A stratification related to characteristic polynomials

James Arthur

Department of Mathematics
University of Toronto
40 St. George Street, Toronto, Ontario, Canada
M5S 2E4

Contents

Introduction 1

1 On the trace formula for general linear groups 7

2 Geometric terms and characteristic polynomials 12

3 The problem of Poisson summation 19

4 Spectral terms and global parameters 32

5 The stratification and the problem of comparison 44

6 On the supplementary terms 58

References 64

Introduction

This paper concerns Beyond Endoscopy, the strategy proposed by Langlands around 2000 for using the trace formula to attack the general principle of functoriality. Our goal is to

Email address: `arthur@math.utoronto.ca` (James Arthur)

Preprint submitted to Elsevier April 18, 2017

describe a hypothetical formula for general linear groups that would display the contribution of nontempered automorphic representations to the geometric side of the trace formula. Such a formula seems to be a necessary prerequisite for recognizing the contribution of the more subtle functorial images to the geometric side.

We recall that functoriality postulates fundamental reciprocity laws among data in automorphic representations. It is a cornerstone of the Langlands program, which has profound implications for number theory, arithmetic geometry and representation theory. (See [La1].) The principle of functoriality has been established in some significant cases. However, what has been done seems less consequential if its compared with the enormous problems that remain.

Some of the progress in functoriality has been a byproduct of the theory of endoscopy. Endoscopy, a theory that had been conjectured by Langlands earlier, is the study of the internal structure of automorphic representations of a reductive group G over a number field F. Its aim is to organize the representations into natural packets that are compatible with their local structure. The packets should in turn be parametrized by suitable homomorphisms from the Galois group Γ_F, or better, the Weil group W_F, or best of all, some large extension of W_F, into the complex L-group ${}^LG = \hat{G} \rtimes W_F$ of G. It is these packets, or equivalently their parameters, to which the principle of functoriality applies. Endoscopy has been established for quasisplit classical groups. (See [Ar7] and [Mok], which rely among other things on the fundamental lemma and transfer [N1] and [W3].) The heart of the proof is a comparison of trace formulas for different groups G.

Beyond Endoscopy aims to combine the trace formula for G with the L-functions attached to automorphic representations π of G. Recall that these are the Dirichlet series of Euler products

$$L^S(s, \pi, r) = \prod_{v \notin S} L_v(s, \pi, r) = \prod_{v \notin S} \det(1 - r(c_v(\pi))q_v^{-s})^{-1},$$

attached to finite dimensional representations

$$r : {}^L G \to \mathrm{GL}(N, \mathbb{C})$$

of ${}^L G$. They converge absolutely for the real part $\mathrm{Re}(s)$ of s sufficiently large, and in the case that π is tempered, are expected to converge for $\mathrm{Re}(s) > 1$. In general, each L-function is expected to have analytic continuation to a meromorphic function of $s \in \mathbb{C}$, which satisfies a functional equation that relates its values at s and $(1 - s)$. (We recall that the functional equation is best stated only after completing $L^S(s, \pi, r)$ with Euler factors $L_v(s, \pi, r)$ at the finite set of ramified valuations $v \in S$.) Functoriality asserts that for any L-homomorphism

$$\rho : {}^L G' \to {}^L G$$

between the L-groups of two groups G and G', and any automorphic representation π' of G', there is an automorphic representation π of G such that

$$L^S(s, \pi, r) = L^S(s, \pi', r \circ \rho)$$

for each r. Since the analytic continuation and functional equation are known to hold for the L-function on the left, in case $G = \mathrm{GL}(N)$ and r is the standard representation of $\hat{G} = \mathrm{GL}(N, \mathbb{C})$, one sees easily from the right hand side that these properties hold in general if the assertion of functoriality is valid. However, they would be difficult to establish directly.

The trace formula is an identity between a geometric expansion and a spectral expansion, both of which depend on a test function $f \in C_c^\infty(G(\mathbb{A}))$ on the adelic group $G(\mathbb{A})$. The spectral side contains the automorphic representations we want to understand, while the geometric side contains explicit and often complex terms that we would like to apply to this end. The core of the spectral side is the trace

$$I_{\mathrm{cusp}}(f) = \mathrm{tr}(R_{\mathrm{cusp}}(f)) = \sum_{\pi \in \Pi_{\mathrm{cusp}}(G)} \mathrm{mult}(\pi)\,\mathrm{tr}(\pi(f))$$

of the right convolution operator by f on the space $L^2_{\mathrm{cusp}}(G(F) \setminus G(\mathbb{A}))$ of square-integrable cusp forms. (If $G(F) \setminus G(\mathbb{A})$ does not have finite volume, one must divide $G(\mathbb{A})$ also by a central subgroup, as we will do for the group $G = \mathrm{GL}(n+1)$ treated in the text.) Indeed, most questions about functoriality and L-functions reduce ultimately to the special case of cuspidal automorphic representations $\pi \in \Pi_{\mathrm{cusp}}(G)$. One suggestion, which might seem unpromising if our experience with endoscopy is to be our guide, would be to transfer the complementary spectral terms to the geometric side, and then to regard the resulting identity as a formula for $I_{\mathrm{cusp}}(f)$ in terms of geometric *and* spectral quantities.

The proposal of Langlands is related to this idea. Very roughly, it amounts to an enrichment of f to a test function

$$f_s, \qquad\qquad s \in \mathbb{C},\ \mathrm{Re}(s) \gg 0,$$

that depends on the automorphic L-functions $L^S(s, \pi, r)$ attached to a fixed finite dimensional representation r. More precisely, given f and r, one could take S to be a large finite set of valuations, and then require that

$$\mathrm{tr}(\pi(f_s)) = \mathrm{tr}(\pi(f))\ L^S(s, \pi, r),$$

for any irreducible (tempered) representation π of $G(\mathbb{A})$ unramified away from S. One could then try to study the expression for $\mathrm{tr}(R_{\mathrm{cusp}}(f_s))$ given by the trace formula as a function of s. Ideally one might try to show that the terms in this expression have analytic continuation to $\mathrm{Re}(s) > 1$, and meromorphic continuation to $s = 1$. The resulting residue could then be regarded as a formula for the distribution

$$\mathrm{res}_{s=1}(\mathrm{tr}(R_{\mathrm{cusp}}(f_s))) = \sum_{\pi\in\Pi_{\mathrm{cusp}}(G)} \mathrm{mult}(\pi)\,\mathrm{tr}(\pi(f))\mathrm{res}_{s=1}(L^S(s,\pi,r)).$$

In particular, it would isolate the cuspidal representations π whose L-function have a pole at $s = 1$. The distribution of such representations, as r varies, will be closely related to groups G' from which π is a functorial image $\pi' \to \pi$ relative to some L-homomorphism ρ.

Langlands' proposal is of course more sophisticated. We have already noted that functoriality is to be regarded as a transfer of packets

$$\{\pi'\} \to \{\pi\}$$

of representations. But the trace formula is sensitive to the representations within a packet. For this reason, one will have to work with the stable trace formula, a more subtle identity derived from the ordinary trace formula, whose spectral side does depend only on the global packets. Another refinement is motivated by a Tauberian theorem, which expresses the residue as a limit of partial sums of Dirichlet coefficients of the L-function $L^S(s,\pi,r)$. In this way, one would need only weight the spectral values of the test function f by these Dirichlet coefficients, rather the actual L-functions. The implication of this is that one considers the question on the geometric side without the assumption of meromorphic continuation above. We refer the reader to the beginnings of the papers [La4] and [FLN] for a more comprehensive discussion. (See also [Ar8, Section 2] and [N3, Section 1].)

The Tauberian limit will not exist for the individual terms in the formula for $I_{\mathrm{cusp}}(f)$, or rather its stable counterpart. The reason is that they include characters of nontempered automorphic representations, whose L-functions can have poles for $\mathrm{Re}(s) > 1$. The bad spectral terms (which include all noncuspidal characters from the discrete part of the spectral expansion) will have to cancel something from the geometric expansion, because we do expect the limit of the difference (being equal to $I_{\mathrm{cusp}}(f)$) to exist. The problem is to find it. We want to isolate some constituents of the geometric expansion (which we hope will be reasonably explicit), whose behaviour matches that of the bad spectral terms. The remaining geometric terms would then in principle be amenable to a study of the Tauberian limit.

It is this problem that we will investigate, in the case G is a general linear group $\mathrm{GL}(n+1)$. The endoscopic packet structure for general linear groups is trivial, so we will be able to work with the basic trace formula (since it is the same as the stable trace formula in this case). We shall use it to introduce candidates for the contribution of the bad spectral terms to the geometric side. We are not in a position to make any kind of quantitative comparison of the two kinds of terms. In fact, the geometric objects we construct will themselves be hypothetical, for they depend on the conjectural extension to $\mathrm{GL}(n+1)$ of the Poisson summation theorem of A. Altug for $\mathrm{GL}(2)$. However these objects are represented by simple and natural formulas. Aesthetic considerations alone suggest that they should be related in some way to the associated spectral terms.

In the preparation of this article, I decided to include a larger expository component than I had originally planned. The ideas related to Beyond Endoscopy that are coming forward from various sources seem compelling to me, even as they remain largely unknown. I hope that this article might bring some of them to a broader audience. Sections 1, 2 and 4 contain elementary descriptions of some aspects of the trace formula, while Sections 3, 4 and 5 include some discussion of the work of Altug. Sections 1 and 2 can also be regarded as motivation for the construction we present in Section 5. I should add, however, that just about everything in this paper, expository or not, is in need of further thought. The paper may in fact be somewhat premature, and I hope that it does not contain too many errors or misstatements.

Langlands' proposals for Beyond Endoscopy are centred around the trace formula. That will be the focus of the paper. There are other approaches to the problems of functoriality, which are sometimes more directly tied to automorphic L-functions. They take motivation from earlier papers [BK1], [BK2] of Braverman and Kazhdan, and sometimes also from ideas related to Ngo's proof [N1], [H] of the fundamental lemma (as does the trace formula itself). We refer the reader to the papers [G], [Laf], [N2] and [N3] in which L-functions have a large role, and to papers [Sak] and [V] for another approach based on relative trace formulas.

I would like to thank colleagues Kumar Murty, Joe Repka and Jacob Tsimerman for enlightening conversations on topics related to this paper.

1. On the trace formula for general linear groups

Our aim is a discussion, at times informal, of some of the first problems that arise in Beyond Endoscopy. As we have noted, Beyond Endoscopy is a strategy proposed by Langlands for using the trace formula to attack the general principle of functoriality. It suggests many interesting and deep problems, which call for new methods across a large area of mathematics. We shall consider some of these questions, as they apply to the simplest cases of the relevant trace formulas.

In this paper, G will be a general linear group $G(n) = \mathrm{GL}(n+1)$, taken over the field of rational numbers. Then $G(\mathbb{Q})$ is a discrete subgroup of the locally compact adelic group $G(\mathbb{A})$. Let

$$Z_+ = Z_{n,+} = Z_{G,+} = A_G(\mathbb{R})^0 \subset G(\mathbb{R}) \subset G(\mathbb{A})$$

be the connected central subgroup of positive real scalar matrices in $G(\mathbb{R})$. The quotient of $G(\mathbb{A})$ by $Z_+G(\mathbb{Q})$ then has finite invariant volume, and supports a Hilbert space of square integrable functions with nontrivial discrete spectrum. In particular, its subspace of cuspidal functions

$$L^2_{\mathrm{cusp}}(Z_+G(\mathbb{Q}) \setminus G(\mathbb{A})) \subset L^2(Z_+G(\mathbb{Q}) \setminus G(\mathbb{A}))$$

is a Hilbert space of functions on which the action of $G(\mathbb{A})$ by right translation decomposes discretely. The irreducible constituents of this unitary representation are the objects of interest. They contain data that among other things govern some of the fundamental workings of the arithmetic world.

One hopes to understand the deeper properties of cuspidal automorphic representations through a sustained analysis of the trace formula. We shall review the primary terms on each side, with the aim of understanding how the analysis proposed in [La4] should proceed. We recall that all of the terms are distributions, depending on a test function f in the relevant space

$$\mathcal{D}(G) = C_c^\infty(Z_+ \setminus G(\mathbb{A})) = C_c^\infty(Z_+ \setminus G(\mathbb{R})) \otimes C_c^\infty(G(\mathbb{A}_{\text{fin}})$$

of smooth compactly supported functions.

The general (invariant) trace formula for G is an identity between a geometric expansion $I_{\text{geom}}(f)$ on one hand and a spectral expansion $I_{\text{spec}}(f)$ on the other. The general terms in these expansions are quite complicated. In this paper, we will consider mainly the primary terms, which are familiar and easy to describe. That is not to say that the more complex secondary terms are less important. The point is rather that they represent extensions of problems in Beyond Endoscopy that are best considered first for the primary terms. The reader might consult [Ar8, (1.1), (1.2)] for a little more perspective, and the general references [Ar1], [Ar4] and [Ar5] for complete descriptions of all of the terms.

The primary terms on the geometric side of the trace formula are those in the *regular elliptic part*

$$I_{\text{ell,reg}}(f) = \sum_{\gamma \in \Gamma_{\text{ell,reg}}(G)} a^G(\gamma) f_G(\gamma), \qquad\qquad f \in \mathcal{D}(G). \tag{1.1}$$

The index of summation is the set $\Gamma_{\text{ell,reg}}(G)$ of conjugacy classes γ in $G(\mathbb{Q})$ that are regular, in the sense that the centralizer G_γ of (any representative of) γ in G is a torus, and elliptic in the sense that the quotient $(Z_+G_\gamma(\mathbb{Q}) \setminus G_\gamma(\mathbb{A}))$ has finite invariant volume. In the case $G = \mathrm{GL}(n+1)$ here, these conditions together specialize to the requirement that the characteristic polynomial p_γ of γ be irreducible over $\mathbb{Q}$. For any γ, the coefficient in the sum is the associated volume

$$a^G(\gamma) = \mathrm{vol}(\gamma) = \mathrm{vol}(Z_+G_\gamma(\mathbb{Q}) \setminus G_\gamma(\mathbb{A})), \tag{1.2}$$

while the distribution is the normalized invariant orbital integral

$$f_G(\gamma) = |D(\gamma)|^{\frac{1}{2}} \operatorname{Orb}(f,\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{A})\setminus G(\mathbb{A})} f(x^{-1}\gamma x)dx, \qquad \gamma \in \Gamma_{\mathrm{ell,reg}}(G), \tag{1.3}$$

of f with respect to the invariant measure on the conjugacy class of γ in $G(\mathbb{A})$. The normalizing factor is given by the Weyl discriminant

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}, \tag{1.4}$$

where $\mathfrak{g}$ and $\mathfrak{g}_\gamma$ are the Lie algebras of G and G_γ respectively. It could have been omitted here, since its global absolute value equals 1, but the Weyl discriminant in general plays a fundamental role in the local aspects of Beyond Endoscopy.

The primary terms on the spectral side comprise what is known as its *discrete part*

$$I_{\mathrm{disc}}(f) = \sum_{\pi \in \Pi_{\mathrm{disc}}(G)} a^G(\pi) f_G(\pi), \qquad f \in \mathcal{D}(G), \tag{1.5}$$

namely that part of the spectral side that decomposes as a discrete linear combination of characters

$$f_G(\pi) = \mathrm{tr}(\pi(f)) = \mathrm{tr}\left(\int_{G(\mathbb{A})} f(x)\pi(x)dx\right), \qquad \pi \in \Pi_{\mathrm{disc}}(G). \tag{1.6}$$

It is defined as the invariant linear form

$$I_{\mathrm{disc}}(f) = \sum_M |W(M)|^{-1} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)|^{-1} \,\mathrm{tr}\left(M_P(w)\mathcal{I}_P(f)\right), \tag{1.7}$$

in which M is summed over conjugacy classes of standard Levi subgroups of G (parametrized in this case by partitions of $(n+1)$) and

$$W(M) = \mathrm{Norm}_G(M)/M$$

is the associated Weyl group (a product of symmetric groups). For any $w \in W(M)$, $\det(w-1)$ is the determinant of $(w-1)$ as a linear operator on the Lie algebra of the real group $Z_{M,+}/Z_{G,+}$, and $W(M)_{\mathrm{reg}}$ is the subset of elements w for which this determinant is nonzero. In the right hand factor, P is the standard parabolic subgroup of G with Levi component M, and $\mathcal{I}_P$ is the representation of $G(\mathbb{A})$ obtained by parabolic induction from the representation of $M(\mathbb{A})$ on the discrete spectrum

$$L^2_{\mathrm{disc}}(Z_{M,+}M(\mathbb{Q}) \setminus M(\mathbb{A})) \subset L^2(Z_{M,+}M(\mathbb{Q}) \setminus M(\mathbb{A})).$$

Finally

$$M_P(w) : \mathcal{I}_P \to \mathcal{I}_P$$

is the global intertwining operator attached to w that is at the heart of Langlands' theory of Eisenstein series.

Notice that the term with $M = G$ in (1.7) is just the trace of f on the discrete spectrum $L^2_{\mathrm{disc}}(Z_+G(\mathbb{Q}) \setminus G(\mathbb{A}))$ of G. In particular, $I_{\mathrm{disc}}(f)$ comes with more terms than just those in the discrete spectrum. In fact, we have a properly embedded chain

$$\Pi_{\mathrm{cusp}}(G) \subset \Pi_2(G) \subset \Pi_{\mathrm{disc}}(G)$$

of families of irreducible unitary representations of $Z_+ \setminus G(\mathbb{A})$, where $\Pi_{\mathrm{cusp}}(G)$ is the set of cuspidal representations, $\Pi_2(G)$ is the set of representations that occur in the discrete spectrum, and $\Pi_{\mathrm{disc}}(G)$ is the set of representations in (1.5) that support the distribution $I_{\mathrm{disc}}(G)$. (We have to use the subscript 2, meaning *square integrable*, for the discrete spectrum, since the subscript *disc* has always stood for the discrete part of the trace formula.) We may as well also introduce distributions

$$I_2(f) = \sum_{\pi \in \Pi_2(G)} f_G(\pi), \tag{1.8}$$

and

$$I_{\mathrm{cusp}}(f) = \prod_{\pi \in \Pi_{\mathrm{cusp}}(G)} f_G(\pi), \tag{1.9}$$

to go along with $I_{\mathrm{disc}}(f)$. They of course stand respectively for the trace of f (as a right convolution operator) on the discrete spectrum and on the cuspidal discrete spectrum. We have not included coefficients $a(\pi)$ in (1.8) and (1.9), which would be required in (1.7) if we were to write the right hand side as a linear combination of characters since it is known for $G = \mathrm{GL}(n+1)$ that representations in the discrete spectrum occur with multiplicity one. It is also known that the representations in $\Pi_{\mathrm{disc}}(G)$ that lie in the complement of $\Pi_2(G)$ support the terms with $M \neq G$ in (1.7). In other words, these supplementary terms contain no characters that occur in the discrete spectrum. (See [Ar7, (1.3)], for example.)

The distribution $I_{\mathrm{disc}}(f)$ is the natural spectral object handed to us by the trace formula.

But it is the trace $I_2(f)$ of f on the discrete spectrum that is of most interest for automorphic representations. We recall that Moeglin and Waldspurger (using Langlands' theory of Eisenstein series) have classified the complement of $\Pi_{\mathrm{cusp}}(G)$ in $\Pi_2(G)$ [MW], so the trace $I_{\mathrm{cusp}}(f)$ of f on the cuspidal discrete spectrum is really the fundamental object. For Beyond Endoscopy, the distinction is critical. The proposals of Langlands for combining the trace formula with (yet to be discovered) techniques from analytic and algebraic number theory can only be applied to cuspidal automorphic representations $\pi \in \Pi_{\mathrm{cusp}}(G)$. This would seem to demand some evidence of the noncuspidal characters from $\Pi_2(G)$ in the geometric terms from $I_{\mathrm{ell,reg}}(f)$. More generally, we would like to have a natural interpretation of the contribution of the difference

$$I_{\mathrm{disc}}(f) - I_{\mathrm{cusp}}(f) \tag{1.10}$$

to the geometric side of the trace formula. This is the problem we want to discuss.

2. Geometric terms and characteristic polynomials

In putting aside the more complex supplementary terms, we are implicitly thinking of the trace formula as an approximate identity between the geometric expansion $I_{\mathrm{ell,reg}}(f)$ and its spectral counterpart $I_{\mathrm{disc}}(f)$. However, we have noted that Langlands' ideas for studying L-functions through the trace formula will apply only to the cuspidal automorphic representations that comprise the smaller spectral expansion $I_{\mathrm{cusp}}(f)$. Does the trace formula lead to a *geometric* formula for $I_{\mathrm{cusp}}(f)$? Or as we put it in the last section, is there a reasonable formula for the contribution of the difference $I_{\mathrm{disc}}(f) - I_{\mathrm{cusp}}(f)$ to $I_{\mathrm{ell,reg}}(f)$. There is no a priori reason why such a formula should exist. Among other things, we are ignoring any possible contributions from the supplementary terms. Nevertheless, we shall derive an expression from $I_{\mathrm{ell,reg}}(f)$ in Section 5 (based on a hypothesis from that section) that seems to be at least formally related to what we imagine to be the contribution from $I_{\mathrm{cusp}}(f)$.

There is a different way to describe the problem. According to the generalized Ramanujan conjecture, the characters of cuspidal automorphic representations $\pi \in \Pi_{\mathrm{cusp}}(G)$ are tempered. In other words, the characters of local constituents π_v of π are continuous linear forms on the associated Schwartz spaces $\mathcal{C}(G(F_v))$ defined by Harish-Chandra. Despite its resolution in important special cases, however, this conjecture is still far from known in general. Langlands' strategy would be to proceed initially as if it were known, and then to deduce it in the process of establishing functoriality. On the other hand, the representations in the complement of $\Pi_{\mathrm{cusp}}(G)$ in $\Pi_2(G)$ really are nontempered. (This follows from the classification of these representations in [MW] and Langlands' use of estimates of Harish-Chandra to define what are now known as Langlands quotients [La3].) Therefore the geometric expansion $I_{\mathrm{ell,reg}}(f)$ should not be tempered. Can one find an explicit formula for its nontempered component?

The general question is a little more subtle. We are trying to separate the contribution to $I_{\mathrm{ell,reg}}(f)$ of the complement of $\Pi_{\mathrm{cusp}}(G)$ in the larger set of representations $\Pi_{\mathrm{disc}}(G)$, some of which remain tempered. However, this can also be formulated in terms of the asymptotic properties of characters, so the basic idea is similar.

Incidentally, the invariant orbital integrals that represent the summands in the expansion (1.1) for $I_{\mathrm{ell,reg}}(f)$ are tempered distributions. This is a consequence of the basic estimates of Harish-Chandra. It is the actual sum in (1.1) that is responsible for the failure of the distribution $I_{\mathrm{ell,reg}}(f)$ to be tempered. The density of the sum $I_{\mathrm{ell,reg}}(f)$ turns out to approximate the integral of f over $G(\mathbb{A})$, a nontempered distribution equal to the character of the trivial, 1-dimensional representation of $G(\mathbb{A})$. This distribution is of course just the leading term in the expansion of $I_2(f)$.

We recall that the space of test functions is a topological direct limit

$$\mathcal{D}(G) = \varinjlim_{V} \mathcal{D}(G_V)$$

where V ranges over finite sets of valuations on $\mathbb{Q}$ that contain the archimedean place $v_{\mathbb{R}} = v_\infty$. By definition,

$$\mathcal{D}(G_V) = C_c^\infty(Z_+ \setminus G(\mathbb{R})) \otimes \left(\widetilde{\bigotimes_{v \in V - (v_\infty)}} C_c^\infty(G(\mathbb{Q}_v)) \right)$$

is a product of finitely many function spaces. It embeds (injectively) into $\mathcal{D}(G)$ under the mapping that sends $f_V \in \mathcal{D}(G_V)$ to the product $f_V f^V$, where f^V is the characteristic function of the standard maximal compact subgroup K^V of $G^V = G(\mathbb{A}^V)$, the subgroup of elements $g \in G(\mathbb{A})$ such that $g_v = 1$ for every $v \in V$. In this paper, we follow the slightly different convention of taking f^V to be a function in the full unramified Hecke algebra

$$\mathcal{H}(G^V, K^V) = C_c^\infty(K^V \setminus G^V / K^V)$$

of K^V-biinvariant functions in $C_c^\infty(G^V)$. In other words, given $f \in \mathcal{D}(G)$, we will work with a decomposition

$$f = f_V f^V, \qquad\qquad f_V \in \mathcal{D}(G_V),\ f^V \in \mathcal{H}(G^V, K^V),$$

for some fixed V. The normalized orbital integral of f in (1.1) over a class $\gamma \in \Gamma_{\mathrm{ell,reg}}(G)$ is then a corresponding product

$$f_G(\gamma) = f_{V,G}(\gamma) f_G^V(\gamma),$$

which will be convenient to rewrite as the product

$$f_G(\gamma) = f_{V,G}(\gamma) \cdot |D(\gamma)|_V^{-\frac{1}{2}} \operatorname{Orb}(f^V, \gamma), \tag{2.1}$$

of the normalized orbital integral $f_{V,G}(\gamma)$ of f_V, the unnormalized orbital integral $\operatorname{Orb}(f^V, \gamma)$

of f^V, and the normalizing factor

$$\prod_{v \notin V} |D(\gamma)|_v^{\frac{1}{2}} = \prod_{v \in V} |D(\gamma)|_v^{-\frac{1}{2}} = |D(\gamma)|_V^{-\frac{1}{2}}.$$

The first factor $f_{V,G}(\gamma)$ in (2.1) is a local distribution. The third factor $\mathrm{Orb}(f^V, \gamma)$, on the other hand, is to be regarded as a global object. However, for any γ, it is a product of *finitely* many orbital integrals of functions in local Hecke algebras (if we assume that f^V itself is an (infinite) product.) This last factor is particularly important. It should be the setting for a fundamental lemma. However, at the moment there seems to be no guess as to what the fundamental lemma would be in Beyond Endoscopy.

An important change of perspective was introduced in [FLN], following Ngo's use of the Hitchin fibration to prove the endoscopic form of the fundamental lemma. The authors there suggested parametrizing the semisimple conjugacy classes that index terms in geometric expansions by points in what they called the base of the Steinberg-Hitchin fibration. In the case $G = \mathrm{GL}(n+1)$ here, this amounts to identifying points $\gamma \in \Gamma_{\mathrm{ell,reg}}(G)$ with their characteristic polynomials $p_\gamma(\lambda)$. The base of the Steinberg-Hitchin fibration becomes a product

$$\mathcal{A}(n) = \mathcal{B}(n) \times \mathbb{G}_m$$

of affine n-space $\mathcal{B}(n)$ with multiplicative group $\mathbb{G}_m = \mathrm{GL}(1)$. There is then a bijection $\gamma \to a$ from $\Gamma_{\mathrm{ell,reg}}(G)$ onto the subset $\mathcal{A}_{\mathrm{irred}}(n, \mathbb{Q})$ of elements

$$a = (a_1, \ldots\ldots\ldots, a_n, a_{n+1})$$

in $\mathcal{A}(n, \mathbb{Q})$ such that the characteristic polynomial

$$p_a(\lambda) = p_\gamma(\lambda) = \lambda^{n+1} - a_1\lambda^n + \cdots + (-1)^n a_n \lambda + (-1)^{n+1} a_{n+1}$$

is irreducible over $\mathbb{Q}$. We can then write

$$f_G(a) = f_G(\gamma)$$

and

$$f_{V,G}(a) = f_{V,G}(\gamma)$$

in (1.1) and (2.1), where a is the image of γ. (There is no call to relabel the other terms in (1.1) and (2.1), since they have first to be transformed in ways that are more directly related to γ.) This different point of view seems harmless enough, but it has far reaching implications.

We might think of the new outlook philosophically as a change in focus from a rational conjugacy class γ in $\mathrm{GL}(n+1,\mathbb{Q})$ whose eigenvalues $\{\gamma_i\}$ we pretend we know, to a monic polynomial p_a in $\mathbb{Q}[\lambda]$ whose roots we assume we do not know. The latter view has a flavour of the old fashioned theory of equations, and its preoccupation with Galois resolvents and other explicit quantities. The discriminant of γ, by which we mean the discriminant $D(p_a)$ of the characteristic polynomial $p_a = p_\gamma$ of γ, is an excellent illustration of the two points of view. For it has the usual two interpretations, one as a product of squares of differences of roots $\{\gamma_i\}$, the other as an integral, multivariable polynomial in the coefficients $\{a_j\}$. With the second interpretation, the discriminant can be regarded as the constant term of a monic quadratic polynomial (in one variable) with no linear term. This polynomial is the most basic Galois resolvent. It has a rational root if and only if the Galois group of the splitting field of the original polynomial p_a is contained in the alternating group $A_n \subset S_n$. The discriminant is important for the deeper study of $I_{\mathrm{ell,reg}}(f)$.

In particular, it is a key ingredient in the structure of orbital integrals. For in the p-adic case, the discriminant provides the leading term in the Shalika germ expansion, namely the germ attached to the regular unipotent class, for which the reader can consult Section 8 of the paper [Re1] of J. Repka. Repka has also calculated the germ attached to the subregular unipotent class. In the case that $G = \mathrm{GL}(3)$, and that γ is elliptic at a p-adic place $v = v_p$ with $p \neq 3$, the subregular germ is related to a different resolvent. In this case, the function $d(\gamma, \mathbb{Q}_v)$ of Repka that provides the new ingredient in his expressions [Re2, (10.1),(10.2)] for the subregular germ has a natural formula in terms of γ at the bottom of [Re2, p. 178]. When the formula is expressed in terms of the coefficients $\{a_j\}$ of $p_a(\lambda)$, it has another interpretation. In precise terms, it equals the cube root of the absolute value of a well known invariant, the linear coefficient in the quadratic Lagrange resolvent for the solution of the cubic equation $p_a(\lambda) = 0$. I do not know how general this phenomenon is, or what if any implications it might have for the study of $I_{\mathrm{ell,reg}}(f)$.

Be that as it may, our new outlook is certainly compatible with a simplification we shall adopt. From now on, we shall follow [La4] and [Al2], taking the set V above to be the singleton $V = \{v_\infty\} = \{v_\mathbb{R}\}$, and the function $f = f_V f^V$ to be a product

$$f = f_\infty \cdot f^\infty = f_\infty \cdot f_p^k \cdot f^{\infty,p} \tag{2.2}$$

for a fixed prime p and nonnegative integer k. Then $f_\infty = f_V$ is a function in $C_c^\infty(Z_+ \setminus G(\mathbb{R}))$, and $f^{\infty,p}$ is the characteristic function of the standard, open, maximal compact subgroup $K^{\infty,p}$ of $G^{\infty,p}$, while f_p^k is the product of $p^{-k/(n+1)}$ with the characteristic function of the open compact subset

$$\Big\{X \in \mathfrak{g}(\mathbb{Z}_p) \ : \ |\det X|_p = p^{-k}\Big\}$$

of $G_p = G(\mathbb{Q}_p)$, for the matrix Lie algebra $\mathfrak{g} = \mathfrak{g}(n+1)$ of G. For any $\gamma \in \Gamma_{\mathrm{ell,reg}}(G)$, the orbital integral $f_G(a) = f_G(\gamma)$ would then vanish unless the coefficients $\{a_j\}$ are all integers, with

$$|a_{n+1}| = |\det \gamma| = p^k.$$

We will thus be dealing with an irreducible, monic polynomial $p_a(\gamma)$ of degree $(n+1)$ with *integral* coefficients. This of course is the basic object of study in the theory of equations.

However, the main reason for working with the coordinates $\{a_j\}$ instead of $\{\gamma_i\}$ is the possibility of applying the Poisson summation formula to the geometric expansion $I_{\mathrm{ell,reg}}(f)$. We are after all trying to interpret $I_{\mathrm{ell,reg}}(f)$ in spectral terms, and it is natural to suppose that an abelian Fourier transform in the indices of summation might introduce spectral parameters. The possibility of introducing a multiplicative Poisson formula on the extension fields

$$E = E_\gamma = \mathbb{Q}[\lambda]/(p_\gamma(\lambda))$$

that contain the elements γ has long been of interest. However, its application has always been elusive. The new idea, introduced and emphasized in [FLN], is to apply the additive Poisson formula to the Steinberg-Hitchin base $\mathcal{A}$. There are serious obstacles to be overcome before this can be carried out. However, a striking solution in the case $G = \mathrm{GL}(2)$ was given by Altug [Al2]. We shall discuss the question in the next section.

The discriminant has a central role in Altug's solution, as we shall see, and is likely to be equally important in the general case. Before we go on, we should recall that there are really three discriminants attached to any element $a \in \mathcal{A}(n, \mathbb{Z})$ with preimage γ in $\Gamma_{\mathrm{ell,reg}}(G)$. There is the discriminant $D(p_a) = D(p_\gamma)$ of the characteristic polynomial, and also the discriminant $D(E_a) = D(E_\gamma)$ of the field extension of $\mathbb{Q}$ attached to γ. They are related by

$$D(p_\gamma) = D(E_\gamma) s_\gamma^2, \tag{2.3}$$

for a positive integer s_γ. In addition, there is the Weyl discriminant $D(\gamma)$ defined by (1.4).

To see where it fits in, we use the elementary fact that the roots $\{\gamma_i\}$ of p_γ are also the eigenvalues of γ. It follows that

$$\begin{aligned}
D(\gamma) &= \prod_{i\neq j}(1-\gamma_i\gamma_j^{-1}) \\
&= \prod_{i<j}\big((\gamma_i\gamma_j)^{-1}(\gamma_j-\gamma_i)(\gamma_i-\gamma_j)\big) \\
&= \det(\gamma)^{-n}(-1)^{\frac{n(n+1)}{2}}\prod_{i<j}(\gamma_i-\gamma_j)^2.
\end{aligned}$$

Consequently,

$$D(\gamma) = (-1)^{\frac{n(n+1)}{2}}(\det\gamma)^{-n}D(p_\gamma) = (-1)^{\frac{n(n+1)}{2}}(a_{n+1})^{-n}D(p_a). \tag{2.4}$$

All three discriminants are important for Altug's solution in the case $G = \mathrm{GL}(2)$.

3. The problem of Poisson summation

Our aim in this section is to discuss the question raised in [FLN], and mentioned at the end of the last section. Is it possible to apply the additive Poisson summation formula to the geometric expansion $I_{\mathrm{ell,reg}}(f)$? A reasonable answer to this question seems to be essential to further progress in Beyond Endoscopy. We shall first describe the problem in general terms. We shall then give a brief, partial review of Altug's solution in the case $G = \mathrm{GL}(2)$. Finally we shall say a few words about the question in higher rank.

We are assuming that our test function f is of the form (2.2). As a function on $G(\mathbb{A})$, it is therefore a product of an archimedean function f_∞ with the nonarchimedean function $f^\infty = f_p^k f^{\infty,p}$. We are interested in the value $f_G(\gamma)$ of the (normalized) orbital integral of f

at any point $\gamma \in \Gamma_{\mathrm{ell,reg}}$. As in (2.1), it is a product, which we now write in the form

$$f_G(a) = f_{\infty,G}(a) \cdot |D(\gamma)|_\infty^{-\frac{1}{2}} \operatorname{Orb}(f^\infty, \gamma), \qquad \gamma \in \Gamma_{\mathrm{ell,reg}}(G), \tag{3.1}$$

where a is the image of γ in $\mathcal{A}(n, \mathbb{Q})$. One sees from the properties of the unnormalized orbital integral

$$\operatorname{Orb}(f^\infty, \gamma) = \operatorname{Orb}(f_p^k, \gamma) \cdot \operatorname{Orb}(f^{\infty,p}, \gamma),$$

and the definition of the function f_p^k, that $f_G(a)$ vanishes unless $\{a_j\}$ is a vector of integers, whose last component $a_{n+1} = \det(\gamma)$ satisfies

$$|a_{n+1}|_v = \begin{cases} p^k, & \text{if } v = v_\infty, \\ p^{-k}, & \text{if } v = v_p, \\ 1, & \text{otherwise.} \end{cases}$$

In other words, $f_G(a)$ vanishes unless the irreducible monic polynomial $p_a(\lambda)$ has integral coefficients with constant term a_{n+1} equal to p^k or $-p^k$, as we noted at the end of the last section.

We introduce the subscheme

$$\mathcal{A}(n, p^k) = \mathcal{B}(n) \times \{\pm p^k\} = \mathcal{B}^{+1}(n) \oplus \mathcal{B}^{-1}(n)$$

of $\mathcal{A}(n)$, where

$$\mathcal{B}^\varepsilon(n) = \mathcal{B}(n) \times \{\varepsilon p^k\}, \qquad \varepsilon \in \{\pm 1\}.$$

For the given f, the corresponding function

$$f_G(a), \qquad\qquad a \in \mathcal{A}_{\mathrm{irred}}(n, \mathbb{Q}),$$

is then supported on the subset

$$\mathcal{A}_{\mathrm{irred}}(n, p^k, \mathbb{Z}) = \mathcal{B}^{+1}_{\mathrm{irred}}(n, \mathbb{Z}) \cup \mathcal{B}^{-1}_{\mathrm{irred}}(n, \mathbb{Z}),$$

where

$$\mathcal{B}^{\varepsilon}_{\mathrm{irred}}(n, \mathbb{Z}) = \mathcal{B}^{\varepsilon}(n) \cap \mathcal{A}_{\mathrm{irred}}(n, \mathbb{Z}), \qquad\qquad \varepsilon \in \{\pm 1\}.$$

If ϕ is any function on $\mathcal{A}_{\mathrm{irred}}(n, p^k, \mathbb{Z})$, we shall write

$$\phi^{\varepsilon}(b) = \phi(b, \varepsilon p^k), \qquad\qquad b \in \mathcal{B}^{\varepsilon}_{\mathrm{irred}}(n, \mathbb{Z}),\ \varepsilon \in \{\pm 1\}. \tag{3.2}$$

The notation (3.2) applies in particular to the function $f_{\infty,G}(a)$ in (3.1). This archimedean component can obviously be regarded as the source of the analysis. Since p and k are fixed, for example, the function $f_G(a)$ on the left side of (3.1) varies only with the function $f_{\infty,G}(a)$ on the right. Following [Al2, p. 7], we can decompose any element x in $G(\mathbb{R}) = \mathrm{GL}(n+1, \mathbb{R})$ into a unique product

$$x = z_x x^1 \varepsilon_x,$$

where x^1 belongs to $\mathrm{SL}(n+1, \mathbb{R})$, z_x equals the positive real number $|\det x|^{\frac{1}{n+1}}$, identified with the corresponding scalar matrix in $\mathrm{GL}(n+1, \mathbb{R})$, and ε_x equals the sign of $\det(x)$, identified with the diagonal matrix

$$\mathrm{diag}(\varepsilon_x, 1, \dots, 1)$$

in $\mathrm{GL}(n+1,\mathbb{R})$. In particular, if $a = (b, \varepsilon p^k)$ is the image in $\mathcal{A}_{\mathrm{irred}}(n, p^k, \mathbb{Z})$ of a rational matrix $x = \gamma$ in $\Gamma_{\mathrm{ell,reg}}(G)$, we see immediately that $\varepsilon_\gamma = \varepsilon$ and $z_\gamma = p^{\frac{k}{n+1}}$. It follows from this that the product (3.1) can be written as

$$f_G^\varepsilon(b) = f_{\infty,G}^\varepsilon(b) \cdot |D(\gamma)|_\infty^{-\frac{1}{2}} \operatorname{Orb}(f^\infty, \gamma),$$

in the notation of (3.1).

We are not ready to relabel the other factors on the right hand side of (3.1) in terms of a or (b, ε). The same goes for the coefficient $\mathrm{vol}(\gamma)$ in the original expansion (1.1) of $I_{\mathrm{ell,reg}}(f)$. Let us therefore just restate the expansion in the following form.

Lemma 3.1. *For our given function f, $I_{\mathrm{ell,reg}}(f)$ is equal to the expression*

$$\sum_{\varepsilon \in \{\pm 1\}} \sum_{b \in \mathcal{B}_{\mathrm{irred}}^\varepsilon(n,\mathbb{Z})} f_{\infty,G}^\varepsilon(b) \cdot |D(\gamma)|_\infty^{-\frac{1}{2}} \cdot \mathrm{vol}(\gamma) \cdot \operatorname{Orb}(f^\infty, \gamma), \tag{3.3}$$

where γ is the preimage in $\Gamma_{\mathrm{ell,reg}}(G)$ of the point $a = (b, \varepsilon p^k)$.

The expansion (3.3) is where any discussion of Poisson summation would have to begin. The inner sum on the right hand side is over a subset of elements in $\mathcal{B}(n,\mathbb{Z})$, a lattice in the real vector space $\mathcal{B}(n,\mathbb{R})$. Is there is a natural extension of the summands to a Schwartz function on $\mathcal{B}(n,\mathbb{R})$, or at least a function to which the Poisson summation applies? The answer to the question, posed in this naive way, is emphatically no.

There are several serious difficulties. The most striking is that the volume $\mathrm{vol}(\gamma)$ is fundamentally dependent on the integral structure of the set $\mathcal{B}_{\mathrm{irred}}^\varepsilon(n,\mathbb{Z})$ that contains b. The same goes for the factor $\operatorname{Orb}(f^\infty, \gamma)$. As a finite product of nonarchimedean orbital integrals, it too depends on the integral structure of the set $\mathcal{B}_{\mathrm{irred}}^\varepsilon(n,\mathbb{Z})$. The archimedean factor $f_{\infty,G}^\varepsilon(b)$ is more amenable. It does have a natural extension to $\mathcal{B}(n,\mathbb{R})$, since it is obtained from the invariant orbital integrals of f_∞, which are defined for the characteristic polynomials of all semisimple conjugacy classes. Actually, in the end one will have to use the invariant distributions attached to weighed orbital integrals when dealing with reducible characteristic polynomials, but we ignore this refinement apart from some comments in Section 6. However, there remains a question with invariant orbital integrals. For even with this case, the resulting function on $\mathcal{B}(n, \mathbb{R})$ can have singularities at the characteristic polynomials

$$p_b^{\varepsilon}(\lambda) = p_{(b, \varepsilon p^k)}(\lambda), \qquad\qquad b \in \mathcal{B}(n, \mathbb{R}),$$

that have repeated factors over $\mathbb{R}$, which is to say when the corresponding discriminants vanish.

The Poisson summation formula of course applies to the *sum* of a suitable function over integral points, not a linear combination. Therefore the function $f_{\infty,G}^{\varepsilon}(b)$ in (3.3) has somehow to be combined with its two coefficients. Altug's solution of the problem for GL(2) comes at the expense of increased complexity, with the addition of some new functions and a further triple summation. The new ingredients turn out to be analytically manageable, however, and are an essential part of the successful application of Poisson summation. The solution is quite subtle, even though G equals GL(2), and we will only be able to review some of the main points here.

Assume for the moment then, that $G = \mathrm{GL}(2)$. We shall describe some of the key steps in Altug's Poisson summation formula. We will not give the final formula, since a precise statement would require a number of new definitions, and would take us too far afield. In Section 5, however, we will introduce a general expression, which gives a less precise version of Altug's formula, but which might also be relevant to higher rank.

The first ingredient in Altug's construction is a formula

$$\mathrm{Orb}(f^{\infty},\gamma) = p^{-\frac{k}{2}} \left\{ \sum_{f|s_\gamma} f \left(\prod_{q|f} \left(1 - \left(\frac{D(E_\gamma)}{q} \right) q^{-1} \right) \right) \right\} \tag{3.4}$$

for the q-adic orbital integral term in (3.3). The outer sum is over divisors f of the positive integer s_γ in (2.3), while the inner sum is over *prime* divisors of f. The innermost bracket is the Kronecker symbol attached to the discriminant $D(E_\gamma)$ of the quadratic extension E_γ and the prime q, so the corresponding factor in the product is equal to the local L-value

$$L_q\left(1, \left(\frac{D(E_\gamma)}{\cdot}\right)\right)$$

of the quadratic L-function for $D(E_\gamma)$. The formula was proved by Langlands in two steps in [La4]. In Section 2.5 of [La4], he notes that the global formula (3.4) is equivalent to the local formula

$$\mathrm{Orb}(f'_q,\gamma) = 1 + \sum_{j=1}^{k_q} q^j \left(1 - \left(\frac{D(E_q)}{q} \right) \right), \tag{3.5}$$

where

$$f'_q = \begin{cases} p^{-\frac{k}{2}} f_p^k, & \text{if } q = p, \\ f_q, & \text{if } q \neq p, \end{cases}$$

is the relevant characteristic function at q, and $k_q = \mathrm{val}_q(s_\gamma)$ is the exponent of q in the prime factorization

$$s_\gamma = \prod_q q^{k_q}$$

of s_γ. Langlands then reduces the local formula (3.5) to the formula for $\mathrm{Orb}(f'_q,\gamma)$ he derived in the earlier [La4, Lemma 1]. (See also [K3, Section 5.9]).

The second ingredient is a formula

$$\mathrm{vol}(\gamma) = |D(E_\gamma)|^{\frac{1}{2}} \cdot L\left(1, \left(\frac{D(E_\gamma)}{\cdot}\right)\right) \tag{3.6}$$

for the volume in (3.3) in terms of the value at 1 of the associated global L-function. It is a consequence of Dirichlet's class number formula for the quadratic field E_γ and some calculations for archimedean measures at the beginning of [La4, Section 2.1].

Following Altug, we substitute (3.4) and (3.6) into the expression (3.3). We then combine three of the factors in the resulting expression, using (2.3) and (2.4) to write

$$\begin{aligned}
&|D(\gamma)|_\infty^{-\frac{1}{2}}\, |D(E_\gamma)|^{\frac{1}{2}}\, p^{-\frac{k}{2}} \\
=&|D(p_\gamma)|^{-\frac{1}{2}} \left|(\det\gamma)^{-1}\right|^{-\frac{1}{2}} |D(E_\gamma)|^{\frac{1}{2}}\, p^{-\frac{k}{2}} \\
=&\left|D(E_\gamma)s_\gamma^2\right|^{-\frac{1}{2}} (p^k)^{\frac{1}{2}} |D(E_\gamma)|^{\frac{1}{2}} p^{-\frac{k}{2}} \\
=&s_\gamma^{-1} p^{\frac{k}{2}} p^{-\frac{k}{2}} = s_\gamma^{-1},
\end{aligned}$$

since n here is equal to 1. The expression (3.3) becomes the sum, over either γ or its image $(b, \varepsilon p^k)$, of terms

$$f_{\infty,G}^{\varepsilon}(G) \cdot L\left(1, \left(\frac{D(E_\gamma)}{\cdot}\right)\right) \sum_{f|s_\gamma} (f/s_\gamma) \prod_{q|f} \left(1 - \left(\frac{D(E_\gamma)}{q}\right) q^{-1}\right).$$

Altug then makes a change of variables $f \to (s_\gamma/f)$ in the last sum over f. The result is a formula

$$L\left(1, \left(\frac{D_\gamma}{\cdot}\right)\right) \sum_{f|s_\gamma} (f/s_\gamma) \prod_{q|f} \left(1 - \left(\frac{D_\gamma}{q}\right) q^{-1}\right) = \sum_{f|s_\gamma} (1/f) L\left(1, \left(\frac{D_a/f^2}{\cdot}\right)\right), \tag{3.7}$$

where we have written $D_\gamma = D(E_\gamma)$ and $D_a = D(p_a)$. The proof of (3.7) is an elementary rearrangement of the sum [Al1, Lemma 2.1.1], which also relies on a basic property of the Kronecker symbol $\left(\frac{D}{q}\right)$ attached to any discriminant D. Namely, if q divides D, $\left(\frac{D}{q}\right) = 0$ by definition, so the local factor $L_q\left(s, \left(\frac{D}{\cdot}\right)\right)$ of the global L-function then equals 1. With the formula in hand, it is then convenient to replace the sum over the divisors f of s_γ on the right with a sum $\sum'$ over the divisors f^2 of the discriminant $D_a = D(p_a)$ for which D_a/f^2 is still a discriminant. The general expansion (3.3) becomes

$$I_{\mathrm{ell,reg}}(f) = \sum_{\varepsilon,b} f^{\varepsilon}_{\infty,G}(b) \cdot \sum_{f^2|D_a}{}' (1/f) L\left(1, \left(\frac{D_a/f^2}{\cdot}\right)\right). \tag{3.8}$$

The new expansion (3.8) is formula (4) of [Al2]. It is a remarkable reformulation of the original elliptic expansion, which was derived in [Al2, Sections 2.2.2-2.2.4]. However, it presents the same arithmetic difficulties, now in the guise of the global L-values. Altug deals with them by means of an approximate functional equation, a technique that in principle can be applied to any L-function with functional equation and good analytic behaviour. He applies it to a nonstandard L-function, defined for any discriminant D as a finite linear combination

$$L(s, D) = \sum_{f^2|D}{}' (1/f^{2s-1}) L\left(s, \left(\frac{D/f^2}{\cdot}\right)\right) \tag{3.9}$$

of ordinary quadratic L-functions. On the one hand, the value of (3.9) at $s = 1$ equals the inner sum in (3.8). On the other, the general values of (3.9) satisfy a standard kind of functional equation [SY, Lemma 2.1], [Al2, Section 3.1], which goes back to Zagier [Z]. Altug uses this in his derivation [Al2, Sections 3.2-3.4] of the approximate functional equation that ultimately leads to a resolution of the arithmetic difficulties in (3.8).

This is the part of the argument I will not try to review in detail. Roughly speaking, Altug uses the approximate functional equation to write $L(s, D)$ as the sum of a "Dirichlet series with variable coefficients" that converges at $s = 1$, together with the value at $(1 - s)$

of a similar series that also converges at $s = 1$ [Al2, Proposition 3.4]. This specializes to a formula for $L(1, D_a)$ as an absolutely convergent sum, rather than the analytic continuation to $s = 1$ of the original Dirichlet series $L(s, D_a)$. The equation (3.8) becomes

$$I_{\mathrm{ell,reg}}(f) = \sum_{\varepsilon,b} f^{\varepsilon}_{\infty,G}(b) L(1, D_a), \qquad\qquad a = (b, \varepsilon p^k), \tag{3.10}$$

where the L-value can now be written in the general form

$$L(1, D_a) = \sum_{f^2 | D_a}{}' \frac{1}{f} \sum_{l=1}^{\infty} \frac{1}{l} \left(\frac{D_a/f^2}{l} \right) J(f, l, D_a). \tag{3.11}$$

The function $J(f, l, D_a)$ here is the expression in the square brackets of formula (11) of [Al2, Corollary 3.5]. The expression contains a somewhat arbitrary constant A, which is later specialized in the statement of [Al2, Theorem 4.2] by setting

$$A = |D_a|^{\alpha}, \qquad\qquad 0 < \alpha < 1.$$

With this understanding, $J(f, l, D_a)$ not only makes the sum over l converge, allowing it for example to be interchanged with the sum over b in (3.10), but also has the property that as a function of $a = (b, \varepsilon p^k)$, it resolves the singularities of the archimedean factor $f^{\varepsilon}_{\infty,G}$. In particular, the product

$$f^{\varepsilon}_{\infty,G}(b) J(f, l, D_a)$$

that will occur in (3.10) extends to a Schwartz function of b in the space $\mathcal{B}(1, \mathbb{R}) = \mathbb{R}$. (See [Al2, Sections 4.1, 4.2]).

There are two points here. One is that the singularities of the invariant orbital integrals $f^{\varepsilon}_{\infty,G}(b)$ at points with $D_a = 0$ disappear. The other is that the integral sum of orbital integrals in (3.10), which is over the subset $\mathcal{B}^{\varepsilon}_{\mathrm{irred}}(1, \mathbb{Z})$ of elements in $\mathcal{B}(1, \mathbb{Z})$ such that the characteristic polynomial p_b^ε is irreducible, can be extended to the full set $\mathcal{B}(1,\mathbb{Z})=\mathbb{Z}$. As Altug points out, this would not have been possible before, since the L-values in (3.8) are not defined by absolutely convergent series. But with the application of the approximate functional equation, it is possible to replace (3.10) by an extended expansion

$$\bar{I}_{\mathrm{ell,reg}}(f)=\sum_{\varepsilon\in\{\pm1\}}\sum_{l=1}^{\infty}\frac{1}{l}\sum_{b\in\mathbb{Z}}f^{\varepsilon}_{\infty,G}(b)\sideset{}{'}\sum_{f^2|D_a}\frac{1}{f}\left(\frac{D_a/f^2}{l}\right)J(f,l,D_a). \tag{3.12}$$

This new expression contains a sum over the lattice $\mathbb{Z}$ in $\mathbb{R}$, which of course is the setting for Poisson summation. But what is the meaning of $\bar{I}_{\mathrm{ell,reg}}(f)$? It is not equal to the original expression $I_{\mathrm{ell,reg}}(f)$, which is what comes from the trace formula. The answer is that the extended expansion is a simplification, to be used in place of the more complicated supplementary terms attached to the complementary elements b. In other words, $\bar{I}_{\mathrm{ell,reg}}(f)$ represents an approximation to the full geometric side of the trace formula. In Altug's hands [Al3], [Al4], however, it has led to some important estimates of the kind expected in Beyond Endoscopy.

But (3.12) is still not quite ready for the application of Poisson summation. The problem is the Kronecker symbol $\left(\frac{D_a/f^2}{l}\right)$, which does not extend to the space of points $b\in\mathbb{R}$. Altug deals with this last difficulty by taking the sum over $b\in\mathbb{Z}$ inside the sum over f^2, and then breaking it into a (finite) sum of arithmetic progressions of modulus $4lf^2$. The Kronecker symbol is easily seen to be invariant under translation of b by an integral multiple of $4lf^2$, allowing Altug then to apply the Poisson summation formula to each of the arithmetic progressions. After rescaling the variables in the lattice dual to $(\mathbb{Z}/4lf^2)$, he arrives at last at a sum of Fourier transforms at points $\xi\in\mathbb{Z}$. This is the desired result. Notice that it comes with a supplementary triple sum over l, f and a set of representatives of the associated arithmetic progressions.

We shall finish this section with some remarks on the possibility of extending the application of Poisson summation to higher rank. Not surprisingly, there are some difficulties. The most significant appears to be the generalization of the formula (3.4) for $\mathrm{Orb}(f^\infty, \gamma)$, the product of q-adic orbital integrals that occurs in (3.3), or what would be equivalent, the corresponding local formula (3.5). The global formula (3.4) for GL(2) seems to have been well known in different guises, as for example in [Shi, Exercise 4.12, p. 106], but perhaps not in connection with orbital integrals. On the other hand, there have been a number of papers on p-adic orbital integrals for general linear groups, either on their Shalika germs or directly on the characteristic functions needed here. (See Repka [Re1], [Re2], Kottwitz [K1], [K2], [K3], Rogawski [Ro1] [Ro2] and Waldspurger [W1] [W2].) Waldspurger has the most complete results, a complex algorithm for any tamely ramified local extension $(E_q/\mathbb{Q}_q)$ that leads in principle to formulas for the general Shalika germs. The problem for us, as I see it, is to express q-adic orbital integrals in terms of formulas that directly generalize (3.5).

Indeed, the structure of (3.5) and its global counterpart (3.4) was essential for Altug. In our review of his work above, we very much needed the sum over f from (3.4). It was used to rewrite the sum over b in terms of the arithmetic progressions that took care of the Kronecker symbol. I have concocted analogues of (3.5) from, say, the formulas for GL(3) in [K2, p. 660]. However, they are not very natural, and they do not seem amenable to generalizations of the change of variables formula (3.7). Some version of this formula would seem also to be essential to any extension of Altug's result. We will return to the question in a moment, but let us first consider the volume term $\mathrm{vol}(\gamma)$ in (3.3).

We are assuming at this point that $G = \mathrm{GL}(n+1)$ is of general rank. The general analogue of the quadratic L-function $L\left(s, \left(\frac{D_\gamma}{\cdot}\right)\right)$ in (3.7) is the L-function

$$L(s, \sigma_{T/G}) = \zeta_E(s)/\zeta_{\mathbb{Q}}(s).$$

The left hand side follows the notation of [FLN], in which $T = T_\gamma$ is the maximal torus (taken up to conjugacy) in $\mathrm{GL}(n+1)$ with $T(\mathbb{Q}) = E^* = E^*_\gamma$. A direct generalization to $\mathrm{GL}(n+1)$ of Altug's application of the approximate functional equation would require Artin's conjecture for the n-dimensional Galois representation $\sigma_{T/G}$. This is known [U], [vdW], [MM, Corollary 2.4.2] for the cases $n = 2, 3$ (in which the image of $\sigma_{T/G}$ is solvable), but seems to be out of reach in general. The inevitable conclusion is that we ought to be working with the full Dedekind zeta function $\zeta_E(s)$ rather than its quotient by $\zeta_{\mathbb{Q}}(s)$. This raises the problem of possible poles introduced by the zeros of $\zeta_{\mathbb{Q}}(s)$. A preliminary review of Altug's arguments suggests that one might be able to sidestep this difficulty, but I have not checked the details. I hope to return to the question elsewhere after a more careful analysis.

If one does work with $\zeta_E(s)$, there will be an immediate substitute for the Kronecker symbol $\left(\frac{D_\gamma}{q}\right)$. It is given by the Kummer-Dedekind theorem, which characterizes the localization of the field

$$E = \mathbb{Q}[\lambda]/(p_a(\lambda))$$

at a prime q in terms of the degrees of the irreducible factors of $p_a(\lambda)$ modulo q. There is a condition on q, namely that it not divide the integer

$$s_\gamma = |D_a/D_\gamma|^{\frac{1}{2}} = |D(p_a)/D(E_\gamma)|^{\frac{1}{2}}.$$

I have not considered whether this represents a problem, but I would expect it to be resolved once we have a suitable generalization of (3.4).

Returning to the problem of (3.4), we can follow our remarks on $\zeta_E(s)$ by pointing out some interesting new ideas in a recent paper by Z. Yun. In [Y], Yun establishes the meromorphic continuation, functional equation and class number formula for the zeta function $\zeta_R(s)$ of an order R in $\mathcal{O}_E$ [Y, Theorem 1.2]. This is the generalization of the Dedekind zeta function $\zeta_E(s)$ in which the ring $\mathcal{O}_E$ is replaced by a subring R (with 1) whose the quotient field remains equal to E. Our interest is in the special case of a "monogenic" order, the ring

$$R = R_\gamma = \mathbb{Z}[\lambda]/(p_\gamma(\lambda)). \tag{3.13}$$

Yun constructs the global zeta function $\zeta_R(s)$ by altering the local factors $\zeta_{E,q}(s)$ of the Dedekind zeta function at finitely many primes q. More precisely, he defines $\zeta_{R,q}(s)$ in general as a Dirichlet series in powers of q^{-s} analogous to the series expansion of the usual local Euler factor in case $R = \mathcal{O}_E$ [Y, (2.3)]. He then shows that the quotients

$$\tilde{J}_{R,q}(s) = |s(R)|_q^{-1}\, \zeta_{R,q}(s)\zeta_{E,q}(s)^{-1}, \tag{3.14}$$

in which $s(R)$ is the positive integer given in terms of the global discriminants of R and $\mathcal{O}_E$ by

$$|D(R)/D(E)| = s(R)^2,$$

have some very nice properties [Y, Theorem 2.5]. In particular, they satisfy a local functional equation

$$\tilde{J}_{R,q}(s) = \tilde{J}_{R,q}(1-s). \tag{3.15}$$

Moreover, if

$$|s(R)|_q^{-1} = q^{\delta_{R,q}},$$

they can be written in the form

$$\tilde{J}_{R,q}(s) = |s(R)|_q^{-1}\, P_{R,q}(q^{-s}), \tag{3.16}$$

for a polynomial $P_{R,q}$ with integral coefficients, constant term 1, and degree equal to $2\delta_{R,q}$. Yun then uses these local results to deduce the analytic properties of the global zeta function

$\zeta_R(s)$ from the corresponding properties of $\zeta_E(s)$ [Y, Section 3].

Suppose that R is the monogenic order R_γ in (3.13) attached to a class $\gamma \in \Gamma_{\mathrm{ell,reg}}(G)$. Then the integer $s(R) = s(R_\gamma)$ is the positive integer s_γ we introduced in (2.3). Yun shows that there is a close, and to my mind quite surprising, relationship between the local zeta factors $\zeta_{\gamma,q} = \zeta_{R_\gamma,q}$ and q-adic orbital integrals at γ. To be precise, let $f_q = f_q^0$ be the characteristic function in $G(\mathbb{Q}_q) = \mathrm{GL}(n+1, \mathbb{Q}_q)$ of the standard open compact subgroup $K_q = \mathrm{GL}(n+1, \mathbb{Z}_q)$. Then Yun establishes an identity

$$\mathrm{Orb}(f_q, \gamma) = \tilde{J}_{\gamma,q}(0) = \tilde{J}_{\gamma,q}(1), \tag{3.17}$$

between the orbital integrals and the quotients (3.14) [Y, Corollary 4.6]. Otherwise said, there is a natural way to extend $\mathrm{Orb}(f_q, \gamma)$ to a function of s of the form (3.16). The functional equation (3.15) then specializes to a symmetry of the function $\mathrm{Orb}(f_q, \gamma)$.

I have looked at these formulas in the case of GL(2). The specialization of (3.16) becomes the formula (3.5) (with $k = 0$). The symmetry given by the local functional equation corresponds to a change of variables $f \to (s_\gamma/f)$ in the formula (3.4). Finally, the global functional equation for $\zeta_\gamma(s) = \zeta_{R_\gamma}(s)$, with n arbitrary, reduces essentially to the functional equation [SY, Lemma 2.1] for the nonstandard quadratic L-function (3.9) mentioned earlier in the section, in case $n = 1$. I will try to study these properties for higher rank elsewhere, with possible application to Poisson summation for $\mathrm{GL}(n+1)$.

4. Spectral terms and global parameters

For the moment, we consider again the special case $G = G(1) = \mathrm{GL}(2)$. We have described how Altug was able to apply Poisson summation to the geometric expansion in this case. We did not give the resulting formula, which is rather complicated. The reader can refer to [Al2, Theorem 4.2] for a precise statement, and to Section 5 here for a less precise extrapolation of Altug's formula, which is part of our preliminary attempt to understand the general case. The main point, however, is clear enough. The formula amounts to an expression for the extended elliptic expansion $\bar{I}_{\text{ell,reg}}(f)$ as a sum of additive Fourier transforms on $\mathbb{R}$, the Steinberg-Hitchin base for $\mathrm{SL}(2,\mathbb{R})$, at integral points ξ in $\mathcal{B}(n,\mathbb{Z}) \simeq \mathbb{Z}$. We shall write this simply as

$$\bar{I}_{\text{ell,reg}}(f) = \sum_{\xi \in \mathbb{Z}} \hat{\bar{I}}_{\text{ell,reg}}(\xi, f), \tag{4.1}$$

where the summand is a distribution in f that depends on ξ.

After establishing an explicit formula (4.1), Altug solved the problem discussed in Section 1 in the case $G = \mathrm{GL}(2)$. More precisely, he showed that the contribution to (4.1) of the difference (1.10) of spectral terms is essentially equal to the summand $\hat{\bar{I}}_{\text{ell,reg}}(0, f)$ with $\xi = 0$. This is a remarkable result, which was conjectured in weaker form in [FLN].

We do need to be careful in our choice of words, as the analysis of these matters is subtle. Altug did not prove that (1.10) actually equals the summand $\hat{\bar{I}}_{\text{ell,reg}}(0, f)$. What he established was that the two quantities are asymptotic to each other. I cannot be precise about this, but my understanding is that the error term, namely the actual difference

$$(I_{\text{disc}}(f) - I_{\text{cusp}}(f)) - \hat{\bar{I}}_{\text{ell,reg}}(0, f),$$

will be sharp enough that it has no bearing on some of the finer estimates one hopes to establish in Beyond Endoscopy [La4, Sections 1.5-1.7], [FLN, Section 1], [Ar8, Section 2]. (For a treatment of the error term, see Proposition 5.5.3 of [Al1], and the remarks preceding its statement.) In other words from the perspective of cuspidal automorphic representations, which would of course be the basic objects of study in Beyond Endoscopy for $\mathrm{GL}(n+1)$, we can often regard the cuspidal part $I_{\text{cusp}}(f)$ of the trace formula for $G = \mathrm{GL}(2)$ as being equal to the complementary sum

$$\sum_{\xi \neq 0} \hat{\hat{I}}_{\mathrm{ell,reg}}(\xi, f)$$

from (4.1).

The result is in fact more elegant than I have indicated. In Proposition 6.1 of [Al2], Altug expands the summand $\hat{\hat{I}}_{\mathrm{ell,reg}}(0, f)$ in (4.1) into a further sum of three terms. Two of these supplementary terms are quite simple and occur naturally as residues, while the third is a more complicated expression that includes the contours from which the residues arise. In Lemma 6.2 of [Al2], Altug then observes that the first residual term equals the value at f of the character of the trivial automorphic representation of $G = \mathrm{GL}(2)$, while the second is the character of an induced automorphic representation of $G = \mathrm{GL}(2)$, up to a scalar multiple that leaves it equal to the summand with $w \neq 1$ in the formula (1.7) for $I_{\mathrm{disc}}(f)$. This means that the two residual terms in the decomposition of $\hat{\hat{I}}_{\mathrm{ell,reg}}(0, f)$ equal

$$I_2(f) - I_{\mathrm{cusp}}(f)$$

and

$$I_{\mathrm{disc}}(f) - I_2(f)$$

respectively. Their sum therefore equals the difference

$$I_{\mathrm{disc}}(f) - I_{\mathrm{cusp}}(f)$$

under consideration. The asymptotic behaviour, with the estimates mentioned above, is a property of the third supplementary term, and therefore represents a separate question.

We return to the general case that $G = G(n) = \mathrm{GL}(n+1)$. Altug's results give a structural explanation for the contribution of the singular representations to the geometric expansion of GL(2). This is what one hopes to generalize to $G = \mathrm{GL}(n+1)$. In general, however, there are many more noncuspidal representations in the discrete part $I_{\mathrm{disc}}(f)$ of the trace formula. These singular representations include the noncuspidal representations $\pi \in \Pi_2(G) \setminus \Pi_{\mathrm{cusp}}(G)$ that occur in the actual discrete spectrum. They are the noncuspidal characters in the term with $M = G$ in the formula (1.7) for $I_{\mathrm{disc}}(f)$. There are also the induced cuspidal representations $\pi \in \Pi_{\mathrm{disc}}(G) \setminus \Pi_2(G)$, which do not lie in the discrete spectrum. They contribute to the terms with $M \neq G$ in (1.7). Finally, there are the singular representations that combine both phenomena, the induced *noncuspidal* representations $\pi \in \Pi_{\mathrm{disc}}(G) \setminus \Pi_2(G)$, which also contribute to the terms with $M \neq G$ in (1.7). Our aim is to consider how these various spectral objects might relate to the expected extension of the dual geometric expansion (4.1) from GL(2) to $G = \mathrm{GL}(n+1)$. For motivation, we shall finish the section with a brief discussion on how one would parametrize the singular constituents of $I_{\mathrm{disc}}(f)$.

Like much else in this area, the parametrization of automorphic representations is hypothetical. It rests on the existence of the automorphic Galois group $L_{\mathbb{Q}}$, a locally compact extension of the global Weil group $W_{\mathbb{Q}}$. The construction of $L_{\mathbb{Q}}$ could be regarded as one of the ultimate aims of Beyond Endoscopy, so its rigorous application here is out of the question. On the other hand, if we believe in this universal group, its expected properties can be very helpful in understanding the behaviour of automorphic representations.

The fundamental property for $L_{\mathbb{Q}}$ should be a canonical bijection between equivalence classes of irreducible unitary (continuous) representations

$$\phi : L_{\mathbb{Q}} \to \mathrm{GL}(n+1, \mathbb{C})$$

of dimension $(n+1)$ and equivalence classes of unitary cuspidal automorphic representations μ of $\mathrm{GL}(n+1)$. Among various requirements, the bijection should be compatible with the local Langlands correspondence, relative to canonical conjugacy classes of embeddings of the local Langlands (Weil-Deligne) groups

$$L_{\mathbb{Q}_v} \to L_{\mathbb{Q}}.$$

Moreover, the determinant of any ϕ should match the central character of μ, as 1-dimensional characters

$$n_\mu : W_{\mathbb{Q}}^{ab} = \mathbb{A}^*/\mathbb{Q}^* \to U(1).$$

More generally, we could relax the conditions that ϕ be irreducible and unitary. Then the correspondence should extend to a bijection from arbitrary $(n+1)$-dimensional representations of $L_{\mathbb{Q}}$ to automorphic representations of $\mathrm{GL}(n+1)$ that are *isobaric* in the sense of [La2, Section 2].

One can describe the classification [MW] by Moeglin and Waldspurger of the representations $\pi \in \Pi_2(G)$ in the discrete spectrum in terms of our hypothetical parameters. Since the representations in $\Pi_2(G)$ are required to be trivial on Z_+, the parameters will have a corresponding condition on their determinants. The classification would then be a bijection

$$\psi \to \pi,$$

from the set of irreducible unitary representations

$$\psi : L_{\mathbb{Q}} \times \mathrm{SU}(2) \to \mathrm{GL}(n+1, \mathbb{C})$$

of dimension $(n+1)$ whose determinant, as a character on the group

$$(L_{\mathbb{Q}} \times \mathrm{SU}(2))^{ab} = L_{\mathbb{Q}}^{ab} = \mathbb{A}^*/\mathbb{Q}^*,$$

is of finite order, onto the set of representations $\pi \in \Pi_2(G)$. Any such ψ extends analytically from the unitary group $U(2)$ to its complexification $\mathrm{SL}(2,\mathbb{C})$. It therefore determines an $(n+1)$ dimensional representation

$$\phi_\psi(u) = \psi\left(u, \begin{pmatrix} |u|^{\frac{1}{2}} & 0 \\ 0 & |u|^{-\frac{1}{2}} \end{pmatrix}\right), \qquad u \in L_{\mathbb{Q}},$$

of $L_{\mathbb{Q}}$ alone, where $|u|$ is the absolute value of the image of u in the Weil group $W_{\mathbb{Q}}$. The representation π attached to ψ will then be the automorphic isobaric representation attached to the parameter ϕ_ψ. This description is in fact a reformulation of the classification of [MW] in terms of the conjectured parameters in [Ar2] for discrete spectra of general groups. For a statement of the Moeglin-Waldspurger classification itself, as well as a discussion of the parameters above, the reader can consult [Ar7, Section 1.3] in addition to the original paper [MW].

As an isobaric representation, the image π of ψ is a Langlands quotient. Let us recall a little more explicitly how it arises. As an irreducible representation of a product of groups, the given parameter equals a tensor product

$$\psi = \phi \otimes \nu, \tag{4.2}$$

for irreducible unitary representations ϕ and ν of $L_{\mathbb{Q}}$ and $\mathrm{SU}(2)$ respectively, of degrees $(m+1)$ and $(d+1)$, where

$$(n+1) = (m+1)(d+1).$$

Then ϕ corresponds to a unitary cuspidal automorphic representation μ of the group $\mathrm{GL}(m+1)$. If P is the standard parabolic subgroup of $G = \mathrm{GL}(n+1)$ corresponding to the partition

$(m+1,\dots,m+1)$ of $(n+1)$, and $\sigma(\mu)=\sigma_\psi(\mu)$ is the representation

$$\mu(x_1)\,|\det x_1|^{d/2}\otimes\mu(x_2)\,|\det x_2|^{d/2-1}\otimes\cdots\otimes\mu(x_{d+1})\,|\det x_{d+1}|^{-d/2}, \tag{4.3}$$

of the standard Levi subgroup

$$M_P(\mathbb{A})=\{x=(x_1,\dots,x_{d+1})\ :\ x_i\in \mathrm{GL}(m+1,\mathbb{A})\}, \tag{4.4}$$

then π is the unique irreducible quotient of the induced representation $\mathcal{I}_P(\sigma(\mu))$. We conclude that

$$I_2(f)=\sum_{m,\mu} f_G(\pi)=\sum_{m,\mu}\mathrm{tr}(\mathcal{J}_P(\sigma(\mu),f)), \tag{4.5}$$

for m and μ as above, $\sigma(\mu)$ again the representation (4.3) of $M_P(\mathbb{A})$, and $\pi=\mathcal{J}_P(\sigma(\mu))$ the irreducible quotient of $\mathcal{I}_P(\sigma(\mu))$.

The formula (4.5) provides the contribution to $I_2(f)$ of the representations that are singular, in the sense that they are not cuspidal. Indeed, they are just given by the summands for which the divisor $(m+1)$ of $(n+1)$ is proper, which is to say that $m\neq n$. This formula does not depend on the hypothetical global parameters ψ, and could easily have been described directly from the classification in [MW]. However, since the complex dual group $\hat{M}_P$ will be implicit in our deliberations, it is always instructive to have the parameters in mind.

The second class of singular representations are the properly induced representations $\pi\in\Pi_{\mathrm{disc}}(G)$ for which the inducing representation is cuspidal. They are expected to be tempered. Their treatment is simpler, apart from the coefficients $a^G(\pi)$ in (1.5) that come with them. The contribution to $I_{\mathrm{disc}}(f)$ of the representations in this second class is a sum

$$\sum_{m,\mu} a^G(\pi) f_G(\pi)=\sum_{m,\mu} a^G(\pi)\,\mathrm{tr}(\mathcal{I}_P(\sigma_0(\mu),f)),$$

for m and μ as in (4.5) but with $m \neq n$, $\sigma_0(\mu)$ the representation (4.3) but taken without the exponents, and π now being the irreducible induced representation $\mathcal{I}_P(\sigma_0(\mu))$. In this case, the global parameter $\psi = \phi$ for π is what we obtain from the original parameter (4.2) if ν is replaced by the trivial representation of SU(2). The coefficients $a^G(\pi)$ are harmless, at least in the case here of $\mathrm{GL}(n+1)$. They are given by

$$a^G(\pi) = a^G(m) = (-1)^{n-m}(n-m)^{-1},$$

and in particular, depend only on m and n.

The third class of singular representations is the set of properly induced nontempered representations $\pi \in \Pi_{\mathrm{disc}}(G)$, which is to say, induced from noncuspidal representations in the discrete spectrum. This contribution to $I_{\mathrm{disc}}(f)$ is a sum

$$\sum_{m,m',\mu} a^G(\pi) f_G(\pi) = \sum_{m,m',\mu} a^G(\pi)\,\mathrm{tr}(\mathcal{I}_{P'}(\sigma_0(\mu'), f)), \tag{4.6}$$

where m and m' represent proper divisors

$$(m+1)|(m'+1)|(n+1), \qquad\qquad m \neq m' \neq n, \tag{4.7}$$

and μ runs over the cuspidal automorphic representations of $\mathrm{GL}(m+1)$ that are trivial on $Z_{m,+}$ (as in (4.5)). The representation

$$\mu' = \mathcal{J}_P^{P'}(\sigma'(\mu))$$

is the Langlands quotient for $\mathrm{GL}(m'+1)$ attached to μ, and

$$\pi = \mathcal{I}_{P'}(\sigma_0(\mu')) = \mathcal{I}_{P'}(\sigma_0(\mathcal{J}_P^{P'}(\sigma'(\mu))))$$

is the associated irreducible representation induced from the parabolic subgroup P' of G attached to the partition $(m'+1, \dots, m'+1)$ of $(n+1)$. In this case the global parameter ψ for π is what we would obtain from the original parameter if ν were replaced by the irreducible representation ν' of $\mathrm{SU}(2)$ of dimension $(d'+1)$, with

$$(n+1) = (m'+1)(d'+1).$$

Finally, the coefficients here satisfy

$$a^G(\pi) = a^G(m') = (-1)^{n-m'}(n-m')^{-1},$$

as before.

This third class of singular characters includes the other two if we remove the condition $m \neq m' \neq n$ that the divisors (4.7) be proper. We restate this for future reference as the following proposition.

Proposition 4.1. *The discrete part of the spectral side of the trace formula can be written as*

$$I_{\mathrm{disc}}(f) = \sum_{m,m',\mu} a^G(\pi) f_G(\pi) = \sum_{m,m',\mu} a^G(m') \, \mathrm{tr}(\mathcal{I}_{P'}(\sigma_0(\mu'), f)), \tag{4.8}$$

for m, m', μ, μ' and π as in (4.6), but without the condition in (4.7) that the divisors be proper.

The ingredients of Proposition 4.1 are well known, even if the notation is not quite standard. I wanted to describe them this way in order to be able to compare the decomposition of $I_{\mathrm{disc}}(f)$ with the stratification of the elliptic terms we will introduce in the next section. It is a question only of a formal comparison in this paper, whereby the two decompositions will be seen to have the same formal structure. The actual comparison, in which one tries to relate the terms in two decompositions as distributions in f, will be much more difficult. We observe here only that the characters of the representations on the right hand side of (4.8) will have to be a part of the study. These characters are elementary in the case of GL(2) solved by Altug [Al2, Lemma 6.2]. They are more complicated for $G = \mathrm{GL}(n+1)$, but well understood nonetheless. (See for example [Ar7, Section 7.5].)

We shall introduce a notational convention that will make the formal comparison quite transparent. For simplicity, we first write

$$\Pi_{\mathrm{disc}}(n) = \Pi_{\mathrm{disc}}(G(n)) = \Pi_{\mathrm{disc}}(\mathrm{GL}(n+1)).$$

It will then be convenient to denote the subset of cuspidal representations by

$$\Pi^0_{\mathrm{disc}}(n) = \Pi_{\mathrm{cusp}}(n) = \Pi_{\mathrm{cusp}}(\mathrm{GL}(n+1)).$$

Since $\Pi_{\mathrm{cusp}}(n)$ in some sense contains "almost all" the representations in $\Pi_{\mathrm{disc}}(n)$, we can think intuitively of $\Pi^0_{\mathrm{disc}}(n)$ as an open dense subset of $\Pi_{\mathrm{disc}}(n)$, even though there is no topology. If $(m+1)$ is a divisor of $(n+1)$, we also write

$$\Pi^0_{\mathrm{disc}}(m,n) = \{\mathcal{I}_{P'}(\sigma_0(\mu')) \ : \ m',\, \mu \text{ as in } (4.6)\}$$

for the set of representations in $\Pi_{\mathrm{disc}}(n)$ attached to the outer summand of m on the right hand side of (4.8). The case $m = n$ is then the subset

$$\Pi^0_{\mathrm{disc}}(n) = \Pi^0_{\mathrm{disc}}(n,n)$$

of cuspidal representations above. If $m < n$, the subset $\Pi^0_{\mathrm{disc}}(m,n)$ can be regarded as a stratum in $\Pi_{\mathrm{disc}}(n)$. The construction of its representations is founded on the set $\Pi^0_{\mathrm{disc}}(m)$ of cuspidal representations for the smaller group $\mathrm{GL}(m+1)$. The original set is then a disjoint union

$$\Pi_{\mathrm{disc}}(n) = \coprod_{(m+1)|(n+1)} \Pi^0_{\mathrm{disc}}(m,n). \tag{4.9}$$

The decomposition of Proposition 4.1 can obviously be written in these terms. We obtain

$$I_{\mathrm{disc}}(f) = \sum_{(m+1)|(n+1)} I^0_{\mathrm{disc}}(m,f), \tag{4.10}$$

where

$$I^0_{\mathrm{disc}}(m,f) = \sum_{\pi\in\Pi^0_{\mathrm{disc}}(m,n)} I^0_{\mathrm{disc}}(\pi,f),$$

for

$$I^0_{\mathrm{disc}}(\pi,f) = a^G(\pi)f_G(\pi) = a^G(m')\,\mathrm{tr}(\mathcal{I}_{P'}(\sigma_0(\mu'),f)), \qquad \pi\in\Pi_{\mathrm{disc}}(m,n), \tag{4.11}$$

in the notation of (4.8). It is this form of the spectral decomposition that will have a direct geometric counterpart.

Proposition 4.1 in fact comes with refinements of (4.9) and (4.10). If m and m' represent the two divisors in (4.7) (without the inequalities at the right), we could write

$$\Pi^0_{\mathrm{disc}}(m,m',n) = \{\mathcal{I}_{P'}(\sigma_0(\mu')) \ : \ \mu \text{ as in (4.6)}\}$$

for the set of representations in $\Pi_{\mathrm{disc}}(n)$ attached to the summand of m and m' on the right hand side of (4.8). Then

$$\Pi^0_{\mathrm{disc}}(m,n) = \coprod_{m'} \Pi^0_{\mathrm{disc}}(m,m',n)$$

so that

$$\Pi^0_{\text{disc}}(n) = \coprod_{m,m'} \Pi^0_{\text{disc}}(m, m', n).$$

This is the refinement of (4.9). Observe also that the set

$$\Pi^0_2(m, n) = \Pi^0_{\text{disc}}(m, n, n).$$

consists of those representations in $\Pi^0_{\text{disc}}(m, n)$ that lie in the subset $\Pi_2(n) = \Pi_2(G(n))$, and hence that

$$\Pi_2(n) = \coprod_{(m+1)|(n+1)} \Pi^0_2(m, n).$$

We thus obtain the refinement

$$I_{\text{disc}}(f) = \sum_m I^0_{\text{disc}}(m, f) = \sum_m \left(\sum_{m'} I^0_{\text{disc}}(m, m', f) \right) \tag{4.12}$$

of (4.10), as well as decomposition

$$I_2(f) = \sum_{(m+1)|(n+1)} I^0_2(m, f) \tag{4.13}$$

of $I_2(f)$, with summands on the right hand sides of (4.12) and (4.13) defined in the obvious way. However it is still the original decomposition (4.10) that will be most pertinent to the next section.

The discussion of this section is valid if f is any function $\mathcal{D}(G)$. We could refine the notation slightly to account for the special nature of the function $f = f_\infty f^\infty$ fixed in Section 2, where f^∞ lies in the global Hecke algebra

$$\mathcal{H}(G^\infty, K^\infty) = \mathcal{H}(\text{GL}(n+1, \mathbb{A}^\infty), \text{GL}(n+1, \hat{\mathbb{Z}})).$$

If $\Pi(G)$ is any set of automorphic representations of $G = \mathrm{GL}(n+1)$ (with central character on Z_+ implicitly understood to be trivial), let us write

$$\Pi(G, \hat{\mathbb{Z}}) = \Pi(\mathrm{GL}(n+1), \mathrm{GL}(n+1, \hat{\mathbb{Z}}))$$

for the subset of representations in $\Pi(G)$ whose restriction to the compact subgroup $\mathrm{GL}(n+1, \hat{\mathbb{Z}})$ of $G(\mathbb{A})$ is nonzero. The function (4.11) then vanishes unless π lies in the subset $\Pi^0_{\mathrm{disc}}(m, n, \hat{\mathbb{Z}})$ of $\Pi^0_{\mathrm{disc}}(m, n)$. The sum over $\Pi^0_{\mathrm{disc}}(m, n)$ in the definition of $I^0_{\mathrm{disc}}(m, f)$ can therefore be restricted to the subset $\Pi^0_{\mathrm{disc}}(m, n, \hat{\mathbb{Z}})$.

5. The stratification and the problem of comparison

The stratification will be for affine n-space $\mathcal{B}(n)$, regarded as a scheme over $\mathbb{Z}$. It is entirely elementary. What is not elementary is the question of its precise implications for the geometric terms in the trace formula, and in particular, for the analogue for $G = \mathrm{GL}(n+1)$ of the expansion (4.1). We shall introduce the stratification first, and then discuss how it ought to apply to the geometric expansion.

Let us denote the affine space by $\Xi(n)$ rather than $\mathcal{B}(n)$ in this context, to indicate that it is to contain the dual variables ξ for which Fourier transforms are defined. The construction is by induction. We assume inductively that we have defined an open subset $\Xi^0(m)$ of $\Xi(m)$ for any proper divisor $(m+1)$ of $(n+1)$. We use this to define a locally closed subset

$$\Xi^0(m, n) = \big\{(\xi_m, 0, \xi_m, 0, \dots, 0, \xi_m) : \xi_m \in \Xi^0(m)\big\} \tag{5.1}$$

of $\Xi(n)$, where if

$$(n+1) = (m+1)(d+1),$$

the vector in the brackets contains $(d+1)$-copies of the smaller vector ξ_m, and d-copies of the component 0. The number of components of this vector therefore equals

$$m(d+1)+d = md+m+d = (n+1)-1 = n,$$

so that $\Xi^0(m,n)$ is indeed a (locally closed) subset of $\Xi(n)$. We then require $\Xi(n)$ to be the (disjoint) union over *all* divisors $(m+1)$ of $(n+1)$ of the subsets $\Xi^0(m,n)$. Since the open subset attached to $m=n$ is just $\Xi^0(n)$, and thus equals

$$\Xi^0(n) = \Xi^0(n,n) = \Xi(n) \setminus \left(\coprod_{m \neq n} \Xi^0(m,n) \right),$$

this completes the inductive definition.

We summarize the construction formally as follows.

Construction 5.1. *There is a stratification*

$$\Xi(n) = \coprod_{\substack{m=0 \\ (m+1)|(n+1)}}^{n} \Xi^0(m,n), \tag{5.2}$$

where $\Xi^0(m,n)$ is the locally closed subset of $\Xi(n)$ obtained from the open subset $\Xi^0(m)$ of $\Xi(m)$ by (5.1), and where $\Xi^0(m)$ is in turn defined inductively by (5.2)

The stratification is simple, and it is obviously compatible with the $\mathbb{Z}$-structure on $\Xi(n)$. That is,

$$\Xi(n,\mathbb{Z}) = \coprod_m \Xi^0(m,n,\mathbb{Z}),$$

where

$$\begin{aligned} \Xi^0(m,n,\mathbb{Z}) &= \Xi^0(m,n) \cap \Xi(n,\mathbb{Z}) \\ &= \left\{ (\xi_m, 0, \xi_m, 0, \ldots, 0, \xi_m) \ : \ \xi_m \in \Xi^0(m,\mathbb{Z}) \right\}. \end{aligned}$$

The question is how it might be related to the spectral contributions to the geometric side of the trace formula.

Suppose that an application of Poisson summation holds for $G = \mathrm{GL}(n+1)$, and thereby gives us an analogue

$$\bar{I}_{\mathrm{ell,reg}}(f) = \sum_{\xi \in \Xi(n,\mathbb{Z})} \hat{\bar{I}}_{\mathrm{ell,reg}}(\xi, f) \tag{5.3}$$

for G of Altug's dual geometric expansion (4.1) for GL(2). We will then have a distribution valued function

$$\hat{\bar{I}}_{\mathrm{ell,reg}}(f) : \xi \to \hat{\bar{I}}_{\mathrm{ell,reg}}(\xi, f), \qquad\qquad \xi \in \Xi(n,\mathbb{Z}),$$

on $\Xi(n,\mathbb{Z})$. Its restriction to the various strata will in turn give us a decomposition

$$\bar{I}_{\mathrm{ell,reg}}(f) = \sum_{(m+1)|(n+1)} \hat{\bar{I}}^0_{\mathrm{ell,reg}}(m, f), \tag{5.4}$$

of the extended geometric expansion, where

$$\hat{\bar{I}}^0_{\mathrm{ell,reg}}(m, f) = \sum_{\xi \in \Xi^0(m,n,\mathbb{Z})} \hat{\bar{I}}_{\mathrm{ell,reg}}(\xi, f). \tag{5.5}$$

Notice the similarity of (5.4) and (5.5) with their spectral counterparts

$$I_{\mathrm{disc}}(f) = \sum_{(m+1)|(n+1)} I^0_{\mathrm{disc}}(m, f) \tag{5.6}$$

and

$$I^0_{\mathrm{disc}}(m, f) = \sum_{\pi \in \Pi^0_{\mathrm{disc}}(m,n,\hat{\mathbb{Z}})} I^0_{\mathrm{disc}}(\pi, f) \tag{5.7}$$

from the end of the last section. The comparison problem may then be stated as follows.

Question 5.2. *Are there identities that relate the terms in the geometric expansion* (5.4) *with those in the spectral expansion* (5.6)*?*

The main term in the decomposition (5.5) is the distribution

$$\hat{\bar{I}}^0_{\mathrm{ell,reg}}(f) = \hat{\bar{I}}^0_{\mathrm{ell,reg}}(n, f) = \sum_{\xi \in \Xi^0(n,\mathbb{Z})} \hat{\bar{I}}_{\mathrm{ell,reg}}(\xi, f). \tag{5.8}$$

This should be the summand that is closest to the cuspidal part

$$I_{\mathrm{cusp}}(f) = I^0_{\mathrm{disc}}(f) = I^0_{\mathrm{disc}}(n, f)$$

of the trace formula for $\mathrm{GL}(n+1)$. It is not likely to actually equal $I_{\mathrm{cusp}}(f)$, if for no reason other than we have excluded the supplementary geometric and spectral terms in the trace formula. What if all these extra terms were added to the mix? Might the enriched analogue $\hat{I}^0_{\mathrm{geom}}(f)$ of $\hat{\bar{I}}^0_{\mathrm{ell,reg}}(f)$ then be (almost) equal to $I_{\mathrm{cusp}}(f)$? Such an identity would be truly remarkable. It would leave $\hat{I}^0_{\mathrm{geom}}(f)$ as the only surviving part of the trace formula to be applied to the refined study of automorphic L-functions, as proposed by Langlands. We shall discuss the missing supplementary terms briefly in Section 6.

There is reason in any case to be hopeful. The decompositions (5.4) and (5.6) are strikingly similar in general, and they are very closely related as distributions in the special case of $G = \mathrm{GL}(2)$. For higher rank, the elliptic regular terms are still to be regarded as the primary geometric ingredients of the trace formula, just as the terms in $I_{\mathrm{disc}}(f)$ are to be considered the primary spectral ingredients. If the general decompositions are to match, in whatever form they ultimately take, there should be some evidence of it in their restrictions (5.4) and (5.6) to the primary terms. We shall add some further remarks on the question.

First, however, let us digress with a brief discussion of the expected sum (5.3). To get a sense of its complexity, even in the special case (4.1) for GL(2), we shall discuss some of its internal structure, extrapolating from GL(2) to the general case $G = \mathrm{GL}(n+1)$.

Where do we stand? We are working with the assumption that some generalization of Altug's Poisson summation formula applies to $G = \mathrm{GL}(n+1)$, and we want to see what it might look like. We are considering only the primary term $I_{\mathrm{ell,reg}}(f)$ of $I_{\mathrm{geom}}(f)$, with the test function $f \in \mathcal{D}(G)$ taken to be unramified at all nonarchimedean places. We are thus thinking of $I_{\mathrm{ell,reg}}(f)$ as a linear form in the function $f_\infty \in C_c^\infty(Z_+ \setminus G(\mathbb{R}))$. A successful application of Poisson summation would lead to an explicit expression in terms of abelian Fourier transforms of the functions $f^\varepsilon_{\infty,G}$ on $\mathcal{B}(n,\mathbb{R})$. Before we describe what it might be, we should first recall a point from Section 3. We assume that in dealing with the arithmetic coefficients in the expansion (3.3), we have also been able to insert the values of $f^\varepsilon_{\infty,G}(b)$ at points b in the complement of $\mathcal{B}^\varepsilon_{\mathrm{irred}}(n,\mathbb{Z})$ in the lattice $\mathcal{B}(n,\mathbb{Z})$. For only then will we have a geometric expression to which we can think of applying Poisson summation. As in Section 3, we write $\bar{I}_{\mathrm{ell,reg}}(f)$ for the hypothetical geometric expression obtained by thus enlarging $I_{\mathrm{ell,reg}}(f)$.

We can now try to imagine what form the geometric expression might take after Poisson summation. The formula will not be particularly simple, for it will have to include the rather complex analytic contributions we get from regularizing the two arithmetic coefficients in (3.3). Following Altug's results for $G = \mathrm{GL}(2)$ ([Al2, Theorem 4.2, formula (13)], we shall build the hypothetical formula in several steps.

For a start, there should be an implicit convolution of the Fourier transform of $f^\varepsilon_{\infty,G}$ with another function. This would appear explicitly as the Fourier transform of a product

$$\int_{\mathcal{B}(n,\mathbb{R})} (f^\varepsilon_{\infty,G}(x)J(x))e(-x\cdot\xi)dx, \qquad\qquad \xi \in \Xi(n,\mathbb{Z}),$$

with

$$e(r) = e^{2\pi i r}, \qquad\qquad r \in \mathbb{R}.$$

The role of the auxiliary function $J(x)$, at least a this stage, is to make the product vanish to high order at the singular points

$$\left\{x \in \mathcal{B}(n, \mathbb{R}) : D_{(x, \varepsilon p^k)} = 0\right\}$$

of $f^{\varepsilon}_{\infty,G}$, and thereby represent a Schwartz function on $\mathcal{B}(n, \mathbb{R})$. (See [Al2, Proposition 4.1].) As the Fourier transform is supposed to come from Poisson summation, there should be a sum over ξ (as well as over the sign ε). The second complication is that each summand should be multiplied by a second auxiliary function

$$K(\xi, \varepsilon p^k), \qquad\qquad \xi \in \Xi(n, \mathbb{Z}),\ \varepsilon \in \{\pm 1\}.$$

It should be a kind of Kloosterman sum, which encodes the arithmetic properties that were stripped from the two coefficients in (3.3). We would then have a sum

$$\sum_{\xi \in \Xi(n,\mathbb{Z})} \left(\sum_{\varepsilon} \int f^{\varepsilon}_{\infty,G}(x) J(x) e(-x \cdot \xi) dx \right) K(\xi, \varepsilon p^k).$$

A third complication is that the two auxiliary functions should depend jointly on an auxiliary variable u, which we take to range over the lattice points $\mathbb{N}^{c+1}$ of the open positive cone in $\mathbb{R}^{c+1}$. (The integer c is equal to 1 in Altug's case of $G = \mathrm{GL}(2)$, and might well remain so in general.) We set $J(x) = J(u, x)$ and $K(\xi, \varepsilon p^k) = K_u(\xi, \varepsilon p^k)$, and then sum the expression above over u. With this step $J(u, x)$ assumes a further role of the coefficient in an infinite series with origins in the arithmetic coefficients in (3.3), coupled with a regulating function that forces it to converge. (See [Al2, Section 4.1].) Finally, we need to adjust the point ξ in the exponential $e(-x\cdot\xi)$. Rather than ξ, we take its image in a lattice obtained from $\Xi(n,\mathbb{Z})$ by rescaling. Continuing to be motivated by [Al2, Section 4.2], we assume that the rescaling factors are given by a vector

$$m(u,p^k) = \left(m_1(u)^{-1}p^{k/n+1}, \dots, m_n(u)^{-1}p^{k/n+1}\right)$$

in the open positive cone of $\Xi(n,\mathbb{R})$, for which the numbers $\{m_j(u)\}$ are monomials in the variables $\{u_1,\dots,u_{c+1}\}$. We then replace ξ (in the exponential but *not* in $K_u(\xi,\varepsilon p^k)$) by the (ring) product

$$m(u,p^k)\xi = \left(m_1(u)^{-1}p^{k/n+1}\xi_1, \dots, m_n(u)^{-1}p^{k/n+1}\xi_n\right).$$

Combining these steps, we obtain our guess for the rough form assumed by the extended geometric expression after Poisson summation. We take it as a hypothesis, in which $J(u,x)$ and $K_u(\xi,\varepsilon p^k)$ are to be regarded as explicit functions.

Hypothesis 5.3. *The extended geometric expansion $\bar{I}_{\mathrm{ell,reg}}(f)$ obtained from $I_{\mathrm{ell,reg}}(f)$ according to the assumption above equals*

$$\sum_u\sum_\xi\left(\sum_\varepsilon\int_{\mathcal{B}(n,\mathbb{R})} f^\varepsilon_{\infty,G}(x)J(u,x)e(-x,m(u,p^k)\xi)dx\right)K_u(\xi,\varepsilon p^k),$$

where u, ξ and ε are summed over $\mathbb{N}^{c+1}$, $\Xi(n,\mathbb{Z})$ and $\{\pm1\}$, and f is the given function in Lemma 3.1.

The hypothesis comes with an implicit assumption that the outer sum over u and ξ is absolutely convergent, and can therefore be taken in any order. We would then obtain an identity of the required form (5.3), in which $\hat{\hat{I}}_{\text{ell}}(\xi, f)$ is an expression of the form

$$\sum_{u}\left(\sum_{\varepsilon}\int_{\mathcal{B}(n,\mathbb{R})} f^{\varepsilon}_{\infty,G}(x)J(u,x)e(-x,m(u,p^k)\xi)dx\right)K_u(\xi,\varepsilon p^k), \tag{5.9}$$

for any $\xi \in \Xi(n,\mathbb{Z})$. This would be the analogue for $G = \text{GL}(n+1)$ of the expansion (4.1) for GL(2).

We can now return to Question 5.2. It will not be simple to resolve, one way or the other. Our reason for the digression that led to Hypothesis 5.3 was simply to suggest that there will still be a good deal of analysis in the terms $\hat{\hat{I}}_{\text{ell,reg}}(\xi, f)$, even after the expected identity (5.3) has been established. Informed by the work of Altug in [Al2, Section 6], we should expect a complex analytic component to this, which is concrete, and comes with interesting contours and residues. Be that as it may, let us consider the summands in (5.4) and (5.6) corresponding to $(m+1)$, where

$$(n+1) = (m+1)(d+1).$$

The dominant terms in the summand $I^0_{\text{disc}}(m, f)$ of (5.6) come from the representations $\pi \in \Pi_2(G)$. We formulated the discussion of these objects in Section 4 in terms of the irreducible unitary representations

$$\psi = \phi \otimes \nu : L_{\mathbb{Q}} \times \text{SU}(2) \to \text{GL}(n+1,\mathbb{C}),$$

where ϕ is an irreducible unitary representation of the hypothetical group $L_{\mathbb{Q}}$ of degree $(m+1)$, and ν is the irreducible representation of SU(2) (or equivalently $\text{SL}(2,\mathbb{C})$) of degree $(d+1)$. We are assuming that ν is fixed, since its degree $(d+1)$ was fixed in the choice of m above. We can then let ϕ vary over the irreducible unitary representations of $L_{\mathbb{Q}}$ of degree $(m+1)$. These would correspond to cuspidal automorphic representations $\mu \in \Pi_{\text{cusp}}(m)$ for $\text{GL}(m+1)$. The group $\mathrm{GL}(n+1,\mathbb{C})$ here of course represents the dual group $\hat{G}$ of $G = \mathrm{GL}(n+1)$. Let $\hat{M}$ be the standard Levi subgroup of $\hat{G}$ corresponding to the partition $(m+1,\ldots,m+1)$ of $(n+1)$. The restriction of ψ to $L_{\mathbb{Q}}$ is the direct sum of $(d+1)$-copies of ϕ, which obviously maps $L_{\mathbb{Q}}$ to $\hat{M}$. It corresponds to the cuspidal automorphic representation of the Levi subgroup $M = M_P$ of $\mathrm{GL}(n+1)$ we denoted by $\sigma_0(\mu)$ in Section 4, namely the (outer) tensor product of $(d+1)$-copies of μ. The representations π range over the Langlands quotients $\mathcal{J}_P(\sigma(\mu))$, where we recall that $\sigma(\mu)$ is the automorphic representation of M obtained by twisting $\sigma_0(\mu)$ with the exponents in (4.3).

The general terms in $I^0_{\mathrm{disc}}(m,f)$ are obtained from factorizations

$$(n+1) = (m+1)(d'+1)(e'+1) = (m+1)(d+1),$$

where $(m'+1) = (m+1)(e'+1)$ is as in Section 4. For a given m', they make up the distribution $I^0_{\mathrm{disc}}(m,m',f)$ given by the summand of m' in the refined decomposition (4.12) of $I_{\mathrm{disc}}(f)$, which is defined in turn as a summand of (m,m') in the formula for $I_{\mathrm{disc}}(f)$ in Proposition 4.1. The goal would be to match it with the corresponding distribution in some parallel decomposition of the summand $\hat{\hat{I}}_{\mathrm{ell,reg}}(m,f)$ in (5.4) we have fixed. The cumbersome notation in Proposition 4.1 was intended to emphasize the origins of the constituents of $\hat{I}^0_{\mathrm{disc}}(m,f)$ in the set

$$\Pi^0_{\mathrm{disc}}(m) = \Pi_{\mathrm{cusp}}(m)$$

of cuspidal automorphic representations for $\mathrm{GL}(m+1)$. Incidentally the irreducible representation ν of $\mathrm{SU}(2)$ of degree $(d+1)$, which is fixed by virtue of the fact that m is fixed, appears explicitly only in the dominant terms with $m' = m$. For the more general terms where m' is arbitrary, the underlying representation ν' of $\mathrm{SU}(2)$ is of degree $(d'+1)$, but it is in a sense still governed by ν.

The summand $\hat{\bar{I}}_{\mathrm{ell,reg}}(m,f)$ in (5.4) is defined in terms of the stratum $\Xi^0(m,n)$ in the affine space $\Xi(n)$. Can we motivate the definition of the stratification, beyond simply observing that the decomposition (5.4) of $\bar{I}_{\mathrm{ell,reg}}(f)$ it provides is formally parallel to the decomposition (5.6) of $I_{\mathrm{disc}}(f)$? We shall at least say a few words on the broader context from which it arises.

In the discussion above, we considered the restriction $\tilde{\phi}$ of the representation $\psi = \phi \otimes \nu$ of $L_{\mathbb{Q}} \times \mathrm{SU}(2)$ to the subgroup $L_{\mathbb{Q}}$. Since it is the second factor ν that is fixed in this exercise, it would be natural to look also at the restriction $\tilde{\nu}$ of ψ to the second subgroup $\mathrm{SU}(2)$. This is obviously the direct sum of $(m+1)$-copies of ν. Regarded as a representation of the group $\mathrm{SL}(2,\mathbb{C})$, it corresponds under the Jacobson-Morozov theorem to a unipotent conjugacy class, namely the unipotent class with Jordan blocks of size $(d+1,\dots,d+1)$. Jordan normal form is of course the standard way to classify unipotent conjugacy classes in general linear groups. However, there is also a second classification. Rather than representing a unipotent class by a *minimal* standard Levi subgroup that it meets, taken up to conjugacy, one characterizes it by a *maximal* standard Levi subgroup, taken again up to conjugacy, such that the unipotent conjugacy class intersects the associated standard unipotent radical. The second Levi subgroup corresponds to the partition dual to the first, which in the case hand is $(m+1,\dots,m+1)$.

Jordan normal form is the specialization to general linear groups of a general classification due to Bala and Carter. The second classification is closely related to, though not actually the same as, the specialization to general linear groups of Dynkin's parametrization of unipotent conjugacy classes by weighted Coxeter diagrams. (See [CM]).

We shall say that a general representation

$$\tilde{\nu} : \mathrm{SU}(2) \to \hat{G}$$

is *discrete* if the quotient

$$\mathrm{cent}(\mathrm{im}(\tilde{\nu}), \hat{G})/Z(\hat{G})$$

of the centralizer of its image by the centre of $\hat{G}$, a priori a complex reductive group, is actually semisimple. This means that $\tilde{\nu}$ should contribute to $I_{\mathrm{disc}}(f)$, in the sense that it can be inflated to a global parameter $\tilde{\psi}$ whose image centralizes only $Z(\hat{G})$. With a moment's thought, we see that $\tilde{\nu}$ is discrete if and only if it is a direct sum of irreducible representations of the same degree. Any discrete representation $\tilde{\nu}$ is then *even*, in the sense that the weighted Coxeter diagram assigned by Dynkin to the unipotent class of $\tilde{\nu}$ has vertices labelled only with the integers 0 and 2, and not 1. For even unipotent classes, Dynkin's parametrization matches their characterization in $\mathrm{GL}(n+1)$ as dense orbits in unipotent radicals.

To be a little more explicit, suppose that

$$\tilde{\nu} = \psi|_{\mathrm{SU}(2)} = \nu \oplus \cdots \oplus \nu,$$

is the discrete representation obtained from our fixed irreducible representation ν of $\mathrm{SU}(2)$ (and a variable parameter ψ). To describe its Dynkin diagram, we need to assign weights to the vertices of the Coxeter diagram

$$\circ - \circ - \circ - \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot - \circ - \circ - \circ$$

of the derived group $\mathrm{SL}(n+1, \mathbb{C})$ of $\hat{G}$. This entails permuting the entries of the diagonal matrix

$$\tilde{\nu}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \nu\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \oplus \cdots \oplus \nu\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad z \in \mathbb{C}^*,$$

in $\mathrm{SL}(n+1,\mathbb{C})$ so that its differential lies in the closure of the positive Weyl chamber. Since $\tilde{\nu}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ is the direct sum of $(m+1)$-copies of the diagonal matrix

$$\mathrm{diag}(z^{d/2}, z^{d/2-1}, \dots, z^{-d/2})$$

in $\mathrm{SL}(d+1,\mathbb{C})$, the required permutation gives us the diagonal matrix

$$\mathrm{diag}(z^{d/2}, \dots, z^{d/2}, z^{d/2-1}, \dots, z^{d/2-1}, \dots\dots\dots, z^{-d/2}, \dots, z^{-d/2})$$

in $\mathrm{SL}(n+1,\mathbb{C})$. The weights are then given by the differences of successive exponents of these diagonal entries. The (weighted) Dynkin diagram of $\tilde{\nu}$ is therefore equal to

$$\overset{0}{\circ} - \cdots - \overset{0}{\circ} - \overset{2}{\circ} - \overset{0}{\circ} - \cdots - \overset{0}{\circ} - \overset{2}{\circ} - \overset{0}{\circ} - \cdots\cdots\cdots - \overset{0}{\circ} - \overset{2}{\circ} - \overset{0}{\circ} - \cdots - \overset{0}{\circ}\,. \tag{5.10}$$

It consists of $(d+1)$ blocks of vertices labelled by the 0-vector in $\mathbb{Z}^m$, and d isolated vertices labelled by 2.

We should now look back at the definition (5.1) of the stratum $\Xi^0(m,n)$ in $\Xi(n)$. We see that it is completely parallel to the (weighted) Dynkin diagram (5.10). The same goes for the corresponding summand $\hat{\bar{I}}^0_{\mathrm{ell,reg}}(m,f)$ in the decomposition (5.4) of $\bar{I}_{\mathrm{ell,reg}}(f)$, since it is defined (5.5) in terms of $\Xi^0(m,n)$. Now there is a canonical bijection from the set of vertices of the Coxeter diagram to the standard basis of $\Xi(n)$. To pass from the Dynkin diagram to the stratum $\Xi^0(m,n)$, we change each vertex with a 2 in (5.10) to the corresponding 0 coordinate in (5.1), and we replace the $(d+1)$-blocks of vertices labelled with the 0-vector in $\mathbb{Z}^m$ in (5.10) with a generic element $\xi_m \in \Xi^0(m)$, embedded diagonally in $\Xi^0(m,n)$ as in (5.1). In particular, the 0-coordinates of the vectors ξ in the sum (5.5) that defines the geometric distribution $\hat{\bar{I}}^0_{\mathrm{ell,reg}}(m,f)$ attached to ν come directly from the Dynkin diagram (5.10) for $\tilde{\nu}$.

How might we then explain the peculiar nature of the nonzero co-ordinates in these vectors ξ?

The Dynkin diagram (5.10) determines a standard parabolic subgroup $\hat{P}$ of $\hat{G}$ such that the unipotent conjugacy class attached to $\tilde{\nu}$ intersects the unipotent radical $U_{\hat{P}}$ in an open dense subset. The unipotent conjugacy class is of course that of the unipotent element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, while $\hat{P}$ is the standard parabolic subgroup whose Levi component $M_{\hat{P}}$ corresponds to the subdiagram of (5.10) obtained by deleting the vertices indexed by 2. The group $\hat{M}_P = M_{\hat{P}}$ is then dual to the Levi subgroup

$$M_P = \mathrm{GL}(m+1) \times \cdots \times \mathrm{GL}(m+1) \tag{5.11}$$

of $G = \mathrm{GL}(n+1)$. Now, the summand $\hat{\hat{I}}^0_{\mathrm{ell,reg}}(m, f)$ in (5.4) is a supposed to match the summand $I^0_{\mathrm{disc}}(m, f)$ in the spectral decomposition (5.6) of $I_{\mathrm{disc}}(f)$. To see explicitly how we do this, we recall that $I^0_{\mathrm{disc}}(m, f)$ was constructed from the cuspidal automorphic representations of $\mathrm{GL}(m+1)$, and their diagonal transfers

$$\mu \to \sigma_0(\mu), \qquad\qquad \mu \in \Pi_2(m), \tag{5.12}$$

to M_P. At some point, there would be an induction assumption that the cuspidal term $I_{\mathrm{cusp}}(f) = I^0_{\mathrm{disc}}(f)$ for $G = \mathrm{GL}(n+1)$ in (5.6) matches (in a way that would need to be specified) the leading term $\hat{\hat{I}}^0_{\mathrm{ell,reg}}(f)$ in the geometric expansion (5.4). Taking this for granted in case n is replaced by m, we conclude that the diagonal transfer (5.12) is to be replaced by the diagonal embedding of the subset $\Xi^0(m, \mathbb{Z})$ of $\Xi(m, \mathbb{Z})$. This explains the definition (5.1). It does not, of course, prove that the distribution $\hat{\hat{I}}_{\mathrm{ell,reg}}(m, f)$ is equal to $I^0_{\mathrm{disc}}(m, f)$.

If this all seems slightly murky, the reader could think about the case that $n = 3$. Then $G = \mathrm{GL}(4)$, and $(m+1)$ ranges over the divisors 1, 2 and 4 of 4. The (weighted) Dynkin diagrams for $m = 0$, 1 and 3 are

$$\begin{array}{ccccc} 2 & & 2 & & 2 \\ \circ & - & \circ & - & \circ \end{array}, \qquad \begin{array}{ccccc} 0 & & 2 & & 0 \\ \circ & - & \circ & - & \circ \end{array}, \qquad \begin{array}{ccccc} 0 & & 0 & & 0 \\ \circ & - & \circ & - & \circ \end{array},$$

respectively, with corresponding Levi subgroups M_P equal to $\mathrm{GL}(1)\times\mathrm{GL}(1)\times\mathrm{GL}(1)\times\mathrm{GL}(1)$, $\mathrm{GL}(2)\times\mathrm{GL}(2)$ and $\mathrm{GL}(4)$. Using the diagrams to display the associated strata $\Xi^0(m,3)$, we write

$$\begin{array}{ccccc} (0 & , & 0 & , & 0) \\ \circ & - & \circ & - & \circ \end{array}, \qquad \begin{array}{ccccc} (\xi_1 & , & 0 & , & \xi_1) \\ \circ & - & \circ & - & \circ \end{array}, \qquad \begin{array}{ccccc} (\xi_1 & , & \xi_2 & , & \xi_3) \\ \circ & - & \circ & - & \circ \end{array},$$

where the vectors $\xi = (\xi_1, \xi_2, \xi_3)$ above the three diagrams partition the set $\Xi(3,\mathbb{Z}) = \mathbb{Z}^3$ into three disjoint subsets. In the first diagram, $\xi = 0$ is just the zero vector. This would be the minimal stratum, which should be attached to the trivial one-dimensional automorphic representation of $G(\mathbb{A})$, the singular representation induced from the trivial representation of the maximal Levi subgroup $\mathrm{GL}(2)\times\mathrm{GL}(2)$ and the singular representation induced from the trivial representation of the minimal Levi subgroup M_P that comprise the distribution $I^0_{\mathrm{disc}}(0,f)$. In the second diagram, ξ_1 ranges over the nonzero integers. The corresponding sum matches the cuspidal discrete spectrum of $\mathrm{GL}(2)$, according to Altug's main theorem [Al2] (Theorem 6.1 together with Lemma 6.2). Its diagonal transfer to $M_P = \mathrm{GL}(2)\times\mathrm{GL}(2)$ is the foundation for the induced representations that lead to the distribution $I^0_{\mathrm{disc}}(1,f)$ with $m = 1$. In the third diagram, ξ ranges over the remaining points in $\mathbb{Z}^3$. The corresponding sum gives the distribution $\hat{\tilde{I}}^0_{\mathrm{ell,reg}}(f)$ we would hope compare with the cuspidal part $I_{\mathrm{cusp}}(f) = I^0_{\mathrm{disc}}(3,f) = I^0_{\mathrm{disc}}(f)$ of $I_{\mathrm{disc}}(f)$.

Let me add one final comment. The diagonal transfer (5.12) represents a twisted endoscopic transfer from the group $G(m) = \mathrm{GL}(m+1)$ to the product M_P, relative to the diagonal embedding

$$\mathrm{GL}(m+1,\mathbb{C}) \hookrightarrow \hat{M}_P = \mathrm{GL}(m+1,\mathbb{C}) \times \cdots \times \mathrm{GL}(m+1,\mathbb{C})$$

of its dual group. Its image is in fact forced on us as the centralizer in $\hat{G}$ of the image of $\tilde{\nu}$. This observation is obvious for the general linear groups we are considering here. However, its analogue for other groups, in which endoscopic transfer is replaced by a more general functorial transfer, is more significant. In fact, the generalization of our constructions to arbitrary groups appears to be quite remarkable. I have not studied all of the details, but I hope to return to them in another paper.

6. On the supplementary terms

We have already suggested that Question 5.2 really needs to be posed for the full trace formula, not just its primary terms. The supplementary terms are certainly more complex, and are accordingly less developed. However they seem to be important in the general case, perhaps more so than for GL(2), and are undoubtedly essential for a full understanding of the problems of this paper. I have not had the chance to study them in the context of Beyond Endoscopy, so I do not have much to say. I will confine the discussion of this section to some general remarks. The reader might first consult the brief discussion in [La4, Section 4.5], which alludes to the general supplementary terms as well as to other questions for higher rank we have considered here.

We continue to take G to be the general linear group $G(n) = \mathrm{GL}(n+1)$ over $\mathbb{Q}$. For the moment, we shall take f to be a general function in the space $\mathcal{D}(G)$ of smooth, Z_+-invariant functions of compact support on $G(\mathbb{A})$. As we noted in Section 1, the full trace formula is an identity

$$I_{\mathrm{geom}}(f) = I_{\mathrm{spec}}(f), \qquad\qquad f \in \mathcal{D}(G), \tag{6.1}$$

between a geometric expansion on the left and a spectral expansion on the right. The supplementary geometric terms make up the difference

$$I_{\mathrm{geom}}(f) - I_{\mathrm{ell,reg}}(f),$$

while the supplementary spectral terms make up the difference

$$I_{\mathrm{spec}}(f) - I_{\mathrm{disc}}(f).$$

We shall not recall their precise definitions, leaving the reader to consult the references cited in Section 1 for the details.

On the geometric side, the main supplementary terms come from orbital integrals that are weighted, in the sense that the integrals over the given (nonelliptic) conjugacy classes are taken with respect to measures that are not invariant. On the spectral side the main supplementary terms come from characters that are weighted, which is to say that they are traces of induced representations composed with operators that are not scalars. Weighted orbital integrals and weighted characters are not invariant distributions. However, they come with a natural remedy. Roughly speaking, one transfers the noninvariant components of the weighted characters on the spectral side to the geometric side, where they can then be combined with the weighted orbital integrals. This leads to more complicated, but invariant, distributions on the geometric side, and simpler residues (literally) of the weighted characters on the spectral side that are also invariant. These are the main supplementary terms in the (invariant) trace formula.

In general, the geometric terms fibre over the set $\Gamma_{\mathrm{ss}}(G)$ of semisimple conjugacy classes

γ in $G(\mathbb{Q})$. As in Section 2, one can take the characteristic polynomials

$$p_a(\lambda) = p_\gamma(\lambda) = \lambda^{n+1} - a_1\lambda^n + \cdots + (-1)^n a_n \lambda + (-1)^{n+1} a_{n+1}$$

of their classes. This gives us a bijection $\gamma \to a$ from $\Gamma_{ss}(\mathbb{Q})$ to the set $\mathcal{A}(n, \mathbb{Q})$ of all $\mathbb{Q}$-points in the Steinberg-Hitchin base. One can therefore take the fibration to be over $\mathcal{A}(n, \mathbb{Q})$. The general spectral terms fibre over the set $\Pi(G)$ of automorphic representations π of G that occur in the spectral decomposition of $L^2(Z_+G(\mathbb{Q}) \setminus G(\mathbb{A}))$, in the precise sense governed by Eisentein series. Both of these statements are slightly misleading, however, since what amount to the "singular" fibres occur naturally as disjoint unions of subsets, and these are best treated separately.

We can of course specialize our test function to the product $f = f_\infty f^\infty$ attached to a given p and k in (2.2). This has implications for the supplementary terms on both sides of the trace formula.

For the geometric side, the invariant distribution attached to a local nonarchimedean weighted orbital integral is equal to the weighted orbital integral itself if it is evaluated at a function in the unramified Hecke algebra. Since the corresponding global invariant distribution satisfies a natural splitting formula in terms of its local components, the supplementary geometric terms over $\gamma \in \Gamma_{ss}(G)$ should then vanish unless the image a of γ in $\mathcal{A}(n, \mathbb{Q})$ is of the form

$$a = (b, \varepsilon p^k), \qquad \varepsilon \in \{\pm 1\},\ b \in \mathcal{B}(n, \mathbb{Z}), \tag{6.2}$$

in the notation of Section 3. This should lead to an analogue for $I_{\mathrm{geom}}(f)$ of the formula (3.3) for $I_{\mathrm{ell,reg}}(f)$ in Lemma 3.1, but where the inner sum in (3.3) replaced by a sum over b in the full lattice $\mathcal{B}(n, \mathbb{Z}) \cong \mathbb{Z}^n$.

There is no reason at this preliminary stage of investigation to write down explicitly what form the general analogue of (3.3) for $I_{\mathrm{geom}}(f)$ would take. For we would then have to introduce further notation, which among other things would be needed to describe the general splitting formulas for the various terms. Let us just say that if the element a in (6.2) comes from a regular class $\gamma \in \Gamma_{\mathrm{ss,\ reg}}(G)$, and if γ is in turn the image of a regular elliptic class γ_M in a standard Levi subgroup M of G, the splitting formula for the product $f = f_\infty f^\infty$ is a sum over Levi subgroups L, with

$$M \subset L \subset G, \tag{6.3}$$

of corresponding products. For any given L, one factor is the weighted orbital integral at γ_M of the unramified function f^∞ relative to the pair (L, M), and the other factor is the invariant distribution attached to the weighted orbital integral of f_∞ at the image γ_L of γ_M in $\Gamma_{\mathrm{ss}}(L)$, relative to the pair (G, L). We would want some analogue of (3.5) for the first factor, and some estimate for the second factor as an approximation of the invariant orbital integral of f_∞ at γ.

For the spectral side, we can consider a general term in the fibre of $\pi \in \Pi(G)$. To do so, we represent π as an induced image

$$\pi = \mathcal{I}_P(\pi_{M,\lambda}), \qquad \pi_M \in \Pi_{\mathrm{disc}}(M),\ \lambda \in \mathfrak{a}^*_{M,G},$$

from a standard Levi subgroup M of G. We are writing $\pi_{M,\lambda}$ for the twist of π_M by an element in the real vector space

$$i\mathfrak{a}^*_{M,G} = (Z_{M,+}/Z_{G,+})^*\,.$$

The term attached to π and M is again associated to a sum over Levi subgroups (6.3) of products. In this case, however, the products are slightly different. For any given L, one factor is a purely global term attached to the pair (L, M), composed of logarithmic derivatives of automorphic L-functions of representations $\pi_{M,\lambda}$. The other factor is the purely local invariant distribution attached to the weighted character of f at the image π_L of $\pi_{M,\lambda}$ in $\Pi(L)$, relative to the pair (G, L). The product of the two factors is then integrated over λ in the space $i\mathfrak{a}^*_{M,L}$. Since f itself is a product of $f_\infty f^\infty$, it appears at first glance that the splitting formula for the purely local factor will induce a further sum over Levi subgroups L_1, with $L \subset L_1 \subset G$, breaking the symmetry of the spectral side with the geometric side. However, the nonarchimedean component of this distribution will vanish at the unramified function ϕ^∞, whenever $L_1 \neq L$. The general spectral expansion will therefore be given by a sum over M and L in (6.3), making it at least in this sense parallel to the general geometric expansion.

I am not sure that my impressionistic description of the supplementary terms is precise enough to be of much use. The reader can consult [Ar1] for the relevant details (geometric in Section 3 and spectral in Section 4 of that paper). My only aim here was to point out the modest simplification in the general trace formula for functions f of the form (2.2). We see that it contains four kinds of supplementary terms. On the geometric side, there are the local nonarchimedean terms attached to pairs (L, M), and the local archimedean terms that correspond to the pairs (G, L). On the spectral side, there are the logarithmic derivatives of global L-functions attached to pairs (L, M), and the local archimedean terms that correspond to the pairs (G, L).

Langlands investigates three of these four supplementary terms for $G = \mathrm{GL}(2)$ in [La4, Sections 2.2–2.4, and Appendix C]. (The fourth kind of supplementary term, namely the invariant distribution attached to a weighted (archimedean) character for the pair (G, L), is elementary for GL(2), in the sense that it vanishes if $L \neq G$, and equals the underlying invariant character if $L = G$.) He considers their contributions to the trace formula, but taken modulo functions that contribute nothing to the relevant Tauberian limit (in this case

[La4, (12)]. In particular, he studies their behaviour as the given prime p becomes large (but with the integer k, which was denoted by m in [La4], still remaining fixed). With this equivalence relation, he observes that the terms simplify considerably as distributions in f_∞. (See also [Al3, Section 3].)

I have not had a chance to think about Langlands' investigations of these terms, let alone their possible extension from GL(2) to GL($n+1$). My present understanding may be quite flawed, but let me nonetheless venture a few more remarks. For there are some interesting techniques available in higher rank, especially in the archimedean case. For example, it would be worthwhile to consider the main theorem [Ar6]. It represents a limit formula satisfied by the invariant distributions attached to archimedean weighted orbital integrals. A similar formula is undoubtedly also valid for p-adic weighted orbital integrals. The question is whether this formula can be combined with the techniques used by Langlands in the special cases for GL(2).

Another question concerns the fourth kind of supplementary terms. For groups of higher rank, their contributions to the spectral side of the trace formula are not always elementary. They are related in a rather complicated way to both the values of characters and certain residues in the complex domain (See [Ar3, Section 10].) Characters are of course something we take seriously, especially when they occur discretely on the spectral side. Which ones survive the Tauberian limits applied by Langlands? Can these then be combined with the character formulas for GL($n+1$) reviewed in [Ar7, Section 7.5] to give information on Question 5.2?

A final comment concerns the basic p-adic function f_p^k, chosen in (2.2) as the p-adic component of our adelic function $f = f_\infty f^\infty$. It was defined as an element in the unramified p-adic Hecke algebra that depends on a fixed positive integer k, and a prime p that has also been fixed, but which will eventually be expected to vary. For GL(2), f_p^k was characterized independently by the identity

$$\operatorname{tr}(\pi_p(f_p^k)) = \operatorname{tr}(\rho^k(c(\pi_p))) \tag{6.4}$$

of [La4, (11)], in which π_p is any unramified representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ with Frobenius-Hecke class $c(\pi_p)$ in $\hat{G}$, and ρ^k is the irreducible representation of $\hat{G} = \mathrm{GL}(2, \mathbb{C})$ of degree $(k+1)$. (See the standard calculation near the beginning of Section 2.1 of [La4].) In this paper, we defined f_p^k simply by the obvious extension to $\mathrm{GL}(n+1)$ of the direct construction for $\mathrm{GL}(2)$ in [La4, Section 2.1], in order to keep our discussion manageable. But for $G = \mathrm{GL}(n+1)$, the irreducible representations ρ of $\hat{G}$ are parametrized by much more than just the positive integers k. To investigate the finer aspects of Beyond Endoscopy, we would presumably want to choose basic functions f_p^ρ that satisfy the analogue of (6.4) for any ρ, and in fact more generally, if ρ is any representation of the full L-group ${}^L G$. I do not know what implications this will have for p-adic integrals, as represented by more complicated versions of the formulas (3.4) and (3.5). Since we do not yet have *any* analogues of these formulas in higher rank, it is hard to know what to expect.

References

[Al1] S. A. Altuğ. *Beyond Endoscopy via the Trace Formula.* Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, 2013.

[Al2] S. A. Altuğ. *Beyond Endoscopy via the trace formula – I: Poisson summation and isolation of special representations.* In: *Compos. Math.* **151.10** (2015), pp. 1791–1820.

[Al3] S. A. Altuğ. *Beyond Endoscopy via the trace formula – II: Asymptotic expansions of Fourier transforms and bounds towards the Ramanujan conjecture.* 2015. eprint: arXiv:1506.08911.

[Al4] S. A. Altuğ. *Beyond Endoscopy via the trace formula – III: The standard representation.* 2015. eprint: arXiv:1512.09249.

[Ar8] J. Arthur. *Problems beyond endoscopy.* Submitted to Proceedings of Conference in Honor of the 70th birthday of Roger Howe.

[Ar4] J. Arthur. *A stable trace formula. I. General expansions.* In: *J. Inst. Math. Jussieu* **1.2** (2002), pp. 175–277.

[Ar5] J. Arthur. *An introduction to the trace formula.* In: *Harmonic Analysis, the Trace Formula, and Shimura Varieties.* Vol. 4. Clay Math. Proc. Amer. Math. Soc., Providence, RI, 2005, pp. 1–263.

[Ar6] J. Arthur. *An asymptotic formula for real groups.* In: *J. Reine Angew. Math.* **601** (2006), pp. 163–230.

[Ar7] J. Arthur. *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups.* Vol. 61. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2013.

[Ar1] J. Arthur. *The invariant trace formula. II. Global theory.* In: *J. Amer. Math. Soc.* **1.3** (1988), pp. 501–554.

[Ar2] J. Arthur. *Unipotent automorphic representations: Conjectures.* In: *Astérisque* **171-172** (1989). Orbites unipotentes et représentations, II, pp. 13–71.

[Ar3] J. Arthur. *On elliptic tempered characters.* In: *Acta Math.* **171.1** (1993), pp. 73–138.

[BK1] A. Braverman and D. Kazhdan. *γ-functions of representations and lifting.* In: *Geom. Funct. Anal.* **Special Volume, Part I** (2000). With an appendix by V. Vologodsky, GAFA 2000 (Tel Aviv, 1999), pp. 237–278.

[BK2] A. Braverman and D. Kazhdan. *γ-sheaves on reductive groups.* In: *Studies in Memory of Issai Schur (Chevaleret/Rehovot, 2000).* Vol. 210. Progr. Math. Birkhäuser Boston, Boston, MA, 2003, pp. 27–47.

[CM] D. H. Collingwood and W. M. McGovern. *Nilpotent Orbits in Semisimple Lie Algebras.* Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[FLN] E. Frenkel, R. Langlands, and B. Ngo. *Formule des traces et fonctorialité: Le début d'un programme*. In: *Ann. Sci. Math. Québec* **34** (2010), pp. 199–243.

[G] J. R. Getz. *Nonabelian Fourier transforms for spherical representations*. 2015. eprint: arXiv:1506.09128.

[H] T. C. Hales. *The fundamental lemma and the Hitchin fibration [after Ngô Bao Châu]*. In: *Séminaire Bourbaki, Astérisque* **348.1035** (2012), pp. 233–263.

[K3] R. E. Kottwitz. *Harmonic analysis on reductive p-adic groups and Lie algebras*. In: *Harmonic Analysis, the Trace Formula, and Shimura Varieties*. Vol. 4. Clay Math. Proc. Amer. Math. Soc., Providence, RI, 2005, pp. 393–522.

[K1] R. E. Kottwitz. *Orbital integrals on* GL_3. In: *Amer. J. Math.* **102.2** (1980), pp. 327–384.

[K2] R. E. Kottwitz. *Unstable orbital integrals on* $\mathrm{SL}(3)$. In: *Duke Math. J.* **48.3** (1981), pp. 649–664.

[Laf] L. Lafforgue. *Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires*. In: *Jpn. J. Math.* **9.1** (2014), pp. 1–68.

[La4] R. P. Langlands. *Beyond endoscopy*. In: *Contributions to Automorphic Forms, Geometry, and Number Theory*. Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 611–697.

[La5] R. P. Langlands. *Singularités et transfert*. In: *Ann. Math. Qué.* **37.2** (2013), pp. 173–253.

[La1] R. P. Langlands. *Problems in the theory of automorphic forms*. In: *Lectures in Modern Analysis and Applications, III*. Vol. 170. Lecture Notes in Math. Springer, Berlin, 1970, pp. 18–61.

[La2] R. P. Langlands. *Automorphic representations, Shimura varieties, and motives. Ein Märchen*. In: *Automorphic Forms, Representations and L-Functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 205–246.

[La3] R. P. Langlands. *On the classification of irreducible representations of real algebraic groups.* In: *Representation Theory and Harmonic Analysis on Semisimple Lie Groups.* Vol. 31. Math. Surveys Monogr. Amer. Math. Soc., Providence, RI, 1989, pp. 101–170.

[MM] M. R. Murty and V. K. Murty. *Non-vanishing of L-Functions and Applications.* Vol. 157. Progress in Mathematics. Birkhäuser Verlag, Basel, 1997.

[Mok] C. P. Mok. *Endoscopic classification of representations of quasi-split unitary groups.* In: *Mem. Amer. Math. Soc.* **235.1108** (2015).

[MW] C. Mœglin and J.-L. Waldspurger. *Le spectre résiduel de* $\mathrm{GL}(n)$. In: *Ann. Sci. École Norm. Sup. (4)* **22.4** (1989), pp. 605–674.

[N3] B. C. Ngô. *Hankel transform, Langlands functoriality and functional equation for automorphic L-functions.* Preprint.

[N1] B. C. Ngô. *Le lemme fondamental pour les algèbres de Lie.* In: *Publ. Math. Inst. Hautes Études Sci.* **111** (2010), pp. 1–169.

[N2] B. C. Ngô. *On a certain sum of automorphic L-functions.* In: *Automorphic Forms and Related Geometry: Assessing the Legacy of I. I. Piatetski-Shapiro.* Vol. 614. Contemp. Math. Amer. Math. Soc., Providence, RI, 2014, pp. 337–343.

[Re1] J. Repka. *Shalika's germs for p-adic* $\mathrm{GL}(n)$*. I. The leading term.* In: *Pacific J. Math.* **113.1** (1984), pp. 165–172.

[Re2] J. Repka. *Shalika's germs for p-adic* $\mathrm{GL}(n)$*. II. The subregular term.* In: *Pacific J. Math.* **113.1** (1984), pp. 173–182.

[Ro1] J. D. Rogawski. *An application of the building to orbital integrals.* In: *Compositio Math.* **42.3** (1980/81), pp. 417–423.

[Ro2] J. D. Rogawski. *Some remarks on Shalika germs.* In: *The Selberg Trace Formula and Related Topics (Brunswick, Maine, 1984).* Vol. 53. Contemp. Math. Amer. Math. Soc., Providence, RI, 1986, pp. 387–391.

[Sak] Y. Sakellaridis. *Beyond endoscopy for the relative trace formula I: Local theory*. In: *Automorphic Representations and L-Functions*. Vol. 22. Tata Inst. Fundam. Res. Stud. Math. Tata Inst. Fund. Res., Mumbai, 2013, pp. 521–590.

[Shi] G. Shimura. *Introduction to the Arithmetic Theory of Automorphic Functions*. Kanô Memorial Lectures, No. 1. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971.

[SY] K. Soundararajan and M. P. Young. *The prime geodesic theorem*. In: *J. Reine Angew. Math.* **676** (2013), pp. 105–120.

[U] K. Uchida. *On Artin L-functions*. In: *Tôhoku Math. J. (2)* **27** (1975), pp. 75–81.

[V] A. Venkatesh. *"Beyond endoscopy" and special forms on GL(2)*. In: *J. Reine Angew. Math.* **577** (2004), pp. 23–80.

[vdW] R. W. van der Waall. *On a conjecture of Dedekind on zeta-functions*. In: *Indagationes Mathematicae (Proceedings)* **78.1** (1975), pp. 83–86.

[W1] J.-L. Waldspurger. *Sur les germes de Shalika pour les groupes linéaires*. In: *Math. Ann.* **284.2** (1989), pp. 199–221.

[W2] J.-L. Waldspurger. *Sur les intégrales orbitales tordues pour les groupes linéaires: Un lemme fondamental*. In: *Canad. J. Math.* **43.4** (1991), pp. 852–896.

[W3] J.-L. Waldspurger. *Le lemme fondamental implique le transfert*. In: *Compositio Math.* **105.2** (1997), pp. 153–236.

[Y] Z. Yun. *Orbital integrals and Dedekind zeta functions*. In: *The Legacy of Srinivasa Ramanujan*. Vol. 20. Ramanujan Math. Soc. Lect. Notes Ser. Ramanujan Math. Soc., Mysore, 2013, pp. 399–420.

[Z] D. Zagier. *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*. In: *Modular Functions of One Variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*. Vol. 627. Lecture Notes in Math. Springer, Berlin, 1977, pp. 105–169.