TRANSFER, THE FUNDAMENTAL LEMMA, AND THE WORK OF WALDSPURGER

JAMES ARTHUR

Automorphic forms are eigenfunctions of natural operators attached to reductive algebraic groups. Their eigenvalues are of great arithmetic significance. In fact, the information they contain is believed to represent a unifying force for large parts of number theory and arithmetic algebraic geometry.

The Langlands program is a collection of interlocking conjectures and theorems that govern the theory of automorphic forms. It explains in precise terms how this theory, with roots in harmonic analysis on algebraic groups, characterizes some of the deepest objects of arithmetic. There has been substantial progress in the Langlands program since its origins in a letter from Langlands to Weil in 1967. In particular, it has had applications to famous problems in number theory, including Artin's conjecture on L-functions, Fermat's Last Theorem, the Sato-Tate conjecture, and the behaviour of Hasse-Weil zeta functions. However, its deepest parts remain elusive.

At the center of the Langlands program is the principle of functoriality, a series of conjectural reciprocity laws among automorphic forms on different groups. There appears to be no direct way to prove it in any but the simplest of cases. One strategy for more general cases has been to compare trace formulas. The general trace formula for a reductive algebraic group G over a number field F is a complex identity, which relates spectral and geometric objects. The spectral side contains the inaccessible data from automorphic forms to which the principle of functoriality applies. The geometric side is more explicit, but also quite complicated. It is a sum of various kinds of integrals over spaces attached to G. The general idea is to compare the spectral data on different groups by establishing relations among the geometric terms in the corresponding trace formulas.

A serious obstruction for over thirty years has been the transfer of test functions between different groups. Given a smooth function of compact support for one group G, one tries to define a function on the second group by requiring that the orbital integrals on the geometric side of its trace formula match those of the first function. The problem is to show that these numbers really do represent the integrals of a smooth function of compact support. The transfer conjecture was formulated precisely by Langlands and Shelstad, and then later by Kottwitz and Shelstad for more general twisted groups. The conjecture included a family of explicit functions, called transfer factors, by which orbital integrals on one group would have to be multiplied in order to be orbital integrals on the other. Transfer factors are themselves a remarkable part of the story. They are natural if complicated objects, whose construction goes to the heart of class field theory.

A test function is defined on the adèlic group $G(\mathbb{A})$. At almost all p-adic places v of F, it is required to be the characteristic function of a maximal compact subgroup of $G(F_v)$. The fundamental lemma is a variant of the transfer conjecture. It asserts that the transfer of such a function must be of the same form, namely the characteristic function of a maximal compact subgroup of the F_v-points of the second group. The fundamental lemma appears at first to be a combinatorial problem. For the orbital integrals of characteristic functions reduce immediately to finite sums of terms that can be calculated explicitly. However, there are infinitely many orbital integrals to be treated, and as they vary, the number of terms in the associated finite sums increases without bound. Various elementary methods have been applied to the fundamental lemma over the years, but they have always met with at best limited success.

In the mid 1990's, Jean-Loup Waldspurger proved that the transfer conjecture would follow from the fundamental lemma. This was quite a surprise. For the fundamental lemma pertains to very special functions at certain p-adic places, while the transfer conjecture applies to general functions at all p-adic places. (The transfer problem for archimedean places had been solved earlier by Shelstad, using the work of Harish-Chandra. In fact, her solution served as a guide for the later construction of general transfer factors.) Waldspurger used global methods, specifically a simple version of the trace formula, to solve what was a local problem. In the past few years, he has also completed a far-reaching study of twisted harmonic analysis, which among other things, reduces the twisted transfer conjecture of Kottwitz and Shelstad to a twisted form of the fundamental lemma.

The breakthrough for the fundamental lemma was provided by Bao Chao Ngo, following his proof with Laumon for the special case of unitary groups. Ngo discovered a striking way to interpret the geometric side of the simple trace formula (or rather its analogue for a global field of positive characteristic). He observed that the entire geometric side could be expressed as a sum over the rational points of an arithmetic Hitchin fibration, the arithmetic analogue of a variety familiar from the theory of G-bundles on a Riemann surface. Earlier, Goresky, Kottwitz and MacPherson had discovered a geometric interpretation for the local terms in the simple trace formula (again for a global field of positive characteristic). Building on their work, and exploiting the interplay of local and global methods in ingenious ways, Ngo was eventually able to establish a general form of the fundamental lemma.

Ngo's results actually apply to a p-adic Lie algebra of positive characteristic. They include a nonstandard fundamental lemma, which Waldspurger was lead to conjecture in his study of twisted harmonic analysis. The theorem needed for the comparison of trace formulas applies to p-adic groups of characteristic 0. The link is provided by two separate results of Waldspurger. A reduction to a Lie algebra of characteristic 0 is included in his work on twisted harmonic analysis. The transition to Lie algebras of positive characteristic is a consequence of a different study, but one which is again based on methods of p-adic harmonic analysis. Another proof of this reduction was subsequently established by Cluckers, Hales and Loeser, by completely different methods of motivic integration.

I should also mention a further generalization of the fundamental lemma which, like it or not, is also needed. It applies to the more exotic weighted orbital integrals, which occur in the general case of the trace formula. This has also been established recently. Exploiting the ideas of Ngo, with among other things, some remarkable new applications of intersection cohomology on which I am not qualified to comment, Chaudouard and Laumon have now proved a general form of the weighted fundamental lemma. It again applies only to a p-adic Lie algebra of positive characteristic. But Waldspurger, working at the same time, has also been able to extend his two theorems of reduction. The most general form of the fundamental lemma is therefore now available in all cases. Thanks to Waldspurger, this in turn implies the general form of the Kottwitz-Langlands-Shelstad transfer conjecture.

I have emphasized the role of transfer in the comparison of trace formulas. This is likely to lead to a classification of automorphic representations for many groups G, according to Langlands' conjectural theory of endoscopy. The fundamental lemma also has other important applications. For example, its proof fills a longstanding gap in the theory of Shimura varieties. For Kottwitz had observed some years ago that the geometric terms in the arithmetic Lefschetz formula for a Shimura variety are twisted orbital integrals. The twisted fundamental lemma now allows a comparison of these terms with corresponding terms in the trace formula. This in turn leads to reciprocity laws between the arithmetic data in the cohomology of many such varieties with the spectral data in automorphic forms.

Let me conclude by saying that Waldspurger has made other major contributions to representation theory, which are quite independent of his pivotal role in the fundamental lemma and transfer. They include a large body of early work on the Shimura correspondence for modular forms that is still very influential, a classification of automorphic discrete spectra for general linear groups (with C. Moeglin), fundamental results on the homogeneity of p-adic characters and Shalika germs, a characterization of the stability properties of unipotent orbital integrals on p-adic classical groups, and a profound study for the group $SO(2n+1)$ of the representations of depth 0 parametrized by Lusztig. As with all of Waldspurger's work, these contributions are marked by their depth, and the application of Waldspurger's extraordinary mathematical power.