The Endoscopic Classification of Representations

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We shall outline a classification [A] of the automorphic representations of special orthogonal and symplectic groups in terms of those of general linear groups. This necessarily includes a classification of local L-packets of representations. It also requires a classification of the extended packets that are the local constituents of nontempered automorphic representations. Our description will be brief. In particular, we will restrict it to *quasisplit*[*] orthogonal and symplectic groups G, even though at least some of the results can be extended (not without effort) to inner twists of G.

The methods rest ultimately on two comparisons of trace formulas. One is the spectral identity that is the end product of the stabilization of the trace formula for G. This was established some years ago [A1], under the assumption of the fundamental lemma. It now holds without condition, thanks to the work of Waldspurger [W1][W2], the recent breakthrough by Ngo [N], and the extensions of Chaudouard and Laumon [CL1] [CL2]. The other is the spectral identity given by the stabilization of the twisted trace formula for $GL(N)$. This formula is still conditional. The relevant twisted fundamental lemmas are now known [W3], [W4], at least up to the twisted variants of [CL1] and [CL2]. The problem is to develop twisted generalizations of the techniques of [A1] and related papers. Until this is done, the results we describe here have also to be regarded as conditional.

We take F to be a local or global field of characteristic 0, and G to be a quasisplit, special orthogonal or symplectic group over F. Then G has a complex dual group $\widehat{G}$, and a corresponding L-group

$$^{L}G = \widehat{G} \rtimes \Gamma_{E/F}.$$

We are taking $\Gamma_{E/F} = \mathrm{Gal}(E/F)$ to be the Galois group of a suitable finite extension E/F. If G is split, for example, the absolute Galois group $\Gamma = \Gamma_F = \Gamma_{\overline{F}/F}$ acts trivially on $\widehat{G}$, and we can take $E = F$.

There are three general possibilities for G, which correspond to the three infinite families of simple groups $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$. They are as follows

Type $\mathbf{B}_n$: $G = SO(2n+1)$ is split, and $\widehat{G} = Sp(2n, \mathbb{C}) = {}^{L}G$.

Type $\mathbf{C}_n$: $G = Sp(2n)$ is split, and $\widehat{G} = SO(2n+1, \mathbb{C}) = {}^{L}G$.

[*]It is understood that G is "classical", in the sense that it is not an outer twist of the split group $SO(8)$ by a triality automorphishm of order 3.

Type $\mathbf{D}_n$: $G = SO(2n)$ is quasisplit, and $\widehat{G} = SO(2n, \mathbb{C})$. In this case, ${}^L G = SO(2n, \mathbb{C}) \rtimes \Gamma_{E/F}$, where E/F is an arbitrary extension of degree 1 or 2 whose Galois group acts by outer automorphisms on $SO(2n, \mathbb{C})$ (which is to say, by automorphisms that preserve a fixed splitting of $SO(2n, \mathbb{C})$). The nontrivial outer automorphism of $SO(2n, \mathbb{C})$ is induced by conjugation by some element in its complement in $O(2n, \mathbb{C})$.

The other infinite family of simple groups is of course $\mathbf{A}_n$. We will regard the split (reductive) group $GL(N)$, with $N = n + 1$, as our representative from this family. Its role is different. For we are treating the representations of $GL(N)$ as known objects, in terms of which we want to classify the representations of G.

1 Statement of the local theorems

Suppose first that F is local. The local Langlands group is the locally compact extension of the local Weil group W_F defined by

$$L_F = \begin{cases} W_F, & \text{if } F \text{ is archimedean,} \\ W_F \times SU(2), & \text{if } F \text{ is } p\text{-adic.} \end{cases}$$

The group G comes with the family

$$\Phi(G) = \{\phi : \ L_F \longrightarrow {}^L G\}$$

of Langlands parameters, and the family

$$\Pi(G) = \{\pi : \ G(F) \longrightarrow GL(V)\}$$

of irreducible (admissible) representations of $G(F)$. We recall that ϕ is an L-homomorphism (which means among other things that it commutes with the projections of L_F and ${}^L G$ onto $\Gamma_{E/F}$), and that it is taken up to conjugacy by $\widehat{G}$ on ${}^L G$. The representation π is of course also taken up to its usual form of equivalence.

We actually work with quotients

$$\widetilde{\Phi}(G) = \Phi(G)/\sim$$

and

$$\widetilde{\Pi}(G) = \Pi(G)/\sim$$

of $\Phi(G)$ and $\Pi(G)$. The equivalence relation $\sim$ is trivial for types $\mathbf{B}_n$ and $\mathbf{C}_n$, but is defined by conjugation by $O(2n)$ (in place of $SO(2n)$) for type $\mathbf{D}_n$. More precisely, if $G = SO(2n)$ is of type D_n, $\widetilde{\Phi}(G)$ is the set of $O(2n, \mathbb{C})/SO(2n, \mathbb{C})$-orbits in $\Phi(G)$ under the action of $O(2n, \mathbb{C})$ by conjugation on ${}^L G$, and $\widetilde{\Pi}(G)$ is the set of $O(2n, F)/SO(2n, F)$ orbits in $\Pi(G)$ under the action of

$O(2n,F)$ by conjugation on $G(F)$. It is only these coarser sets that are related to the analogous sets for $GL(N)$. Among them, a special role is played by the smaller sets

$$\widetilde{\Phi}_{\mathrm{bdd}}(G) = \{\phi \in \widetilde{\Phi}(G) : \ \mathrm{im}(\phi) \text{ is bounded}\}$$

of parameters of bounded image, and

$$\widetilde{\Pi}_{\mathrm{temp}}(G) = \{\pi \in \widetilde{\Pi}(G) : \ \pi \text{ is tempered}\}$$

of irreducible representations whose characters (and matrix coefficients) are tempered with respect to Harish-Chandra's Schwartz space $\mathcal{C}(G)$.

Each of the three pairs of sets above has a natural role in the local classification of representations. We have one more pair of local sets to introduce, but its role is primarily global. It consists of the set

$$\widetilde{\Psi}(G) = \{\psi : \ L_F \times SU(2) \longrightarrow {}^L G, \ \mathrm{im}(\psi) \text{ is bounded}\}$$

of (orbits of) L-homomorphisms from the product $L_F \times SU(2)$ to ${}^L G$, and the set

$$\widetilde{\Pi}_{\mathrm{unit}}(G) = \{\pi \in \widetilde{\Pi}(G) : \ \pi \text{ is unitary}\}$$

of (orbits of) irreducible representations of $G(F)$ that are unitary. This pair of sets is especially important for us. For we shall see that it governs the local constituents of automorphic representations.

We can regard $\widetilde{\Psi}(G)$ as an intermediate set between two families of Langlands parameters. The earlier set $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ can obviously be identified with the subset of parameters in $\widetilde{\Psi}(G)$ that are trivial on the second factor $SU(2)$. On the other hand, any $\psi \in \widetilde{\Psi}(G)$ maps to a parameter

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right), \qquad w \in L_F,$$

in $\widetilde{\Phi}(G)$, where $|w|$ is the pullback to L_F of the absolute value on W_F, and ψ is identified with its analytic extension from $L_F \times SU(2)$ to the group $L_F \times SL(2,\mathbb{C})$. It follows from the boundedness condition on ψ, and the fact that a homomorphism from $SL(2,\mathbb{C})$ to a complex group is determined by its diagonal weight, that the mapping $\psi \to \phi_\psi$ from $\widetilde{\Psi}(G)$ to $\widetilde{\Phi}(G)$ is injective. There is consequently a chain

$$\widetilde{\Phi}_{\mathrm{bdd}}(G) \subset \widetilde{\Psi}(G) \subset \widetilde{\Phi}(G)$$

of parameter sets. We also have a parallel claim

$$\widetilde{\Pi}_{\mathrm{temp}}(G) \subset \widetilde{\Pi}_{\mathrm{unit}}(G) \subset \widetilde{\Pi}(G)$$

of sets of irreducible representations, since any tempered representation is unitary.

For any $\psi \in \widetilde{\Psi}(G)$, we have the centralizer

$$S_\psi = \mathrm{Cent}\big(\mathrm{im}(\psi), \widehat{G}\big)$$

in $\widehat{G}$ of the image of (a representative of) ψ. This is of course a complex reductive subgroup of $\widehat{G}$. We write

$$\overline{S}_\psi = S_\psi / Z(\widehat{G})^\Gamma$$

for its quotient by the group Galois invariants in the center of $\widehat{G}$. The group

$$\mathcal{S}_\psi = \pi_0(\overline{S}_\psi) = \overline{S}_\psi / \overline{S}_\psi^0$$

of connected components of $\overline{S}_\psi$ is then a finite abelian 2-group.

Theorem 1 [A, Theorem 1.5.1] (F local). (a) *For any $\psi \in \widetilde{\Psi}(G)$, there is a finite set $\widetilde{\Pi}_\psi$ over $\widetilde{\Pi}_{\mathrm{unit}}(G)$, together with a mapping*

$$\pi \in \widetilde{\Pi}_\psi \longrightarrow \langle \cdot, \pi \rangle \in \widehat{\mathcal{S}}_\psi,$$

from $\widetilde{\Pi}_\psi$ to the group of (linear) characters on $\mathcal{S}_\psi$, both of which are canonically determined by endoscopic character relations.

(b) *Suppose that $\phi = \psi$ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$. Then the elements in $\widetilde{\Pi}_\phi$ are tempered and multiplicity free. Moreover, the mapping from $\widetilde{\Pi}_\phi$ to $\widehat{\mathcal{S}}_\phi$ is injective in general, and bijective in case F is p-adic. Finally*

$$\widetilde{\Pi}_{\mathrm{temp}}(G) = \coprod_{\phi \in \widetilde{\Phi}_{\mathrm{bdd}}(G)} \widetilde{\Pi}_\phi.$$

Remarks. 1. By a set $\widetilde{\Pi}_\psi$ over $\widetilde{\Pi}_{\mathrm{unit}}(G)$, we understand simply a mapping

$$\widetilde{\Pi}_\psi \longrightarrow \widetilde{\Pi}_{\mathrm{unit}}(G).$$

of sets. The multiplicity free assertion in (b) means that the fibres are trivial if $\phi = \psi$ lies in $\widetilde{\Phi}_{\mathrm{bdd}}(G)$, and that we can therefore regard $\widetilde{\Pi}_\phi$ as a subset of $\widetilde{\Pi}_{\mathrm{temp}}(G)$.

2. The last assertion of (b) is that any element in $\widetilde{\Pi}_{\mathrm{temp}}(G)$ lies in a unique packet $\widetilde{\Pi}_\phi$. This is the local Langlands correspondence for G, or rather a slightly weaker version in the case $G = SO(2n)$ that classifies orbits in $\Pi_{\mathrm{temp}}(G)$ under a group of order 2, rather than individual representations in $\Pi_{\mathrm{temp}}(G)$.

We will need to be precise about the endoscopic character relations that determine the packets and the resulting classification. Since this is slightly more technical, we shall postpone it until after we have stated the global theorems, and listed some of the preliminary applications.

In preparation for the global theorems, we have to introduce a larger set $\widetilde{\Psi}^+(G)$ of local parameters. It will be used to represent the local constituents of automorphic representations in a way that provides for the possible failure of the generalized Ramanujan conjecture for $GL(N)$. We define $\widetilde{\Psi}^+(G)$ in the same way as $\widetilde{\Psi}(G)$, but without the condition that the image of the parameter ψ be bounded. The set so obtained can be regarded as a natural complexification of $\widetilde{\Psi}(G)$. Theorem 1 then extends by analytic continuation to parameters ψ in $\widetilde{\Psi}^+(G)$, as will become evident once we have described the endoscopic character formulas on which it is based. The price to pay for this generalization is that the representations in a packet $\widetilde{\Pi}_\psi$ need no longer be irreducible or unitary. However, for the subset of parameters that are local constituents of automorphic representations (in the discrete spectrum, say), the representations are unitary, and most probably also irreducible.

2 Statement of the global theorems

Suppose now that F is global. In this report, we shall state the global theorems in terms of the hypothetical global Langlands group L_F. This group is expected to be a locally compact extension

$$1 \longrightarrow K_F \longrightarrow L_F \longrightarrow W_F \longrightarrow 1$$

of the global Weil group W_F by a compact connected group K_F. It should come with a conjugacy class of local embeddings

$$\begin{array}{ccc} L_{F_v} & \longrightarrow & W_{F_v} \\ \downarrow & & \downarrow \\ L_F & \longrightarrow & W_F \end{array}$$

over the corresponding Weil groups, for any valuation v of F. The essential property of L_F is that its irreducible N-dimensional representations should be in canonical bijection with the cuspidal automorphic representations of $GL(N, \mathbb{A})$.

The structure of L_F is far from known, in contrast to the simple formula we have for the local Langlands group. Its existence is deeper than the theory of endoscopy, to which the results we are describing pertain, and deeper even than the principle of functoriality. However, we shall assume here that L_F does exist in order to simplify the discussion. We cannot of course do this in the volume [A]. The only choice there is to formulate global results for G in terms of self-dual, cuspidal automorphic representations of $GL(N, \mathbb{A})$ rather than irreducible self-dual N-dimensional representations of L_F. This leads to some significant technical complications.

Assuming the existence of L_F, we define the global parameter sets $\widetilde{\Phi}(G)$, $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ and $\widetilde{\Psi}(G)$ as in the local case above. For example,

$$\widetilde{\Psi}(G) = \Psi(G)/\sim,$$

where $\sim$ is the equivalence relation above, and $\Psi(G)$ is the set of $\widehat{G}$-conjugacy classes of L-homomorphisms ψ from the product $L_F \times SU(2)$ to ${}^LG$ such that the image of ψ is bounded.

For any $\psi \in \widetilde{\Psi}(G)$, and any valuation v, we obtain a localization ψ_v by pulling ψ back to the local Langlands group L_{F_v}. We may as well treat this object as a local parameter in the larger set $\widetilde{\Psi}^+(G_v)$, following what must be done in [A]. In the context of L_F, this amounts to our taking on the weaker assumption necessitated by the possible failure of the generalized Ramanujan conjecture, that L_{F_v} embeds only in a natural complexification $L_{F,\mathbb{C}}$ of L_F.

For any $\psi \in \widetilde{\Psi}(G)$, we have the centralizer S_ψ in $\widehat{G}$ of the image of ψ, and the 2-group defined by its quotient

$$\mathcal{S}_\psi = S_\psi / S_\psi^0 \, Z(\widehat{G})^\Gamma = \overline{S}_\psi / \overline{S}_\psi^0.$$

For any v, we then have a localization mapping

$$x \longrightarrow x_v, \qquad\qquad x \in \mathcal{S}_\psi,$$

from $\mathcal{S}_\psi$ to $\mathcal{S}_{\psi_v}$. Given Theorem 1, we define the global packet

$$\widetilde{\Pi}_\psi = \Big\{ \pi = \bigotimes_v \pi_v : \ \pi_v \in \widetilde{\Pi}_{\psi_v}, \ \langle \cdot, \pi_v \rangle = 1 \text{ for almost all } v \Big\}.$$

Any representation $\pi = \bigotimes \pi_v$ in $\widetilde{\Pi}_\psi$ then has a character

$$\langle x, \pi \rangle = \prod_v \langle x_v, \pi_v \rangle, \qquad\qquad x \in \mathcal{S}_\psi,$$

on $\mathcal{S}_\psi$.

Our main global theorem gives a decomposition of the automorphic discrete spectrum

$$L^2_{\text{disc}}\big(G(F)\backslash G(\mathbb{A})\big) \subset L^2\big(G(F)\backslash G(\mathbb{A})\big).$$

The problem is to describe this space explicitly as a module over the global Hecke algebra

$$\mathcal{H}(G) = \widetilde{\bigotimes_v} \mathcal{H}(G_v),$$

which is of course defined as a restricted tensor product of local Hecke algebras on the groups $G_v = G(F_v)$. But because we are dealing with orbits of local parameters in the case $G = SO(2n)$, we have to be content with a slightly weaker result. For any v, we can write $\widetilde{\mathcal{H}}(G_v)$ for the full local Hecke algebra $\mathcal{H}(G_v)$ if G is of type $\mathbf{B}_n$ or $\mathbf{C}_n$. If G is of the type $\mathbf{D}_n$ however, we must take $\widetilde{\mathcal{H}}(G_v)$ to be the proper subalgebra of symmetric functions in $\mathcal{H}(G_v)$ under a suitable fixed automorphism of $G(F_v)$ of order 2 attached to the nontrivial outer automorphism of G. We can then form the locally symmetric subalgebra

$$\widetilde{\mathcal{H}}(G) = \widetilde{\bigotimes_v} \widetilde{\mathcal{H}}(G_v)$$

of the global Hecke algebra $\mathcal{H}(G)$.

Theorem 2 [A, Theorem 1.5.2] (F global). *There is an $\widetilde{\mathcal{H}}(G)$-module isomorphism*

$$L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big) \cong \bigoplus_{\psi\in\widetilde{\Psi}_2(G)} \bigoplus_{\pi\in\widetilde{\Pi}_\psi(\varepsilon_\psi)} m_\psi \pi,$$

where

$$\widetilde{\Psi}_2(G) = \big\{\psi \in \widetilde{\Psi}(G) :\ |S_\psi| < \infty\big\},$$

and m_ψ equals 1 or 2, while

$$\varepsilon_\psi :\ \mathcal{S}_\psi \longrightarrow \{\pm 1\}$$

is a linear character defined explicitly in terms of symplectic ε-factors, and

$$\widetilde{\Pi}_\psi(\varepsilon_\psi) = \big\{\pi \in \widetilde{\Pi}_\psi :\ \langle \cdot, \pi\rangle = \varepsilon_\psi\big\}$$

is the subset of the global packet $\widetilde{\Pi}_\psi$ attached to ε_ψ.

The integer m_ψ is the order of the set $\Psi(G,\psi)$ of $\widehat{G}$-orbits of L-homomorphisms from $L_F\times SU(2)$ to ${}^LG$ that map to the (possibly larger) orbit ψ. It equals 2 if G is of type $\mathbf{D}_n$ and the degrees of the irreducible constituents of ψ (as an N-dimensional representation) are all even. In all other cases, it equals 1.

The sign character ε_ψ is more interesting. We define a natural representation

$$\tau_\psi :\ S_\psi \times L_F \times SU(2) \longrightarrow GL(\widehat{\mathfrak{g}})$$

of the product of S_ψ with $L_F \times SU(2)$ on the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ by setting

$$\tau_\psi(s,g,h) = \mathrm{Ad}\big(s\cdot\psi(g,h)\big), \qquad\qquad s\in S_\psi,\ (g,h)\in L_F\times SU(2),$$

where Ad is the adjoint representation of ${}^LG$ on $\widehat{\mathfrak{g}}$. Let

$$\tau_\psi = \bigoplus_\alpha \tau_\alpha = \bigoplus_\alpha (\lambda_\alpha \otimes \mu_\alpha \otimes \nu_\alpha)$$

be its decomposition into irreducible representations λ_α, μ_α and ν_α of the groups S_ψ, L_F and $SU(2)$. We then define

$$\varepsilon_\psi(x) = \prod_\alpha{}' \det\big(\lambda_\alpha(s)\big), \qquad\qquad s\in S_\psi,$$

where x is the image of s in $\mathcal{S}_\psi$, and $\prod'$ denotes the product over those indices α with μ_α symplectic and

$$\varepsilon\left(\frac{1}{2},\mu_\alpha\right) = -1.$$

Theorem 2 is the main global theorem. However, to prove it, one must also establish the following result.

Theorem 3 [A, Theorem 1.5.3] (F global). (a) *Suppose that ϕ belongs to the subset*

$$\widetilde{\Phi}_{\mathrm{sim}}(G) = \{\phi \in \widetilde{\Phi}(G) : \ \overline{S}_\psi = S_\psi / Z(\widehat{G})^\Gamma = 1\}$$

of simple generic parameters in $\widetilde{\Psi}(G)$. Then the dual group $\widehat{G}$ is orthogonal if and only if the symmetric square L-function $L(s, \phi, S^2)$ has a pole at $s = 1$, while $\widehat{G}$ is symplectic if and only if it is the skew-symmetric square L-function $L(s, \phi, \Lambda^2)$ that has a pole at $s = 1$.

(b) *Suppose $\phi_1 \in \widetilde{\Phi}_{\mathrm{sim}}(G_1)$ and $\phi_2 \in \widetilde{\Phi}_{\mathrm{sim}}(G_2)$, for two of our groups G_1 and G_2. Then the corresponding Rankin-Selberg ε-factor satisfies*

$$\varepsilon\left(\tfrac{1}{2}, \phi_1 \times \phi_2\right) = 1,$$

if $\widehat{G}_1$ and $\widehat{G}_2$ are either both orthogonal or both symplectic.

The conclusions of Theorem 3 are familiar from the work of Cogdell, Kim, Piatetskii-Shapiro, and Shahidi [CKPS1], [CKPS2], Ginzburg, Rallis and Soudry [GRS], and Lapid [Lap] on representations with Whittaker models.

3 Initial applications

Let us describe a few of the initial applications of the three theorems. Some of these are included in the assertions of the theorems, others are proved in [A], while a couple of others will require further thought. We shall list them, with little comment, according to whether F is local or global.

F local

(i) The representations $\pi \in \widetilde{\Pi}_\psi$ in the packets attached to local parameters $\psi \in \widetilde{\Psi}(G)$ are *unitary*. This is included in the assertion of Theorem 1(a).

(ii) The *local Langlands correspondence* is valid if the quasi split group G equals $SO(2n+1)$ or $Sp(2n)$, and in the slightly weaker form given by $O(2n)$-orbits in the remaining case that G equals $SO(2n)$. This is Theorem 1(b), together with the local refinement Theorem $1'$ that we will state in the next section. (In the third case $G = SO(2n)$, there is a further refinement in [A, §8.4] that falls just short of the full Langlands correspondence.)

F local or global

(iii) If F is local (respectively global), the packet $\widetilde{\Pi}_\phi$ attached to any parameter ϕ in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$ contains a representation π that is locally (resp. globally) *generic*, and such that the character $\langle \cdot, \pi \rangle$ on $\mathcal{S}_\phi$ equals 1. This is established in [A, Proposition 8.3.2], using the results of Ginzburg, Rallis and Soudry. (The Whittaker datum with respect to which π is generic is determined by the underlying Langlands-Shelstad transfer factors, a topic we will not review in this report.)

F global

(iv) *Rankin-Selberg L-functions* for orthogonal and symplectic groups (that is, pairs of quasi split groups (G_1, G_2) of the kind we are considering) have analytic continuation and functional equation. In fact, their analytic behaviour coincides with that of classical Rankin-Selberg L-functions for general linear groups. This is just the *principle of functoriality* for the standard representations of the L-groups ${}^L G_1$ and ${}^L G_2$, which follows from Theorem 2 and the refinement of Theorem 1 we will state presently. In other words, Rankin-Selberg L-functions for pairs (G_1, G_2) are among the original L-functions studied in [JPS].

(v) Theorem 2 will also give us some control of the *symmetric square* and *skew symmetric square L-functions*. On the one hand, these L-functions govern normalizing factors of global intertwining operators for maximal (Siegel) parabolic subgroups of our groups G. On the other, Langlands' theory of Eisenstein series attaches representations in the automorphic discrete spectrum of G to any poles in the right half plane of the intertwining operators. But Theorem 2 should rule out such representations, and therefore also such poles. This question needs to be examined more carefully, and is under consideration by Shahidi.

(vi) Our group G has no *embedded eigenvalues*, in the sense of families of Hecke eigenvalues

$$c^S = \{c_v : \ v \notin S\}$$

(rather than its analogue from mathematical physics for eigenvalues of Laplace operators at archimedean places). As usual, c represents a family of semisimple conjugacy classes in ${}^L G$, taken up to the equivalence relation $(c')^{S'} \sim c^S$ if $c'_v = c_v$ for almost all v. (If $\widehat{G} = SO(N, \mathbb{C})$ for N even, conjugacy is assumed to be taken with respect to the full orthogonal group $O(N, \mathbb{C})$.) The assertion here is that if ψ belongs to the complement of $\widetilde{\Psi}_2(G)$ in $\widetilde{\Psi}(G)$, there is no representation π in the automorphic discrete spectrum of G such that the associated two equivalence classes of families $c(\pi)$ and $c(\psi)$ of Hecke eigenvalues are equal. This follows again from Theorem 2, together with the corresponding property [JS] for $GL(N)$.

(vii) Automorphic representations π of $G(\mathbb{A})$ attached to a generic parameter ϕ in the set

$$\widetilde{\Phi}_2(G) = \widetilde{\Phi}_{\mathrm{bdd}}(G) \cap \widetilde{\Psi}_2(G)$$

occur in the discrete spectrum with *multiplicity* 1 or 2. Here π is intended to be an actual representation of $G(\mathbb{A})$, which represents some orbit in the global packet $\widetilde{\Pi}_\phi$. More precisely, π occurs with multiplicity 1 unless

(a) G is of type $\mathbf{D}_n$,

(b) the global integer m_ϕ defined in §2 equals 2, and

(c) each of the corresponding local integers m_{ϕ_v} equals 1, in which case it occurs with multiplicity 2. (See the remarks surrounding (8.3.8) in [A], which are based on Corollaries 6.6.6 and 6.7.3 of [A].)

(viii) *Symplectic ε-factors* govern which (orbits of) representations in the global packet $\widetilde{\Pi}_\psi$ attached to a general parameter $\psi \in \widetilde{\Psi}_2(G)$ actually do occur in the discrete spectrum. This "application" falls into the "requires further thought" category. What are the implications of the sign character ε_ψ (which is trivial if ψ lies in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$), as it occurs in the multiplicity formula of Theorem 2?

4 On endoscopic character relations

We return to the case that F is local, and more specifically, the endoscopic character relations referred to in Theorem 1(a). We shall state a more precise supplement to Theorem 1(a) that characterizes the packets $\widetilde{\Pi}_\psi$ and the associated families of linear characters $\langle \cdot, \pi \rangle$ on $\mathcal{S}_\psi$.

We write N for the rank of the general linear group attached to the dual $\widehat{G}$ of G. In other words, N equals $2n$, $2n+1$ or $2n$, according to whether the given group G lies in the family $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{D}_n$. There is then a natural embedding of the L-group ${}^LG$ of G into $GL(N, \mathbb{C})$. It is also convenient to introduce the connected component

$$\widetilde{G}(N) = GL(N) \rtimes \widetilde{\theta}(N)$$

in the semidirect product $\widetilde{G}(N)^+$ of $GL(N)$, for the standard automorphism

$$\widetilde{\theta}(N):\; g \longrightarrow \widetilde{J}\, {}^t g^{-1} \widetilde{J}^{-1}, \qquad\qquad g \in GL(N),\; \widetilde{J} = \widetilde{J}(N) = \begin{pmatrix} & 0 & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{N+1} & & & 0 \end{pmatrix},$$

of order 2. This twisted object comes with analogues of the parameter sets $\widetilde{\Psi}(G)$ and $\widetilde{\Phi}(G)$. They consist of the set $\widetilde{\Psi}(N) = \Psi\big(\widetilde{G}(N)\big)$ of (equivalence classes of) self-dual, N-dimensional, unitary representations of the product $L_F \times SU(2)$, and the set $\widetilde{\Phi}(N) = \Phi\big(\widetilde{G}(N)\big)$ of (equivalence classes of) self-dual, N-dimensional, not necessarily unitary representations of L_F.

The embedding of ${}^LG$ into $GL(N, \mathbb{C})$ gives a mapping from $\widetilde{\Psi}(G)$ to $\widetilde{\Psi}(N)$, which can be seen to be injective. There is also an injective mapping from $\widetilde{\Psi}(N)$ to $\widetilde{\Phi}(N)$, obtained by pulling back a representation ψ of $L_F \times SU(2)$ to the representation

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right), \qquad\qquad w \in L_F,$$

of L_F. Finally, we can restrict the local Langlands correspondence for $GL(N)$ (which we recall was established by Harris-Taylor and Henniart, and more recently by Scholze) to the set of self-dual representations of L_F. This gives a bijection from $\widetilde{\Phi}(N)$ to the set $\widetilde{\Pi}(N) = \Pi(\widetilde{G}(N))$ of (equivalence classes of) irreducible, self-dual representations of $GL(N, F)$. The composition

$$\widetilde{\Psi}(G) \hookrightarrow \widetilde{\Psi}(N) \hookrightarrow \widetilde{\Phi}(N) \xrightarrow{\sim} \widetilde{\Pi}(N)$$

of the three mappings then provides an injection

$$\psi \longrightarrow \pi_\psi, \qquad \psi \in \widetilde{\Psi}(G),$$

from $\widetilde{\Psi}(G)$ into $\widetilde{\Pi}(N)$. The representation π_ψ of $GL(N, F)$ is called a *Speh representation.* It is known to be unitary.

For any $\psi \in \widetilde{\Psi}(G)$, the self-dual representation π_ψ is θ-stable. It can therefore be extended to the semidirect product

$$\widetilde{G}(N, F)^+ = GL(N, F) \rtimes \langle \widetilde{\theta}(N) \rangle.$$

There are actually two extensions, which differ by a sign on the coset $\widetilde{G}(N, F)$ of $\widetilde{\theta}(N)$. However, the theory of Whittaker models leads to a *canonical* extension $\widetilde{\pi}_\psi$ of π_ψ to the group $\widetilde{G}(N, F)^+$. (One takes $\widetilde{\pi}_\psi(\widetilde{\theta}(N))$ to be a quotient of the intertwining operator of order 2 for the standard representation ρ_ψ attached to ψ that stabilizes a Whittaker vector.) This in turn gives us a $GL(N, F)$-invariant linear form on the Hecke bi-module $\widetilde{\mathcal{H}}(N) = \widetilde{\mathcal{H}}(\widetilde{G}(N))$ of functions on the coset $\widetilde{G}(N, F)$. It is the twisted invariant character

$$\widetilde{f}_N(\psi) = \mathrm{tr}(\widetilde{\pi}_\psi(\widetilde{f})), \qquad \widetilde{f} \in \widetilde{\mathcal{H}}(N),$$

where $\widetilde{\pi}_\psi(\widetilde{f})$ is the operator

$$\int_{\widetilde{G}(N,F)} \widetilde{f}(x) \widetilde{\pi}_\psi(x) dx.$$

The given group G has its own Hecke algebra, and of more relevance to us here, the subalgebra $\widetilde{\mathcal{H}}(G)$ of symmetric functions in $\mathcal{H}(G)$ defined prior to the statement of Theorem 2. It also comes with an essential set of algebraic objects. We write $\mathcal{E}(G)$ for the set of (isomorphism classes of) endoscopic data G' for G [LS, (1.2)], [KS, 2.1].

Here is our supplement to Theorem 1.

Theorem 1$'$ [A, Theorem 2.2.1] (F local). (a) *For any $\psi \in \widetilde{\Psi}(G)$, there is a unique stable linear form*

$$f \longrightarrow f^G(\psi), \qquad f \in \widetilde{\mathcal{H}}(G),$$

on $\widetilde{\mathcal{H}}(G)$ such that

$$\widetilde{f}^G(\psi) = \widetilde{f}_N(\psi), \qquad \widetilde{f} \in \widetilde{\mathcal{H}}(N),$$

where $\widetilde{f}^G$ is the Kottwitz-Shelstad (twisted) transfer of $\widetilde{f}$ from $\widetilde{G}(N,F)$ to $G(F)$.

(b) *For any $\psi \in \widetilde{\Psi}(G)$, the packet $\widetilde{\Pi}_\psi$ and the mapping $\pi \to \langle \cdot, \pi \rangle$ from $\widetilde{\Pi}_\psi$ to $\widehat{\mathcal{S}}_\psi$ are defined canonically by the relation*

$$f'(\psi') = \sum_{\pi \in \widetilde{\Pi}_\psi} \langle s_\psi x, \pi \rangle \operatorname{tr}\bigl(\pi(f)\bigr), \qquad s \in S_{\psi,\mathrm{ss}},\ f \in \widetilde{\mathcal{H}}(G),$$

where x is the image in $\mathcal{S}_\psi$ of the semisimple point $s \in S_{\psi,\mathrm{ss}}$, and s_ψ is the ψ-image in S_ψ or $\mathcal{S}_\psi$ of the point $1 \times \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $L_F \times SU(2)$, while

$$(G', \psi'), \qquad G' \in \mathcal{E}(G),\ \psi' \in \widetilde{\Psi}(G'),$$

is the endoscopic pair that corresponds bijectively with the given pair (ϕ, s), and $f' = g^{G'}$ is the Langlands-Shelstad transfer of f from $G(F)$ to $G'(F)$.

This theorem requires further comment. We have written f^G at the beginning of Part (a) to mean the *equivalence class* of f, relative to the equivalence relation $f_1 \sim f_2$ if $S(f_1) = S(f_2)$ for every stable linear form on $\widetilde{\mathcal{H}}(G)$. (We recall that a stable linear form is one that lies in the linear span of the set of strongly regular, stable orbital integrals, which is to say, orbital integrals over intersections of $G(F)$ with strongly regular geometric conjugacy classes.) The Kottwitz-Shelstad transfer $\widetilde{f}^G$ of $\widetilde{f}$ is defined only as an equivalence class in $\widetilde{\mathcal{H}}(G)$, but its value $\widetilde{f}^G(\psi)$ at the putative stable linear form $f \to f^G(\psi)$ still makes sense. Part (a) asserts that the stable linear form is uniquely determined by the requirement that this value be equal to the linear form $\widetilde{f}_N(\psi)$ on $\widetilde{\mathcal{H}}(N)$ defined above.

Part (b) is predicated on the bijective correspondence

$$(G', \psi') \ \leftrightarrow\ (\psi, s), \qquad s \in S_{\psi,\mathrm{ss}},$$

that has always (at least in the case of parameters in the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of $\widetilde{\Psi}(G)$) been the motivational heart of Langlands' conjectural theory [L] of endoscopy. (See [A, §1.4], for example.) One verifies that the construction of the stable linear form described for (G, ψ) in (a) specializes to an endoscopic pair (G', ψ'). If G' lies in the important subset $\widetilde{\mathcal{E}}_{\mathrm{ell}}(G)$ of elliptic endoscopic data, for example, there is a decomposition

$$(G', \psi') = (G'_1 \times G'_2, \psi'_1 \times \psi'_2)$$

into groups G'_1 and G'_2 of the kind we are considering, and if $f' = f'_1 \times f'_2$ also is decomposable, we obtain a product

$$f'(\psi') = f'_1(\psi'_1)\, f'_2(\psi'_2)$$

of stable linear forms defined as in (a). For every pair (ψ, s), we thus arrive at a linear form

$$f'(\psi'), \qquad f \in \widetilde{\mathcal{H}}(G),$$

on $\widetilde{\mathcal{H}}(G)$. Part (b) is then an explicit definition of the packets $\widetilde{\Pi}_\psi$ and pairings $\langle x, \pi\rangle$ in terms of these linear forms.

Theorem $1'$ thus gives a canonical construction of the objects in Theorem 1. They in turn characterize the global objects of Theorems 2 and 3. Taken together, the theorems thus reduce the representation theory of our groups G to that of general linear groups $GL(N)$.

5 On the comparison of trace formulas

The proof of the theorems we have stated rests ultimately to a comparison of trace formulas. The comparison has many ramifications, which occupy much of the volume [A]. We shall say just a few words, if only to be able to state a supplementary global theorem. For an elementary discussion of the comparison that is a little more comprehensive, the reader can look at the introductory paper [A2] on the embedded eigenvalue problem.

We are assuming now that F is global. The starting point is actually a definition, rather than a formula. It is the linear form

$$I^G_{\mathrm{disc}}(f) = \sum_{\{M\}} |W(M)|^{-1} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)|^{-1}\,\mathrm{tr}\big(M_P(w)\,\mathcal{I}_P(f)\big), \qquad f \in \widetilde{\mathcal{H}}(G),$$

on the locally symmetric global Hecke algebra $\widetilde{\mathcal{H}}(G)$, whose terms will be familiar from the trace formula. The outer sum is over conjugacy classes of Levi subgroups M of (parabolic subgroups $P = NM$ of) G. The inner sum is over the set

$$W(M)_{\mathrm{reg}} = \{w \in W(M):\ \det(w-1) \neq 0\}$$

of regular elements in the Weyl group

$$W(M) = \mathrm{Norm}(A_M, G)/M$$

of (G, A_M), where the determinant refers to the action of w on the Lie algebra of the F-split component A_M of the centre of M. We have written $\mathcal{I}_P$ for the representation of $G(\mathbb{A})$ induced parabolically from the discrete spectrum

$$L^2_{\mathrm{disc}}\big(M(F)\,A^+_{M,\infty}\backslash M(\mathbb{A})\big), \qquad A^+_{M,\infty} = (R_{F/\mathbb{Q}}A_M)(\mathbb{R})^0,$$

of $M(\mathbb{A})$. Finally, $M_P(w)$ is the self-intertwining operator of $\mathcal{I}_P$ attached to w that plays a central role in Langlands' theory of Eisenstein series.

The linear form $I^G_{\mathrm{disc}}(f)$ is known as the *discrete part* of the trace formula. It is composed of those terms on the spectral side that are discrete in the relevant spectral variables. The term with $M = G$ corresponds to the actual discrete spectrum to which Theorem 2 applies. However, its contribution to the trace formula is not easily separated from that of the terms with $M \neq G$. The most fundamental spectral variables are the Hecke families $c = \{c_v\}$ mentioned in the application (vi) in §3. They have further properties, which we recall as follows.

The locally symmetric Hecke algebra is a direct limit

$$\widetilde{\mathcal{H}}(G) = \varinjlim_{S} \widetilde{\mathcal{H}}(G, K^S),$$

where S ranges over finite sets of valuations outside of which G is unramified, and $\widetilde{\mathcal{H}}(G, K^S)$ is the subspace of functions in $\widetilde{\mathcal{H}}(G)$ that are bi-invariant under a "hyperspecial" maximal compact subgroup

$$K^S = \prod_{v \notin S} K_v$$

of $G(\mathbb{A}^S)$. For any such S, the *unramified* (locally symmetric) Hecke algebra

$$\widetilde{\mathcal{H}}^S_{\mathrm{un}} = \widetilde{\mathcal{H}}\big(K^S \backslash G(\mathbb{A}^S)/K^S\big)$$

acts by multipliers

$$(f, h) \longrightarrow f_h, \qquad\qquad f \in \widetilde{\mathcal{H}}(G, K^S),\ h \in \widetilde{\mathcal{H}}^S_{\mathrm{un}},$$

on $\widetilde{\mathcal{H}}(G, K^S)$. The function $f_h \in \widetilde{\mathcal{H}}(G, K^S)$ is characterized by property that

$$\pi(f_h) = \widehat{h}\big(c^S(\pi)\big)\pi(f),$$

for any π in the subset $\widetilde{\Pi}(G, K^S)$ of representations in $\widetilde{\Pi}(G)$ with K^S-fixed vectors. The function $\widehat{h}$ is the Satake transform of h, which is defined on the set of S-families

$$c^S = \{c_v :\ v \notin S\}$$

of Hecke eigenvalues, while $c^S(\pi)$ is the family attached to π.

Since it is discrete in all of the spectral variables, the linear form $I^G_{\mathrm{disc}}(f)$ can be written as a sum of Hecke eigenforms. More precisely, if f belongs to $\widetilde{\mathcal{H}}(G, K^S)$, there is a canonical decomposition

$$I^G_{\mathrm{disc}}(f) = \sum_{c^S} I^G_{\mathrm{disc},c^S}(f)$$

where

$$I^G_{\mathrm{disc},c^S}(f_h) = \widehat{h}(c^S)\, I^G_{\mathrm{disc},c^S}(f),$$

for any $h \in \mathcal{H}^S_{\mathrm{un}}$. Taking a direct limit over S, we obtain a decomposition

$$I^G_{\mathrm{disc}}(f) = \sum_{c} I^G_{\mathrm{disc},c}(f), \qquad\qquad f \in \widetilde{\mathcal{H}}(G),$$

in which c ranges over equivalence classes of families c^S. This sum, incidentally, is infinite, but convergence is assured by the work [Mu] of Muller. (In [A], we indexed $I^G_{\mathrm{disc}}(f)$ by a discrete set of positive numbers t that parametrize the norms of the imaginary parts of the internal archimedean spectral parameters. For any t and f, there are then only finitely many c with $I^G_{\mathrm{disc},t,c}(f) \neq 0$.)

Suppose that ψ lies in the global set $\widetilde{\Psi}(N)$ of self-dual, unitary, N-dimensional representations of the group $L_F \times SU(2)$. Since the global parameter ψ is unramified at almost all places, it comes with a natural equivalence class $c(\psi)$ of families of Hecke eigenvalues. The following proposition from [A] is an important starting point for the comparison.

Proposition [A, Proposition 3.4.1]. *Suppose that c is an equivalence class of families of Hecke eigenvalues. Then*

$$I^G_{\mathrm{disc},c}(f) = 0, \qquad f \in \widetilde{\mathcal{H}}(G),$$

unless $c = c(\psi)$ for some $\psi \in \widetilde{\Psi}(N)$.

We write

$$I^G_{\mathrm{disc},\psi}(f) = I^G_{\mathrm{disc},c(\psi)}(f),$$

for any $\psi \in \widetilde{\Psi}(N)$. The proposition then implies that

$$I^G_{\mathrm{disc}}(f) = \sum_{\psi \in \widetilde{\Psi}(N)} I^G_{\mathrm{disc},\psi}(f), \qquad f \in \widetilde{\mathcal{H}}(G).$$

To study I^G_{disc}, and in particular the term with $M = G$ in I^G_{disc} that corresponds to the automorphic discrete spectrum of G, one has only to study the linear form $I^G_{\mathrm{disc},\psi}$ attached to any ψ. Notice however that the proposition does not assert that ψ belongs to $\widetilde{\Psi}(G)$. It is in fact an important step along the way to show $I^G_{\mathrm{disc},\psi}(f)$ vanishes unless ψ does lie in the subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$.

At this point, our formula for the linear form $I^G_{\mathrm{disc},\psi}$ is only a definition. The stabilization of the trace formula for G provides a separate description of $I^G_{\mathrm{disc},\psi}$. It is a decomposition

$$I^G_{\mathrm{disc},\psi}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,G')\, \widehat{S}^{G'}_{\mathrm{disc},\psi}(f'), \qquad f \in \widetilde{\mathcal{H}}(G),$$

of $I^G_{\mathrm{disc}\psi}(f)$ into a linear combination of linear forms parametrized by the set $\mathcal{E}_{\mathrm{ell}}(G)$ of isomorphism classes of (global) elliptic endoscopic data G' for G, with explicit coefficients $\iota(G,G')$. For any G', the linear form

$$f \longrightarrow \widehat{S}^{G'}_{\mathrm{disc},\psi}(f'), \qquad f \in \widetilde{\mathcal{H}}(G'),$$

is the composition of a stable linear form

$$S^{G'}_{\mathrm{disc},\psi} = \sum_{\{\psi' :\, \psi' \to \psi\}} S^{G'}_{\mathrm{disc},\psi'}$$

on $\widetilde{\mathcal{H}}(G')$ (or rather the associated form $\widehat{S}^{G'}_{\mathrm{disc},\psi}$ on the space of equivalence classes in $\widetilde{\mathcal{H}}(G')$ defined locally in the last section) with the global Langlands-Shelstad transfer

$$f \longrightarrow f' = f^{G'}$$

of f from $\widetilde{\mathcal{H}}(G)$ to $\widetilde{\mathcal{H}}(G')$. The stable linear form $\widehat{S}^{G'}_{\mathrm{disc},\psi'}(f')$ depends only on the elliptic endoscopic group

$$G' = G'_1 \times G'_2$$

and the corresponding global parameter

$$\psi' = \psi'_1 \times \psi'_2,$$

and equals

$$\widehat{S}^{G'}_{\mathrm{disc},\psi'}(f') = \widehat{S}^{G'_1}_{\mathrm{disc},\psi'_1}(f'_1)\, \widehat{S}^{G'_2}_{\mathrm{disc},\psi'_2}(f'_2),$$

if $f' = f'_1 \times f'_2$ is decomposable. It is therefore defined inductively for G as the difference

$$S^G_{\mathrm{disc},\psi}(f) = I^G_{\mathrm{disc},\psi}(f) - \sum_{G' \neq G} \iota(G,G')\, \widehat{S}^{G'}_{\mathrm{disc},\psi}(f'), \qquad f \in \widetilde{\mathcal{H}}(G).$$

For any ψ, the definition of $I^G_{\mathrm{disc},\psi}(f)$ specializes to a formula

$$I^G_{\mathrm{disc},\psi}(f) = \sum_{\{M\}} |W(M)|^{-1} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)|^{-1}\, \mathrm{tr}\big(M_{P,\psi}(w)\,\mathcal{I}_{P,\psi}(f)\big), \qquad f \in \widetilde{\mathcal{H}}(G),$$

for its ψ-component. Here $\mathcal{I}_{P,\psi}$ is the direct sum of those subrepresentations π of $\mathcal{I}_P$ such that $c(\pi) = c(\psi)$, while $M_{P,\psi}(w)$ is the corresponding restriction of the intertwining operator $M_P(w)$. We must also have a concrete formula for the terms in the stabilization of $I^G_{\mathrm{disc},\psi}(f)$ if we hope to compare the two decompositions. This is provided by the following supplement to Theorem 2, in which the identity (a) is called the stable multiplicity formula in [A].

Theorem 2′ [A, Theorem 4.1.2 and Corollary 4.1.3] (F global). (a) *Given any $\psi \in \widetilde{\Psi}(N)$, we have a formula*

$$S^G_{\mathrm{disc},\psi}(f) = |\mathcal{S}_\psi|^{-1} m_\psi\, \varepsilon_\psi(s_\psi)\, \sigma(\overline{S}{}^0_\psi)\, f^G(\psi), \qquad f \in \widetilde{\mathcal{H}}(G),$$

with a product

$$f^G(\psi) = \prod_v f^G_v(\psi_v), \qquad f = \prod_v f_v,$$

of local stable linear forms defined in Theorem 1′(a), and coefficients $|\mathcal{S}_\psi|^{-1}$, m_ψ, $\varepsilon_\psi(s_\psi)$ and $\sigma(\overline{S}{}^0_\psi)$ that vanish unless ψ lies in the subset $\widetilde{\Psi}(G)$ of $\widetilde{\Psi}(N)$.

(b) *The linear form attached to G' in the stabilization of $I^G_{\mathrm{disc},\psi}(f)$ satisfies*

$$\widehat{S}^{G'}_{\mathrm{disc},\psi}(f') = \sum_{\{\psi' \in \Psi(G') :\, \psi' \to \psi\}} |\mathcal{S}_\psi|^{-1} \varepsilon_{\psi'}(s_{\psi'})\, \sigma(\overline{S}{}^0_{\psi'})\, f'(\psi').$$

We again need to append some comments. The first three coefficients $|\mathcal{S}_\psi|^{-1}$, m_ψ and $\varepsilon_\psi(s_\psi)$ in Part (a) have already been discussed. We note that prescriptions of m_ψ and ε_ψ following the statement of Theorem 2 carry over to general parameters $\psi \in \widetilde{\Psi}(G)$, so the value

$$\varepsilon_\psi(s_\psi) = \varepsilon_\psi\left(\psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right)$$

is defined. The fourth coefficient $\sigma(\overline{S}_\psi^0)$ is something else . It is the value at $\overline{S}_\psi^0$ of a certain $\mathbb{Q}$-valued function σ, whose domain is the set of all isomorphism classes of complex, connected reductive algebraic groups S_1. The function σ is defined inductively by a combinatorial identity, which mimics the identity between the two decompositions of $I^G_{\mathrm{disc},\psi}(f)$. With this analogy, invariant linear forms correspond to the values of a $\mathbb{Q}$-valued function i on the larger domain of connected components S of (isomorphism classes of) general complex reductive groups, while as we have said, stable linear forms correspond to the values of σ on the subset of connected complex groups. We have no call to review the combinatorial construction here, referring the reader instead to [A, Proposition 4.1.1] and [A2, Theorem 4]. Part (b) of the theorem is considerably simpler. It follows directly from the specialization to G' of the formula for G in (a).

Theorem $2'$(a) is one of the main results of [A]. We cannot expect to prove it from the two decompositions of $I^G_{\mathrm{disc},\psi}(f)$ alone. For we have seen that the spectral decomposition serves only as a definition for $I^G_{\mathrm{disc},\psi}(f)$, while the endoscopic decomposition amounts to an inductive definition of the stable linear form $S^G_{\mathrm{disc},\psi}(f)$. To have a chance of proving any of the theorems, one needs to combine these with something else. The extra ingredients are two similar decompositions from the twisted trace formula for $GL(N)$, which is to say, the trace formula for the connected component

$$\widetilde{G}(N) = GL(N) \rtimes \widetilde{\theta}(N)$$

of the group $\widetilde{G}(N)^+$ over F.

For any $\psi \in \widetilde{\Psi}(N)$, the ψ-discrete part of the (twisted) trace formula for $\widetilde{G}(N)$ is the (twisted) invariant linear form

$$\widetilde{I}^N_{\mathrm{disc},\psi}(\widetilde{f}) = \sum_{\{M\}} |W(M)|^{-1} \sum_{w \in W^N(M)_{\mathrm{reg}}} |\det(w-1)|^{-1}\, \mathrm{tr}\big(M_{P,\psi}(\widetilde{f})\,\mathcal{I}_{P,\psi}(\widetilde{f})\big), \qquad \widetilde{f} \in \widetilde{\mathcal{H}}(N),$$

on the Hecke bi-module $\widetilde{\mathcal{H}}(N)$ of functions on $\widetilde{G}(N,\mathbb{A})$. The terms on the right hand side are analogues for $\widetilde{G}(N)$ of terms in the corresponding expression for $I_{\mathrm{disc},\psi}(f)$ ([A, §4.1], [A2, §3]). They serve again simply to define the linear form on the left. However, they are now to be regarded as known, since they pertain to the representations of $GL(N)$ that are to characterize representations of our groups G. The same goes for the stabilization of $\widetilde{I}^N_{\mathrm{disc},\psi}$. It is an endoscopic decomposition

$$\widetilde{I}^N_{\mathrm{disc},\psi}(\widetilde{f}) = \sum_{G \in \widetilde{\mathcal{E}}_{\mathrm{ell}}(N)} \widetilde{\iota}(N,G)\, \widehat{S}^G_{\mathrm{disc},\psi}(\widetilde{f}^G), \qquad \widetilde{f} \in \widetilde{\mathcal{H}}(N),$$

in terms of the stable linear forms $S^G_{\mathrm{disc},\psi}$ that have already been defined. The indexing set $\widetilde{\mathcal{E}}_{\mathrm{ell}}(N)$ consists of the isomorphism classes of (global) elliptic endoscopic data for $\widetilde{G}(N)$. It includes our groups G, which comprise the subset $\widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$ of simple endoscopic data, as well as composite groups

$$G = G_S \times G_O,$$

in which $\widehat{G}_S$ is symplectic and $\widehat{G}_O$ is orthogonal, and for which $\widehat{S}^G_{\mathrm{disc},\psi}(\widetilde{f}^G)$ is expressed in terms of the relevant products.

The twisted trace formula has now been established in complete generality [LW]. Its stabilization is what remains conditional. However, there is reason to be optimistic that the stabilization of the general twisted trace formula will be established in the not too distant future.

The proof of the theorems is complex, even with the two trace formulas and their stabilizations. I hope I have given some idea of its foundations. Let me just reiterate that the technical supplement Theorem $2'$ of Theorem 2 is a fundamental part of the argument. It provides a concrete formula for the stable linear forms $S^G_{\mathrm{disc},\psi}$ in the endoscopic expansion of $\widetilde{I}^N_{\mathrm{disc},\psi}(\widetilde{f})$, in addition to the stable forms $S^{G'}_{\mathrm{disc},\psi'}$ in the endoscopic expansion of of $I^G_{\mathrm{disc},\psi}(f)$.

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