark that the distributions $J^T_{\mathfrak{o}}(f_\xi)$ and $J^T_\chi(f_\xi)$ are independent of our minimal parabolic subgroup so there is no further need to fix P_0. If A is any $\boldsymbol{Q}$-split torus in G, let $\mathscr{P}(A)$ denote the set of parabolic subgroups with split component A. They are in bijective correspondence with the chambers in A. In fact, if P_0 is a minimal parabolic subgroup contained in an element P of $\mathscr{P}(A)$, then

$$\mathscr{P}(A) = \bigcup_{A_1} \bigcup_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} w_s^{-1} P_1 w_s.$$

If $P' \bullet \mathscr{P}(A)$, and $P' = w_s^{-1} P_1 w_s$, define

$$(M_{P'|P}(\Lambda)\Phi)(x) = (M(s, \Lambda)\Phi)(w_s x), \qquad \Phi \in \mathscr{H}(P).$$

Then $M_{P'|P}(\Lambda)$ is a map from $\mathscr{H}(P)$ to $\mathscr{H}(P')$, which is independent of P_0. In fact, if $\operatorname{Re}\Lambda \in \rho_P + \mathfrak{a}_P^+$,

$$(1)\qquad (M_{P'|P}(\Lambda)\Phi)(x) = \int_{N(A)\cap N'(A)\backslash N'(A)} \Phi(nx)\exp(\langle \Lambda + \rho_P, H_P(nx)\rangle) \cdot \exp(-\langle \Lambda + \rho_{P'}, H_{P'}(x)\rangle)\, dn.$$

We have changed our notation to agree with that of **[8]**.

A is said to be a *special subgroup* of G if $\mathscr{P}(A)$ is not empty. Suppose that A and A_1 are special subgroups, with $A \supset A_1$. We write $\Omega^{M_1}(\mathfrak{a}, \mathfrak{a})_{\text{reg}}$ for the set of elements $s \in \Omega(\mathfrak{a}, \mathfrak{a})$ whose space of fixed vectors in $\mathfrak{a}$ is $\mathfrak{a}_1$. Suppose that $P_1 \in \mathscr{P}(A_1)$ and $Q \in \mathscr{P}^{M_1}(A)$, the set of parabolic subgroups of M_1 with split component A. Then there is a unique group in $\mathscr{P}(A)$, which we denote by $P_1(Q)$, such that $P_1(Q) \subset P_1$ and $P_1(Q) \cap M_1 = Q$.

LEMMA 1. *Suppose that A and A_1 are as above and that $P = P_1(Q)$ for some $P_1 \in \mathscr{P}(A_1)$ and $Q \in \mathscr{P}^{M_1}(A)$. Then if $\Lambda \in i\mathfrak{a}_1$, the limit as λ approaches 0 of*

$$\sum_{P_1 \in \mathscr{P}(A_1)} M_{P_1(Q)|P}(\Lambda)^{-1} M_{P_1(Q)|P}(\Lambda + \lambda)\Big(\prod_{\alpha \in \Phi_{P_1}} \langle \lambda, \alpha\rangle\Big)^{-1}$$

exists as an operator on $\mathscr{H}_P$. We denote it by $M(P, A_1, \Lambda)$.

This lemma follows from **[8]**. □

Fix a maximal special subgroup A_0 of G. From Langlands' inner product formula, quoted at the end of Part II, one can prove

LEMMA 2. *For any $\chi \in \mathscr{S}_E(G)$, $J_\chi(f_\xi)$ equals the sum over all special subgroups A_1 and A of G, with $A_1 \subset A \subset A_0$, and over $s \in \Omega^{M_1}(\mathfrak{a}, \mathfrak{a})_{\text{reg}}$ of*

$$c_s \int_{i\mathfrak{a}_1^G} \sum_{\Phi \in \mathscr{B}_{P,\chi}} (M(P, A_1, \Lambda_\xi) M(s, 0)\, I_P(\Lambda_\xi, f)\Phi, \Phi)\, d\Lambda.$$

Here c_s is the product of

$$\left(\frac{1}{2\pi i}\right)^{\dim(A_1/Z)} n^M(A_0)\cdot n(A_0)^{-1}\cdot \left|\det(1 - \operatorname{Ad}(s))_{\mathfrak{a}/\mathfrak{a}_1}\right|^{-1}$$

with the volume of $\mathfrak{a}_P^1$ modulo the lattice generated by Φ_P^1, and P is any element in $\mathscr{P}(A_1)$ which contains some group in $\mathscr{P}(A)$. □

Recall the decomposition $I_P(\Lambda) = \bigoplus_\iota \bigotimes_v I_P(\sigma_{v,\Lambda}^\iota)$. If Φ_v is a smooth vector in $\mathscr{H}_P(\sigma_v^\iota)$ and $\operatorname{Re}\Lambda \in \rho_P + \mathfrak{a}_P^+$, define

$$(M_{P'|P}(\sigma_{v,\Lambda}^\iota)\Phi_v)(x) = \int_{N(Q_v)\cap N'(Q_v)\backslash N'(Q_v)} \Phi_v(nx)\exp(\langle \Lambda + \rho_P, H_P(nx)\rangle) \cdot \exp(-\langle \Lambda + \rho_{P'}, H_{P'}(x)\rangle)\, dn,$$

for $x \in G(\mathbf{Q}_v)$. This is the usual unnormalized intertwining operator for a group over a local field. Then

$$M_{P'|P}(\Lambda) = \bigoplus_\iota \bigotimes_v M_{P'|P}(\sigma_{v,\Lambda}^\iota).$$

LEMMA 3. *If σ_v is an irreducible unitary representation of $M(\mathbf{Q}_v)$, we can define meromorphic functions $r_{P'|P}(\sigma_{v,\Lambda})$, $P, P' \in \mathscr{P}(A)$, $\Lambda \in \mathfrak{a}_C$, so that the operators*

$R_{P'|P}(\sigma_{v,\Lambda}) = M_{P'|P}(\sigma_{v,\Lambda}) \cdot r_{P'|P}(\sigma_{v,\Lambda})^{-1}$ *can be analytically continued in* Λ, *with the following functional equations holding*:

$$R_{P''|P}(\sigma_{v,\Lambda}) = R_{P''|P'}(\sigma_{v,\Lambda})R_{P'|P}(\sigma_{v,\bar{\Lambda}}), \qquad P, P', P'' \in \mathscr{P}(A),$$

and

$$R_{P'|P}(\sigma_{v,\Lambda})^* = R_{P|P'}(\sigma_{v,-\bar{\Lambda}}).$$

Moreover if σ_v *is of class* 1 *and* Φ_v *is the* K_v*-invariant function*,

$$R_{P'|P}(\sigma_{v,\Lambda})\Phi_v = \Phi_v. \quad \square$$

Suppose that $\sigma = \otimes_v \sigma_v$ is an irreducible unitary representation of $M(A)$. If $\Phi = \otimes_v \Phi_v$ is a smooth vector in $\mathscr{H}_P(\sigma) = \otimes_v \mathscr{H}_P(\sigma_v)$, define

$$R_{P'|P}(\sigma)\Phi = \bigotimes_v R_{P'|P}(\sigma_v)\Phi_v.$$

For almost all v, the right-hand vector is the characteristic function of K_v. Define M^1 to be the kernel of the set of rational characters of M defined over $\boldsymbol{Q}$. Then M^1 is defined over $\boldsymbol{Q}$, and $M(A) = M^1(A)A(A)$. Let f be a function in $C_c^\infty(G(\boldsymbol{A}))^K$, as in Part II.

LEMMA 4. *There is a function* $\phi_A(f)$ *in* $C^\infty(M(\boldsymbol{A}))^{K\cap M(\boldsymbol{A})}$ *such that*

$$(m, a) \to \phi_A(f, ma), \qquad m \in M^1(\boldsymbol{A}),\ a \in A(\boldsymbol{A}),$$

is compactly supported in m *and a Schwartz function in* a, *so that the following property holds. If* $\sigma = \otimes_v \sigma_v$ *is any irreducible unitary representation of* $M(\boldsymbol{A})$,

$$\operatorname{tr} \sigma(\phi_A(f)) = \lim_{\lambda \to 0} \sum_{P' \in \mathscr{P}(A)} \Big(\prod_{\alpha \in \Phi_{P'}} \langle \lambda, \alpha \rangle \Big)^{-1} \cdot \operatorname{tr}(R_{P'|P}(\sigma)^{-1} R_{P'|P}(\sigma_\lambda) I_P(\sigma, f)).$$

(*The limit on the right exists and is independent of the fixed group* $P \in \mathscr{P}(A)$.) $\square$

Suppose that $\mathfrak{o}$ is an equivalence class in $G(\boldsymbol{Q})$. Then $M(\boldsymbol{Q}) \cap \mathfrak{o}$ is a finite union (possibly empty), $\mathfrak{o}_1^M \cup \cdots \cup \mathfrak{o}_n^M$, of equivalence classes relative to the group $M(\boldsymbol{Q})$. If $F_{\mathfrak{o}^M}$ is a function defined on the equivalence classes relative to the group $M(\boldsymbol{Q})$, let us write

$$F_{\mathfrak{o}} = n^M(A_0)n(A_0)^{-1} \sum_{i=1}^n F_{\mathfrak{o}_i^M}.$$

Now we shall define an invariant distribution $I_{\mathfrak{o}}$ for each $\mathfrak{o} \in \mathscr{C}$. The definition is inductive; we assume that the invariant distributions $I_{\mathfrak{o}^M}^M$ have been defined for each special subgroup A, with $Z \subsetneq A \subset A_0$. We then define

$$I_{\mathfrak{o}}(f_\xi) = J_{\mathfrak{o}}(f_\xi) - \sum_{\{A: Z \subsetneq A \subset A_0\}} I_{\mathfrak{o}}^M(\phi_A(f)_\xi).$$

LEMMA 5. $I_{\mathfrak{o}}$ *is invariant.*

This is essentially Theorem 5.3 of [8]. Note that this lemma is necessary for our inductive definition, since $\phi_A(f)$ is only defined up to conjugation by an element in $M(A)$. $\square$

Suppose that $\sigma = \otimes_v \sigma_v$ is a unitary *automorphic* representation of $M(A)$. Then

$$r_{P'|P}(\sigma) = \prod_v r_{P'|P}(\sigma_v), \qquad P, P' \in \mathscr{P}(A),$$

is essentially a quotient of two Euler products and is defined by analytic continuation. If $\Lambda \in i\mathfrak{a}$, let $r_{P'|P}(\Lambda)$ be the operator on $\mathscr{H}_P$ which acts on the subspace determined by $I_P(\sigma')$ by the scalar $r_{P'|P}(\sigma'_v)$. If $A \supset A_1$ define the operator $r(P, A_1, \Lambda)$ on $\mathscr{H}_P$ by the limit in Lemma 1, with $M_{P_1(Q)|P}(\Lambda)^{-1}M_{P_1(Q)|P}(\Lambda + \lambda)$ replaced by $r_{P_1(Q)|P}(\Lambda)^{-1}r_{P_1(Q)|P}(\Lambda + \lambda)$. It commutes with the action of $G(\boldsymbol{A})$. Finally, for $\chi \in \mathscr{S}_E(G)$, define $i_\chi(f_\xi)$ by the formula for $J_\chi(f_\xi)$ in Lemma 2, with $M(P, A_1, \Lambda_\xi)$ replaced by $r(P, A_1, \Lambda_\xi)$. Then i_χ is an invariant distribution.

THEOREM 1. *For any* $f \in C_c^\infty(G(\boldsymbol{A}))^K$,

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\mathfrak{o} \in \mathscr{C}} I_{\mathfrak{o}}(f_\xi) - \sum_{\chi \in \mathscr{S}_E(G)} i_\chi(f_\xi). \quad \square$$

REMARK. It follows from formula (2) of Part II that if $\mathfrak{o}$ consists entirely of semisimple elements, $I_{\mathfrak{o}}$ is one of the invariant distributions studied in **[8]**. Moreover since $G = \mathrm{GL}_3$, it is possible to show that for *any* $\mathfrak{o}$, $I_{\mathfrak{o}}$ is a sum of limits of the invariant distributions in **[8]**. Suppose that $f = \prod_v f_v$ and that for *two* places v,

$$\int_{T(\boldsymbol{Q}_v)\backslash G(\boldsymbol{Q}_v)} f_v(x^{-1}tx)\, dx = 0, \qquad t \in T(\boldsymbol{Q}_v)_{\mathrm{reg}},$$

for all maximal tori T in G such that $Z(\boldsymbol{Q}_v)\backslash T(\boldsymbol{Q}_v)$ is not compact. It follows from the last theorem of **[8]** that if there exists an element γ in a given $\mathfrak{o}$ which is $\boldsymbol{Q}$-elliptic mod Z,

$$I_{\mathfrak{o}}(f_\xi) = \mathrm{vol}\big(Z(\boldsymbol{R})^0 \cdot G_\gamma(\boldsymbol{Q})\backslash G_\gamma(\boldsymbol{A})\big) \int_{G_\gamma(\boldsymbol{A})\backslash G(\boldsymbol{A})} f_\xi(x^{-1}\gamma x)\, dx,$$

and that $I_{\mathfrak{o}}(f_\xi) = 0$ if no such γ exists. Moreover, it is easy to see that each $i_\chi(f_\xi) = 0$. From this it follows that if $\{\gamma\}$ is a set of representatives of $G(\boldsymbol{Q})$-conjugacy classes of elements in $G(\boldsymbol{Q})$ which are elliptic mod Z,

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\{\gamma\}} \mathrm{vol}\big(Z(\boldsymbol{R})^0 \cdot G_\gamma(\boldsymbol{Q})\backslash G_\gamma(\boldsymbol{A})\big) \int_{G_\gamma(\boldsymbol{A})\backslash G(\boldsymbol{A})} f_\xi(x^{-1}\gamma x)\, dx,$$

for f as above.

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