An Introduction to Langlands Functoriality

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The Principle of Functoriality has long been regarded as the centre of the Langlands Program. More recently, it has had to share the spotlight with Reciprocity, Langlands' conjecture that relates automorphic representations with motives from algebraic geometry. However, the two principles are closely related, and in any case, Reciprocity came at the end of the decade that followed the years 1960–1967 that are the focus of this volume.

Functoriality famously had its roots in the seventeen-page letter that Langlands gave to André Weil in 1967 [L2]. He wrote some of the details shortly afterwards in the article he dedicated to Salomon Bochner [L3]. It represented a very different direction for Langlands after his monumental volume [L1] on Eisenstein series, which was largely analysis. Langlands credits Bochner, an analyst himself, with directing him towards number theory, especially, I believe, class field theory and its long-sought nonabelian generalization.

There are several ways to introduce functoriality to a general reader. One is as a series of identities (reciprocity laws) that relate families of conjugacy classes

$$c = \{c_p : p \nmid N\}$$

in different complex groups. As objects with complex coordinates, but parametrized by prime numbers, these families are easy to imagine as gateways to higher arithmetic. There are already a number of introductions to functoriality from this point of view. (See for example [A, §4].) We shall take a slightly different approach here, one that is closer to the way Langlands originally presented functoriality[1] in [L3]. We shall describe it as a fundamental property of L-functions, and especially the arithmetic L-functions introduced and studied by Emil Artin. The reader can refer also to the paper [S] by Shahidi in this volume, which is an introduction to Langlands' L-functions that is largely complementary to what is contained here.

The article is intended to be a short historical introduction to functoriality. I have not taken up a well-founded suggestion of Matthew Sunohara to break the paper into sections, which I have formulated as:

1. Introduction
2. From Euler's product to Artin reciprocity to Tate's thesis

[1]The name *functoriality* was introduced only later, along other terms such as *automorphic representation*

3. Artin L-functions and Godement–Jacquet L-functions
4. On Langlands' seven questions
5. Four applications

This would have added clarity, but I have preferred to keep the narrative as informal as possible. I would like to thank Matthew for this suggestion and for other thoughtful comments, most of which I have adopted.

L-functions have a long history, with roots in both analytic and algebraic number theory. For the former, one can look back to Euler. He introduced the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for real numbers $s > 1$, proved that it had what is now called an Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \tag{1}$$

and studied its behaviour near $s = 1$. For algebraic number theory, one thinks of Gauss and his famous law of quadratic reciprocity. This formula anticipated what we look for even today in the study of number fields, and can be used in this connection to define the coefficients of the first L-functions. These in some sense represent an early model for the general automorphic L-function of Langlands.

Recall that a Dirichlet series is an infinite series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s},$$

for complex coefficients a_n and a complex variable s. If the coefficients satisfy a bound

$$|a_n| \leq Cn^{\alpha}, \quad n \in \mathbb{N},$$

for a positive number α, the series converges in the right half-plane $\mathrm{Re}(s) > \alpha+1$. The original model is of course Riemann's extension

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1,$$

of Euler's series. It converges to an analytic function of s in the right half-plane $\mathrm{Re}(s) > 1$. It also has analytic continuation to a meromorphic function of $s \in \mathbb{C}$, whose only singularity is a simple pole at $s = 1$, and which satisfies a functional equation relating its values at s and $1 - s$. In addition, the Riemann zeta function has an Euler product. By the fundamental theorem of arithmetic, it can be represented as a product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \prod_p \left(\sum_{k=0}^{\infty} (p^k)^{-s} \right)$$

of Dirichlet series attached to prime numbers.

An *L-function* is a Dirichlet series with supplementary properties. There seems to be no universal agreement as to the definition, but let us say that an L-function is a Dirichlet series that converges in some right half-plane, and that has an Euler product of the general form[2]

$$L^{\infty}(s) = \prod_{p} \left(1 + \sum_{k=1}^{\infty} c_{p,k} p^{-ks} \right),$$

for complex numbers $c_{p,k}$. We will not insist on analytic continuation and functional equation, simply because this has not been established for many of the L-functions that arise naturally, even though it is widely expected to hold.

Much of the history of nineteenth-century number theory concerns L-functions, explicitly or implicitly. Not surprisingly, it was Dirichlet who introduced the first L-functions after the original Euler product. These were the series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{p} (1 - \chi(p) p^{-s})^{-1}$$

attached to Dirichlet characters χ on $(\mathbb{Z}/N\mathbb{Z})^{\times}$, for any positive integer N. It is understood that $\chi(n)$ depends only on the congruence class of n modulo N, and that $\chi(n) = 0$ if n has any prime factors that divide N. The case that $N = 1$ is of course Euler's product (1). Dirichlet studied these objects as functions of a positive real variable s. The series converges absolutely if $s > 1$, but Dirichlet showed that if $\chi \neq 1$, the series converges conditionally for $0 < s \leq 1$ and that $L(1,\chi) \neq 0$. He used this in 1837 to prove that there are infinitely many primes in any arithmetic series

$$\{a + nd : n > 0\}, \quad (a,d) = 1.$$

Twenty years later, Riemann studied Euler's series as a function of a complex variable s. He proved that it has meromorphic continuation to the complex plane, as we have already noted, and that the product

$$L(s,1) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

satisfies the functional equation

$$L(s,1) = L(1-s,1).$$

He also showed that $L(s,1)$ is entire, apart from a simple pole at $s = 1$. Finally, he introduced his hypothesis that the only zeros of $L(s,1)$ lie on the line $\mathrm{Re}(s) = 1/2$. This would imply a very strong asymptotic estimate for the number

$$\pi(x) = |\{p \leq x\}|$$

[2] The superscript ∞ follows the conventions of Langlands. It is used to indicate that the L-function is incomplete, in the sense that the product is missing an archimedean factor that would greatly simplify an expected functional equation if it were included.

of primes less than or equal to a large number x.

The Riemann hypothesis is of course completely open today. What is less widely discussed is a common belief that the analogue of the Riemann hypothesis holds for all arithmetic L-functions, apart from certain obvious exceptions. In particular, it is thought to hold for the general Langlands L-functions $L(s, \pi, \rho)$ that are the topic of this article. This would imply equally strong asymptotic estimates for the arithmetic data that go into the coefficients of these Dirichlet series.

Later nineteenth-century contributions to the developing theory of L-functions include Kummer's 1851 generalization of Dirichlet L-functions to cyclotomic fields, Dedekind's 1893 generalization of the Riemann zeta function to arbitrary number fields, and Weber's 1897 generalization of Dirichlet L-functions to arbitrary fields. Weber then used the Dedekind zeta function to prove what was called the first inequality (later demoted to the second inequality!), a fundamental early step in the development of abelian class field theory. About this time, the turn of the century, Hilbert was laying the foundations of what would become the modern outline of class field theory. He worked with the Hilbert class field of a given number field F, the maximal *unramified* abelian extension of F, rather than the maximal abelian extension. However, his framework offered a new perspective, and anticipated what would be used in the general ramified case. He also rewrote Gauss' Law of quadratic reciprocity as a product formula for the Hilbert symbol. This made quadratic reciprocity the foundation of the simplest case of class field theory (with its corresponding Dirichlet series), the quadratic extensions of the field $F = \mathbb{Q}$.

We can now go on to Artin L-functions, which we are treating as a foundation for the L-functions introduced by Langlands. For a short but systematic account of the history of class field theory, the reader can consult [Con]. The article [Cog] is an interesting informal introduction to Artin L-functions.

Suppose that F is a number field, and that K/F is a finite Galois extension. Recall that almost all prime ideals $\mathfrak{p}$ in F are unramified in K, and that for any such $\mathfrak{p}$, the Frobenius class $\mathrm{Frob}_{\mathfrak{p}} = \Phi_{\mathfrak{p}}$ is a canonical conjugacy class in the Galois group

$$\Gamma_{K/F} = \mathrm{Gal}(K/F).$$

We thus obtain a family

$$\{\Phi_{\mathfrak{p}} : \mathfrak{p} \notin S\}$$

of conjugacy classes, parametrized by the prime ideals $\mathfrak{p}$ outside some chosen finite set that includes all the ramified primes. This is a fundamental datum attached to K/F, which is given entirely in terms of $\mathfrak{p}$. Recall also that $\mathfrak{p}$ is said to *split completely* in K if $\Phi_{\mathfrak{p}}$ is the identity element 1 in $\Gamma_{K/F}$. It is then a well-known fact that the map

$$\{K/F\} \longrightarrow \mathrm{Spl}(K/F), \tag{2}$$

from the set of finite Galois extensions K/F of F to the set of families $\mathrm{Spl}(K/F) =$

$\mathrm{Spl}^S(K/F)$ of primes $\mathfrak{p} \notin S$ that split completely in K, is injective[3]. Therefore the map parametrizes the finite Galois extensions K/F in terms the data $\mathrm{Spl}(K/F)$.

Suppose for example that $F = \mathbb{Q}$. We can represent K as the splitting field of an irreducible monic polynomial $f(x) \in \mathbb{Z}[x]$ of degree n. There is then an embedding of the Galois group $\Gamma_{K/\mathbb{Q}}$ into the symmetric group S_n, which is canonical up to conjugacy. The conjugacy classes in S_n correspond to partitions of n. In particular, if $\Gamma_{K/\mathbb{Q}}$ equals S_n, which is what happens generically, we can identify the various Frobenius classes for $K/\mathbb{Q}$ with partitions of n. In fact, it follows from the basic theory that the Frobenius class of an unramified prime p is the partition defined by the irreducible factors of $f(x)$ modulo p. This gives a concrete realization of a deep phenomenon. In particular, even without the restriction on $\Gamma_{K/F}$, $\mathrm{Spl}(K/F)$ is the set of primes p such that $f(x)$ breaks into linear factors modulo p.

Emil Artin used the families of conjugacy classes $\Phi_{\mathfrak{p}}$ to construct the L-functions that bear his name. For the coefficients, he had to attach complex parameters to the conjugacy classes in $\Gamma_{E/F}$. His idea was to take not just the Galois extension K of F, but also a finite-dimensional complex representation

$$r : \Gamma_{K/F} \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

of its Galois group. The conjugacy classes in $\Gamma_{K/F}$ would then be mapped to semisimple conjugacy classes in $\mathrm{GL}(n, \mathbb{C})$ (of finite order), which could then be parametrized by their characteristic polynomials. The local Artin L-function at an unramified prime $\mathfrak{p} \notin S$ of F is defined in these terms as

$$L_{\mathfrak{p}}(s, r) = \det(1 - (N\mathfrak{p})^{-s} r(\Phi_{\mathfrak{p}}))^{-1}.$$

The (unramified) global Artin L-function for r is then the Euler product

$$L^S(s, r) = \prod_{\mathfrak{p} \notin S} \det(1 - (N\mathfrak{p})^{-s} r(\Phi_{\mathfrak{p}}))^{-1}.$$

If we identify $\mathfrak{p}$ with its normalized valuation v, as is convenient, we can also write

$$L^S(s, r) = \prod_{v \notin S} L_v(s, r) = \prod_{v \notin S} \det(1 - q_v^{-s} r(\Phi_v))^{-1}, \tag{3}$$

where q_v is the order of the residue class field of $\mathfrak{p}$, and S is now understood to include the finite set S_∞ of normalized archimedean valuations of F as well as the non-archimedean valuations at which r ramifies.

An L-function is not just a way to package arithmetic data. It should also lead ultimately to fundamental asymptotic properties of these data. A necessary condition for this would be that the L-function have meromorphic continuation

[3]The choice of S is immaterial. One could take $\mathrm{Spl}(K/F)$ to be the set of equivalence classes of families $\mathrm{Spl}^S(K/F)$, in which $\mathrm{Spl}^S(K/F)$ is equivalent to $\mathrm{Spl}^{S'}(K/F)$ if the intersection of the two sets has finite complement in each one. The mapping (2) then remains injective with this interpretation of the right-hand side

to a suitable function on the complex plane. Artin conjectured that for any r, $L^S(s,r)$ could be completed with a suitable contribution $L_S(s,r)$ from the places in S so that the resulting product

$$L(s,r) = L_S(s,r)L^S(s,r) \tag{4}$$

has meromorphic continuation, and satisfies a functional equation

$$L(s,r) = \varepsilon(s,r)L(1-s,r^\vee), \tag{5}$$

for the contragredient representation

$$r^\vee(\sigma) = {}^t r(\sigma^{-1}), \quad \sigma \in \Gamma_{K/F},$$

and a monomial

$$\varepsilon(s,r) = ab^s, \quad a \in \mathbb{C}^\times, b \in \mathbb{C}. \tag{6}$$

He conjectured further that for irreducible r, $L(s,r)$ is entire unless r is the trivial 1-dimensional representation (in which case his L-function is just the completed Riemann zeta function

$$L(s,1) = L_\infty(s,1)\zeta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

which is entire apart from a simple pole at $s=1$). It is the last statement that is the deepest. It is known today simply as *the* Artin conjecture.

Artin proved the earlier assertions of his conjecture in a way that became part of the motivation for Langlands' principle of functoriality. The heart of what he established was the case that K/F is abelian. One might imagine that there would be a direct proof of the Artin conjecture in this case. However, that is not the way the mathematical world was put together. Artin gave a decidedly indirect proof that even today seems extraordinary. He showed that every abelian Artin L-function was a Hecke L-function, the class of concrete L-functions that arose from analysis, and for which Hecke had been able to establish the desired analytic properties. This was class field theory. Artin studied what was known, and extended it to what was required for his purposes. The result was the Artin Reciprocity Law, which we state in adelic terms as follows.

Artin Reciprocity Law. *Suppose that K/F is an abelian extension of the number field F. Then there is a canonical isomorphism*

$$\theta_{K/F} : C_F / \mathrm{N}_{K/F}(C_K) \xrightarrow{\sim} \Gamma_{K/F} \tag{7}$$

We recall here that the *adèle* ring of F is a topological direct limit

$$\mathbb{A}_F = \varinjlim_S \Big(\prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_S \Big),$$

where S ranges over finite sets of valuations of F that contain the set S_∞ of archimedean valuations. Then $\mathbb{A}_F$ is a locally compact ring (commutative, with

1), which contains the diagonal image of F as a discrete, co-compact subring. It was introduced by Artin and Whaples (1945), following the earlier introduction of its group of units, the *idèle* group

$$I_F = \mathbb{A}_F^\times = \mathrm{GL}(1, \mathbb{A}_F),$$

by Chevalley (1940). The *idèle class group* is the quotient

$$C_F = I_F / F^\times,$$

while its quotient

$$C_F / \mathrm{N}_{K/F}(C_K) = I_F / F^\times \, \mathrm{N}_{K/F}(I_K) \tag{8}$$

is the domain of the Artin map $\theta_{K/F}$ in (7). The norm map $\mathrm{N}_{K/F} : C_K \to C_F$ is built in the obvious way from the usual norm maps between local and global fields.

The definition of the Artin map is built on local class field theory, which asserts that there is a canonical isomorphism

$$\theta_{K_w/F_v} : F_v^\times / \mathrm{N}_{K_w/F_v}(K_w^\times) \longrightarrow \Gamma_{K_w/F_v},$$

for any completions F_v of F and K_w of K over F_v. Since $\Gamma_{K/F}$ is abelian, this provides a canonical embedding of Γ_{K_w/F_v} into $\Gamma_{K/F}$ that depends only on F_v (and not the choice of the field K_w over F_v). The product over v of these embeddings then gives a well-defined mapping

$$\theta_{K/F} : I_F / \mathrm{N}_{K/F}(I_K) \longrightarrow \Gamma_{K/F}.$$

The deepest property, the one that makes this global mapping a "reciprocity law", is the fact that its kernel equals the image of the subgroup $F^\times$ of I_F. Therefore $\theta_{K/F}$ descends to a mapping on the domain (8) of (7). This seems to have been a point that late nineteenth-century number theorists struggled with. It was eventually clarified (at least for unramified extensions K/F) by Hilbert.

We have formulated the Artin reciprocity law in this detail so as to serve as a foundation for Langlands' nonabelian generalization. Our description does rely on local class field theory, which was an important advance in its own right. However, one could set up the Artin map $\theta_{K/F}$ in a more elementary, if less elegant, way by restricting the factors θ_{K_w/F_v} of $\theta_{K/F}$ to the unramified places of K/F, where they can be defined in terms of Frobenius elements in $\Gamma_{K/F}$. The adelic formulation itself could be replaced by the more concrete (but also more cumbersome) classical description in terms of "moduli" and "conductors". For further information a reader might consult the Wikipedia articles, "Artin reciprocity law" and "Symbols (number theory)", from June 3, 2020. The second of these is like a one-page history of class field theory, in the form of a list of symbols for the evolving reciprocity maps, from the *Legendre symbol* of quadratic reciprocity to the *Hilbert symbol* for Kummer extensions, and then finally, to the *Artin symbol* for arbitrary abelian extensions.

An abelian Artin L-function of degree 1 is defined by a character ξ on the abelian Galois group $\Gamma_{K/F}$ on the right hand side of (7). A character χ on the domain at the left in (7) defines a Hecke L-function. The two definitions can be seen to match under the isomorphism $\theta_{K/F}$, thereby giving an identity

$$L(s,\xi) = \prod_v L_v(s,\xi_v) = \prod_v L_v(s,\chi_v) = L(s,\chi) \tag{9}$$

of the two kinds of abelian L-functions. Hecke used harmonic analysis to show that his abelian L-functions satisfied analogues of all the conditions conjectured by Artin. Therefore the abelian L-functions of Artin also satisfy the assertions of his conjecture. It was in its form of a correspondence $\xi \to \chi$ of abelian global characters, and the resulting identity (9) of L-functions, that Langlands generalized Artin reciprocity.

Artin used his reciprocity law and the Hecke L-functions $L(s,\chi)$ it provided to prove some of the assertions of his general conjecture. The idea was to decompose a general representation r of $\Gamma_{K/F}$ into a virtual linear combination

$$r = \sum_i a_i \, \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\xi_i), \quad \Gamma = \Gamma_{K/F},$$

for one-dimensional representations ξ_i of cyclic subgroups Γ_i of $\Gamma = \Gamma_{K/F}$. Artin proved that this could be done for rational numbers a_i. Standard properties of L-functions then provided a corresponding product decomposition

$$L(s,r) = \prod_i L(s,\xi_i)^{a_i}$$

of $L(s,r)$ into abelian L-functions over cyclic extensions K/F_i. Artin was then able to use this to establish the analytic continuation and functional equation of $L(s,r)$. (See [Cog, p.10] for further remarks on this, including its relation to the later Brauer induction theorem.) What the decomposition did *not* give was Artin's conjectural assertion that $L(s,r)$ is entire. The problem is the contribution to the product of the negative numbers a_i, from which the zeros of $L(s,\xi_i)$ could contribute poles to $L(s,r)$. This crosses into the domain of the Riemann hypothesis and its analogues for Hecke L-functions. The phenomenon is certainly interesting but, by itself at least, does not offer any help with the last assertion of the conjecture, the assertion now known as the Artin conjecture.

Hecke studied the L-functions $L(s,\chi)$ for what he called Grössencharaktere (now known simply *Hecke characters*). They amount to characters on the full idèle class group C_F, not just those that descend to $C_F/\mathrm{N}_{K/F}(C_K)$, for a general number field F, although the ideles were not introduced until twenty years later. Hecke's L-functions represent a major generalization of the Dirichlet L-functions and the Dedekind zeta functions, and indeed, all of the extensions of these functions that had previously been studied. To establish their analytic properties Hecke relied on the classical Mellin transform and classical Poisson summation. In this, he was following Riemann, but with arguments that were by necessity considerably more sophisticated.

Thirty years later, Tate's 1950 Ph.D. thesis [T1] gave a different way of looking at both Hecke's proofs and his results. Tate had the advantage of being able to work with the ideles $I_F = \mathbb{A}_F^{\times}$ that had been introduced ten years earlier by Chevalley, and in terms of which we stated the Artin reciprocity law. The heart of his proof was the application of the Poisson summation formula for the discrete subgroup F of the adèle ring $\mathbb{A}_F$. The simplicity of Tate's arguments led to an important refinement of the functional equation

$$L(s,\chi) = \varepsilon(s,\chi)L(1-s,\overline{\chi})$$

for the given L-function

$$L(s,\chi) = L_S(s,\chi)L^S(s,\chi).$$

This was a decomposition of the global monomial in the equation into a product

$$\varepsilon(s,\chi) = \prod_{v\in S} \varepsilon(s,\chi_v,\psi_v)$$

of local monomials

$$\varepsilon(s,\chi_v,\psi_v) = \varepsilon(\chi_v,\psi_v)q_v^{-n_v(s-\frac{1}{2})},$$

where q_v equals the residual degree of F_v if v is non-archimedean, and equals 1 if v is archimedean, and ψ is a nontrivial additive character on the quotient $\mathbb{A}_F/F$.

We are working towards the 1968 preprint [L3] of Langlands, and the seven questions it posed, in our attempt to understand the origins of functoriality. The logical next step in our exposition here is the generalization of Tate's thesis from GL(1) to GL(n) by Jacquet and Godement [GJ]. It was not published until 1972, well after [L3], but its future existence was clearly part of Langlands' thinking. He was in regular communication with Godement throughout the 1960s, and he mentions the extension of Tate's thesis in his paper [L3] as an essential premise for his conjectures.

Suppose for a moment that G is any reductive group over a number field F, and that π is an automorphic representation of G (which is to say, of the locally compact group $G(\mathbb{A}_F)$). This term was not in use at the time of [L3], as we observed in the footnote 1. Its formal definition did not come until the Corvallis conference [B7], [L5] ten years later. Langlands simply referred to π as an irreducible representation of $G(\mathbb{A}_F)$ that "occurs in" $L^2(G(F)\backslash G(\mathbb{A}_F))$, a description that gives a good idea of the concept, even if it is also somewhat more restrictive than what became the general definition. Langlands also took for granted that an automorphic representation π has a unique (restricted) tensor product decomposition

$$\pi = \widetilde{\bigotimes_v} \pi_v, \quad \pi_v \in \Pi(G_v),$$

into irreducible representations π_v of the local components $G_v = G(F_v)$ of $G(\mathbb{A}_F)$, almost all of which are unramified. The formal proof of this by Flath also came ten years later at the Corvallis conference [F].

We recall that an irreducible representation π_v of G_v is *unramified* if G_v is quasi-split over F_v and split over some unramified extension E_v of F_v, and if the restriction of π_v to a suitable (hyperspecial) maximal compact subgroup K_v of $G_v = G(F_v)$ contains the trivial 1-dimensional representation of K_v. Langlands introduced this notion in [L3] (again without the name). He then observed that there was a bijective correspondence $\pi_v \to c(\pi_v)$ from the unramified representations π_v of G_v to the semisimple conjugacy classes c_v in the L-group[4]

$$^LG_v = \hat{G}_v \rtimes \Gamma_{E_v/F_v} \hookrightarrow {}^LG = \hat{G} \rtimes \Gamma_{E/F}, \quad \hat{G}_v = \hat{G},$$

whose image in Γ_{E_v/F_v} projects onto the Frobenius class Φ_v. This is a consequence of the classification of complex-valued homomorphisms on the Hecke algebra $C_c(K_v\backslash G_v/K_v)$ (under convolution) or equivalently, the description of the unramified principal series for G_v. The automorphic representation π of G thus gives rise to a family

$$c(\pi) = \{c_v(\pi) = c(\pi_v) : v \notin S\}$$

of semisimple conjugacy classes in ^LG, where $S \supset S_\infty$ is again a finite set of valuations of F outside of which π_v is unramified.

Returning to the volume of Godement–Jacquet, we take G equal to $\mathrm{GL}(n)$ over F. An automorphic representation π of G now gives a family

$$c(\pi) = \{c_v(\pi) = c(\pi_v) : v \notin S\}$$

of semisimple conjugacy classes in the complex group $^LG = \hat{G} = \mathrm{GL}(n, \mathbb{C})$. A very special case of Langlands' general definitions (which we will come to presently) is then the associated family

$$L_v(s, \pi) = L(s, \pi_v) = \det(1 - q_v^{-s} c_v(\pi))^{-1}, \quad v \notin S,$$

of unramified local L-functions, and the unramified global Euler product

$$L^S(s, \pi) = \prod_{v \notin S} L_v(s, \pi), \tag{3'}$$

which converges for $\mathrm{Re}(s)$ in some right half-plane.

The main theorem of [GJ] applies to the basic case that π is cuspidal. It asserts that $L^S(s, \pi)$ can be expanded by a finite product

$$L_S(s, \pi) = \prod_{v \in S} L_v(s, \pi) = \prod_{v \in S} L(s, \pi_v)$$

[4]Langlands had earlier introduced the fundamental notion he called the *associate group* for G, and that Borel later named the L-group in [B1]. The notation here is due to Kottwitz [K].

of ramified local L-functions so that the resulting completion

$$L(s,\pi) = L_S(s,\pi)L^S(s,\pi) \tag{4'}$$

has meromorphic continuation, and satisfies a functional equation

$$L(s,\pi) = \varepsilon(s,\pi)L(1-s,\pi^\vee), \tag{5'}$$

for the contragredient representation $\pi^\vee = {}^t\pi(x^{-1})$, and a finite product

$$\varepsilon(s,\pi) = \prod_{v\in S} \varepsilon(s,\pi_v,\psi_v) \tag{6'}$$

of local monomials

$$\varepsilon(s,\pi_v,\psi_v) = \varepsilon(\pi_v,\psi_v)q_v^{-n_v(s-\frac{1}{2})}$$

that depend on the local components ψ_v of a nontrivial additive character ψ on $\mathbb{A}_F/F$. Moreover, $L(s,\pi)$ is entire unless $n = 1$ and $\pi(x) = |x|^u$ for some $u \in \mathbb{C}$. This theorem is the natural generalization of the theorem of Tate for $n = 1$, itself a refinement of the fundamental results of Hecke. The restriction to cuspidal π is not a serious impediment. With techniques from Langlands' theory of Eisenstein series [L1], the theorem can be extended to arbitrary automorphic representations π of G.

With the Galois L-functions $L(s,r)$ of degree n and the automorphic L-functions $L(s,\pi)$ for $\mathrm{GL}(n)$, our exposition has acquired a certain symmetry. The Galois representation r can in fact be made independent of the finite Galois extension K/F, simply by taking it to be a continuous, complex representation of the absolute Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$. For it would then automatically factor through a finite quotient $\Gamma_{K/F}$. The other discrepancy between the two theories is more significant. It is the lack of a local factorization for Artin's Galois ε-factor $\varepsilon(s,r)$ that would match the canonical factorization (6$'$) of the automorphic ε-factor $\varepsilon(s,\pi)$ attached to ψ_v. For Langlands, this was a serious deficiency, given the local classification he had in mind for Question 6 of [L3], which we will come to presently. He worked hard in the late 1960s to establish a local construction of Artin ε-factors $\varepsilon(s,r_v,\psi_v)$. He eventually succeeded, but did not include all of the details in his long treatise [L4]. Soon afterwards, Deligne was able to find a simpler global solution of the problem [T2].

The upshot is that the two theories are completely parallel. Taken together, they very much resemble what had been established in the abelian case of $\mathrm{GL}(1)$. Given the Artin reciprocity law, a reader might well wonder whether every Artin L-function of a degree n representation r of Γ_F is a Godement–Jacquet L-function of an automorphic representation π of $\mathrm{GL}(n)$. That is, whether there is an injective correspondence $r \to \pi$ such that

$$L(s,r) = \prod_v L_v(s,r) = \prod_v L_v(s,\pi) = L(s,\pi). \tag{9'}$$

We would then have a reciprocity law that amounted to nonabelian class field theory. If so, would it then be the final word on the subject?

There are three points to consider in regard to the last question. One would be the uncomfortable prospect of having to prove such a broad nonabelian reciprocity law, given the historical difficulty in establishing just the abelian theory. Nonabelian class field theory, whatever form it might take, was obviously going to be very deep. It would be reassuring to think that the problem at least had some further structure. A second point concerns this last possibility. Suppose that r' is an irreducible Galois representation of degree n', and that ρ' is an irreducible n-dimensional representation of $\mathrm{GL}(n', \mathbb{C})$. The composition

$$r : \Gamma_F \xrightarrow{r'} \mathrm{GL}(n', \mathbb{C}) \xrightarrow{\rho'} \mathrm{GL}(n, \mathbb{C})$$

is then a Galois representation (frequently irreducible) of degree n. The Frobenius classes that define the Artin L-functions $L^S(s, r')$ and

$$L^S(s, r) = L^S(s, \rho' \circ r')$$

satisfy the obvious relation

$$r(\Phi_v) = (\rho' \circ r')(\Phi_v), \quad v \notin S.$$

How could this be reflected in the corresponding automorphic representations? Finally, the work of Harish-Chandra has taught us that representations should be studied uniformly for all groups. If some interesting phenomenon is discovered in one group, or one family of groups such as $\{\mathrm{GL}(n)\}$, it should be investigated for all groups. What are the implications of this for automorphic L-functions?

These considerations were undoubtedly part of the thinking of Langlands that led up to the Principle of Functoriality. However, perhaps the most decisive hints were in his theory of Eisenstein series. They came from the L-functions that he discovered in the global intertwining operators $M(w, \lambda)$ from his functional equations for Eisenstein series. Thus informed by his general results on Eisenstein series, as well as his study of Artin L-functions and abelian class field theory, and perhaps above all, his earlier study of the work of Harish-Chandra, Langlands put his ideas together in the letter to Weil and the paper [L3]. It was clear to him that the theory should indeed encompass the automorphic representations π of an arbitrary reductive group G over F. He actually took F to be any global field, but we shall continue to assume that it is a number field.

At the beginning of [L3], Langlands introduced the L-group. This was a sweeping new idea in its own right. He then defined the semisimple conjugacy classes $c_v(\pi) = c(\pi_v)$ in ${}^LG$ attached to the unramified constituents of an automorphic representation π, which of course also gave the family $c(\pi) = \{c_v(\pi)\}$ we have described above. But he wanted also to attach L-functions to these objects. This was not immediately clear, since unramified L-functions had always been defined as characteristic polynomials, and the L-group ${}^LG$ that contains the conjugacy classes from π does not usually come with a general linear group.

Langlands' solution was simple and elegant. It was to attach another datum to π, a finite-dimensional representation

$$\rho : {}^L G \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

of the L-group. The unramified local and global L-functions could then be defined as

$$L_v(s, \pi, \rho) = L(s, \pi_v, \rho_v) = \det(1 - q_v^{-s} \rho_v(c_v(\pi)))^{-1}$$

and

$$L^S(s, \pi, \rho) = \prod_{v \notin S} L_v(s, \pi, \rho). \tag{3''}$$

Langlands formulated his ideas as a series of questions. The first was designed to frame the entire discussion in terms of L-functions. It asked whether the unramified L-functions $L^S(s, \pi, \rho)$ above have the same analytic properties as in the special case of Godement–Jacquet.

Question 1. *Given G/F, π and ρ as above, is it possible to define local L-functions*

$$L_v(s, \pi, \rho) = L(s, \pi_v, \rho_v)$$

and epsilon factors

$$\varepsilon_v(s, \pi, \rho, \psi) = \varepsilon(s, \pi_v, \rho_v, \psi_v) = \epsilon(\pi_v, \rho_v, \psi_v) q_v^{-n_v(s - \frac{1}{2})}$$

at the ramified (and archimedean) places $v \in S$ so that if

$$L_S(s, \pi, \rho) = \prod_{v \in S} L_v(s, \pi, \rho)$$

and

$$\varepsilon(s, \pi, \rho) = \prod_{v \in S} \varepsilon_v(s, \pi, \rho, \psi), \tag{6''}$$

then the completed global L-function

$$L(s, \pi, \rho) = L_S(s, \pi, \rho) L^S(s, \pi, \rho) \tag{4''}$$

has meromorphic continuation to the complex plane with only finitely many poles, and satisfies the functional equation

$$L(s, \pi, \rho) = \varepsilon(s, \pi, \rho) L(1 - s, \pi, \rho^\vee), \tag{5''}$$

for $\rho^\vee = {}^t\rho(g^{-1})$?

Langlands then alluded to the case that G equals $\mathrm{GL}(n)$ and ρ is the standard n-dimensional representation of $\hat{G} = \mathrm{GL}(n, \mathbb{C})$. In this case, $L(s, \pi, \rho) = L(s, \pi)$ is the Godement–Jacquet L-function. Referring to ongoing work of Godement, Langlands offered the expectation that the assertions of Question 1 would be answered affirmatively in this case. The other special case of immediate interest was for G equal to the trivial group $\{1\}$. This of course forces the automorphic representation to be trivial, but $\rho = r$ can still be an arbitrary complex representation of the L-group

$$^{L}G = \{1\} \rtimes \Gamma_F,$$

or in other words, a continuous representation of the absolute Galois group Γ_F. In this case, $L(s, \pi, \rho) = L(s, r)$ is an arbitrary Artin L-function. By introducing the further L-functions $L(s, \pi, \rho)$, with the plausible hope that they too have the desired analytic properties, Langlands does indeed impose further structure on the general problem of relating $L(s, r)$ to $L(s, \pi)$. This becomes more vivid as we go along.

After stating Question 1, Langlands wrote, "The idea that led Artin to the general [abelian] reciprocity law suggests that we try to answer [Question 1] in general by answering a further series of questions." It would be very interesting to trace through the details of Artin's proof with Langlands' questions as a guide, but I have not done so. The remaining six questions are divided into three pairs, each consisting of a local and a global version of a question. Question 2 and 3 concern how the L-functions behave under inner twists. It is Question 4 and 5 that introduce local and global functoriality, our main topic. Questions 6 and 7 represent a generalization of parts of functoriality, with the Weil group W_F in place of the Galois group Γ_F.

Recall that an arbitrary reductive group G over F can be obtained uniquely from a quasi-split group G^* (a group that contains a Borel subgroup B^* over F) by twisting the Galois action on G^* by inner automorphisms. The L-group $^{L}G^*$ of G^* is then equal to that of G. Questions 2 and 3 ask whether the automorphic representation theory of G is similar to that of G^*. More precisely, is there a correspondence (binary relation) $\pi_v \to \pi_v^*$ of representations over each localization F_v such that $L(s, \pi_v, \rho_v)$ equals $L(s, \pi_v^*, \rho_v)$? Then if $\pi = \tilde{\bigotimes}_v \pi_v$ is automorphic, is $\pi^* = \tilde{\bigotimes}_v \pi_v^*$ also automorphic, thereby giving an identity $L(s, \pi, \rho) = L(s, \pi^*, \rho)$ of automorphic L-functions for different groups? The answers to these questions are turning out to be interesting and subtle. The representation theory of inner twists is now treated as part of a different theory, Langlands' conjectural theory of endoscopy, which began to evolve in the 1970s. This means that for these questions on L-functions and functoriality, one usually takes G to be quasi-split over F.

For Questions 4 and 5 on functoriality, we take G' and G to be two quasi-split groups over the number field F, related by an L-homomorphism

$$\rho' : {}^{L}G' \longrightarrow {}^{L}G$$

between their L-groups. Question 4 asks whether there is a local correspondence $\pi_v' \to \pi_v$ between the irreducible representations π_v' and π_v of $G'(F_v)$ and $G(F_v)$ such that

$$L(s, \pi_v, \rho_v) = L(s, \pi_v', \rho_v \circ \rho_v')$$

and

$$\varepsilon(s, \pi_v, \rho_v, \psi_v) = \varepsilon(s, \pi'_v, \rho_v \circ \rho'_v, \psi_v)$$

for every complex finite-dimensional representation ρ_v of ${}^L G_v$ and every non-trivial additive character ψ_v on F_v. This is **local functoriality.** Question 5 then asks if $\pi' = \tilde{\bigotimes}_v \pi'_v$ is automorphic for G', and $\pi'_v \to \pi_v$ for every v, whether $\pi = \tilde{\bigotimes}_v \pi_v$ is automorphic for G. This is **global functoriality.** It implies the identity

$$L(s, \pi, \rho) = L(s, \pi', \rho \circ \rho')$$

of global L-functions for every complex, finite-dimensional representation ρ of ${}^L G$. At first glance, it might in fact seem like a harmless assertion. This perhaps accounts for how long it took to be accepted by the mathematical community for what it was, a revolutionary change in our understanding of number theory.

The last two questions concern the special case of functoriality in which $G' = \{1\}$. Then ρ' is an L-homomorphism from the Galois group to the dual group ${}^L G$ of the given quasi-split group. For these questions, Langlands replaced the Galois groups Γ_{F_v} and Γ_F by the local and global Weil groups W_{F_v} and W_F. My understanding is that he learned of these objects in his discussions with Weil, and that he was very happy to discover that they would become a natural part of his theory. Weil had introduced his groups[5] in 1951, as objects that behaved very much like Galois groups. In particular, he was able to attach L-functions (local or global) to finite-dimensional representations ϕ of the relevant Weil group, thereby providing an important generalization of Artin L-functions.

Question 6 asks whether there is a correspondence $\phi_v \to \pi_v$, which takes L-homomorphisms $\phi_v : W_{F_v} \to {}^L G_v$ to irreducible representations π_v of $G(F_v)$, such that

$$L(s, \pi_v, \rho_v) = L(s, \rho_v \circ \phi_v)$$

and[6]

$$\varepsilon(s, \pi_v, \rho_v, \psi_v) = \varepsilon(s, \rho_v \circ \phi_v, \psi_v),$$

for every complex finite-dimensional representation ρ_v of ${}^L G_v$, and every non-trivial additive character ψ_v of F_v. Question 7 then asks if $\phi : W_F \to {}^L G$ is an L-homomorphism, and $\pi = \tilde{\bigotimes}_v \pi_v$ for local images $\phi_v \to \pi_v$ of the correspondence, whether π is automorphic for G. This would imply the identity

$$L(s, \pi, \rho) = L(s, \rho \circ \phi)$$

of global L-functions attached to complex, finite-dimensional representations ρ of ${}^L G$. The local Question 6 has turned out to be particularly important. Langlands later wrote Π_{ϕ_v} for the set of images π_v of a given ϕ_v under the

[5] We recall that the Weil group (over F_v or F) is a locally compact group, with a canonical mapping into the corresponding Galois group (over F_v or F), whose image is dense. The pullback of the mapping then gives an injection $r \to \phi$ from Galois representations to Weil group representations. The Weil L-function for the image of r of course then coincides with the Artin L-function of r. (See [T].)

[6] Langlands was anticipating the results of [L4] here, which included the existence of local ε-factors for Weil groups.

correspondence $\phi_v \to \pi_v$. The local *Langlands classification*, or *local Langlands correspondence*, is the conjecture that the L-packets Π_{ϕ_v} are finite, disjoint sets, whose union over ϕ_v is the set of all[7] irreducible representations of $G(F_v)$. This is now known for quasi-split classical groups, and for all real groups, but otherwise remains largely open. It is now treated as part of Langlands theory of endoscopy.

Let us add a couple more remarks to our description of the questions of [L3]. We have tried to motivate the Principle of Functoriality according to the presentation of Langlands, as a natural outgrowth of the theory of L-functions. More narrowly, we could think of it simply as an attempt to understand Artin L-functions, and to prove the Artin conjecture that $L(s,r)$ is entire. We think back to the simple question we raised on the possible correspondence $r \to \pi$ from Galois representations of degree n to automorphic representations of $\mathrm{GL}(n)$ (with its associated matching $(9')$ of L-functions). The three points we raised then are clearly accounted for in the greatly expanded theory encompassed by Langlands' seven questions. For a start, the automorphic L-functions $L(s,\pi,\rho)$ of Question 1 are attached to automorphic representations π of a general group G, not just $\mathrm{GL}(n)$. Secondly, the seven questions reveal a vast, previously hidden, structure that surrounds the original two L-functions $L(s,r)$ and $L(s,\pi)$. And finally, the automorphic interpretation of the Artin L-function $L(s,r) = L(s,r',\rho')$ attached to an n-dimensional representation ρ' of $\mathrm{GL}(n',\mathbb{C})$ is just the existence of an automorphic representation π of $\mathrm{GL}(n)$ attached to the given automorphic representation π' of $\mathrm{GL}(n')$ such that $L(s,\pi) = L(s,\pi',\rho')$. This is functoriality itself, or rather the special case of it for general linear groups.

As we have noted at the beginning, one can also motivate functoriality simply as a set of reciprocity laws among concrete arithmetic data. Recall that an automorphic representation π of a reductive group G over F comes with a family

$$c(\pi) = \{c_v(\pi) = c(\pi_v) : v \notin S\}$$

of semisimple conjugacy classes in ${}^LG$. Suppose that G', π', G and ρ' are as in the statement of functoriality from Langlands Questions 4 and 5. Then functoriality asserts the existence of an automorphic representation π of G such that

$$c(\pi) = \rho'(c(\pi')).$$

In other words, for each v outside a finite set S, the conjugacy class in ${}^LG$ that contains $\rho'(c(\pi_v))$ equals $c(\pi_v)$. These data should among other things govern the fundamental structure of arithmetic algebraic varieties, and their motivic components. It seems truly remarkable that they should satisfy such concrete relations.

We conclude by recalling the four fundamental applications of functoriality sketched by Langlands at the end of his paper [L3].

[7] This is correctly stated here only if the local field F_v is archimedean. If F_v is non-archimedean, one must replace W_{F_v} with the larger group $W_{F_v} \times \mathrm{SU}(2)$ in order to account for the Steinberg representation π_v of $G(F_v)$, and more generally, what are called the special representations.

(i) *Artin L-functions and nonabelian class field theory*: We have already commented on this, but it is worth repeating, since it is by any measure what mathematicians have been searching for ever since Artin. It is the case of functoriality with $G' = \{1\}$, $\rho' = r$ an n-dimensional representation of the L-group group ${}^L G' = \Gamma_F$, and $\rho = \mathrm{St}_n$ the standard representation of $\mathrm{GL}(n, \mathbb{C})$. The assertion of functoriality is that there is an automorphic representation π of $\mathrm{GL}(n)$ such that

$$L(s, r) = L(s, \pi).$$

This is the original desired identity $(9')$ that we have just been discussing. It characterizes the arithmetic data that classify Galois extensions of F in analytic terms. It also tells us any irreducible Artin L-function is a cuspidal Godement–Jacquet L-function for $\mathrm{GL}(n)$, and hence entire.

(ii) *Analytic continuation and functional equation*: This is a generalization of (i) to an arbitrary automorphic L-function $L(s, \pi, \rho)$, attached to an automorphic representation π of G and an N-dimensional representation ρ of ${}^L G$. Functoriality asserts that there is an automorphic representation π_N of $G_N = \mathrm{GL}(N)$ such that $L(s, \pi, \rho_N \circ \rho)$ equals $L(s, \pi_N, \rho_N)$, for any complex representaiton ρ_N of ${}^L G_N$. If we take ρ_N to be the standard representation St_N of G_N, the assertion becomes

$$L(s, \pi, \rho) = L(s, \pi_N).$$

In other words, any automorphic L-function is Godement–Jacquet L-function. It therefore has meromorphic continuation and functional equation, with only finitely many poles.

(iii) *Generalized Ramanujan conjecture*: The generalized Ramanujan conjecture asserts that a cuspidal automorphic representation $\pi = \tilde{\bigotimes}_v \pi_v$ of $\mathrm{GL}(n)$ is *tempered*. This means that the character

$$f_v \longrightarrow \mathrm{tr}(\pi(f_v)), \quad f_v \in C_c^\infty(\mathrm{GL}(n, F_v)),$$

of each local constituent π_v of π is tempered, in the sense that it extends to a continuous linear form on the Schwartz space $\mathcal{C}(\mathrm{GL}(n, F_v))$ on $\mathrm{GL}(n, F_v)$ defined by Harish-Chandra. We recall that the classical Ramanujan conjecture applies to the case $n = 2$, and π comes from the cusp form of weight 12 and level 1. It was proved by Deligne, who established more generally (for $n = 2$) that the conjecture holds if π is attached to any holomorphic cusp form. The case that π comes from a Maass form remains an important open problem. Langlands observed that functoriality, combined with expected properties of the correspondence $\pi' \to \pi$, would imply the generalized Ramanujan conjecture for $\mathrm{GL}(n)$. His representation theoretic argument is strikingly similar to Deligne's geometric proof.

(iv) *Sato–Tate conjecture*: The Sato–Tate conjecture for the distribution of the numbers $N_p(E)$ of solutions $\pmod p$ of an elliptic curve E over $\mathbb{Q}$ has a general analogue for automorphic representations. Suppose for example that π is a cuspidal automorphic representation of $\mathrm{GL}(n)$. The generalized Ramanujan conjecture of (iii) asserts that the conjugacy classes, represented by diagonal S_n-orbits

$$c_p(\pi) = S_n \cdot \begin{pmatrix} c_{p,1}(\pi) & & 0 \\ & \ddots & \\ 0 & & c_{p,n}(\pi) \end{pmatrix},$$

have eigenvalues of absolute value 1. The generalized Sato–Tate conjecture describes their distribution in the maximal compact torus $\mathrm{U}(1)^n$ of the dual group $\mathrm{GL}(n, \mathbb{C})$. If π is *primitive* (a notion that requires functoriality even to define), the distribution of these classes should be given by the weight function in the Weyl integration formula for the unitary group $\mathrm{U}(n)$. Langlands sketched a rough argument for establishing such a result from general functoriality. Clozel, Harris, Shepherd-Barron and Taylor followed this argument in their proof of the original Sato–Tate conjecture, but using base change for $\mathrm{GL}(n)$ and deformation results in place of functoriality.

I would like to express my gratitude to Robert Langlands, for his friendship and encouragement over the many years since I first met him in 1968, and also for what he has given to everyone in his beautiful and profound mathematical contributions. They offer inspiration for all of us in these troubled times when we are most in need of it.

Figure 1: Jim Arthur and Bob Langlands,
courtesy of the Simons Foundation

References

[A] J. Arthur, *The work of Robert Langlands*, to appear in *The Abel Prize 2018–2022*, Helge Holden, Ragni Piene editors.

[B1] A. Borel, *Formes automorphes et séries de Dirichlet (d'après R. P. Langlands)*, Séminaire Bourbaki, n° 466, 1975.

[BJ] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, in *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math. vol. 33, Part 1, Amer. Math. Soc. 1979, 189–202.

[Cog] J. Cogdell, *On Artin L-functions*, `https://people.math.osu.edu/cogdell.1/artin-www.pdf`

[Con] K. Conrad, *History of class field theory*, `https://kconrad.math.uconn.edu/blurbs/gradnumthy/cfthistory.pdf`

[F] D. Flath, *Decomposition of representations into tensor products*, in *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math. vol. 33, Part 1, Amer. Math. Soc. 1979, 179–184.

[GJ] R. Godement and H. Jacquet, *Zeta Functions of Simple Algebras*, Lecture Notes in Math. 260, Springer, New York, 1972.

[K] R. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), 611–650.

[L1] R. P. Langlands, *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Math. 544, Springer, New York, 1976 (from original manuscript completed in 1965).

[L2] ———, *Letter to A. Weil*, 1967, `https://publications.ias.edu/letter-to-Weil`

[L3] ———, *Problems in the theory of automorphic forms*, Lecture Notes in Math. 170, Springer, New York, 1970, 18–61 (from original manuscript completed in 1968).

[L4] ———, *On the Functional Equation of the Artin L-functions*, 1970, `https://publications.ias.edu/rpl/paper/61`

[L5] ———, *On the notion of an automorphic representation. A supplement to the preceding paper*, in *Automorphic forms, Representations and L-functions*, Proc. Sympos. Pure Math. vol. 33, Part 1, Amer. Math. Soc. 1979, 203–207.

[S] F. Shahidi, *Automorphic L-functions*, in this volume.

[T1] J. Tate, *Fourier Analysis in Number Fields and Hecke's Zeta-Functions* (Thesis 1950), in *Algebraic Number Theory*, Thompson Book Company, 1967, 305–347.

[T2] ———, *Number theoretic background*, in Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math. vol. 33, Part 2, Amer. Math. Soc. 1979, 3–26.