AN ASYMPTOTIC FORMULA FOR REAL GROUPS

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Contents

* Supported in part by NSERC Operating Grant A3483.

Introduction

This paper is the first of a pair of articles on real harmonic analysis. The objects we study are tempered distributions on real groups that occur on the geometric side of the trace formula. In this paper, we shall study the distributions attached to a fixed reductive group $G(\mathbb{R})$. We shall establish explicit formulas for their values at data that approach infinity. In the next paper, we shall use the asymptotic formulas to establish identities among the distributions attached to different groups.

Suppose for a moment that G is abelian. The trace formula is then just the Poisson summation formula, which applies to any discrete cocompact subgroup of $G(\mathbb{R})$ and any Schwartz function f on $G(\mathbb{R})$. The geometric side is a sum of distributions $f(\gamma)$, where γ varies over elements in the discrete group. The spectral side is a sum of Fourier transforms

$$\tau(f) = \int_{G(\mathbb{R})} f(x)\tau(x)dx = \widehat{f}(\tau^{-1}),$$

where τ varies over characters on $G(\mathbb{R})$ that are trivial on the discrete group. Our interest is in nonabelian analogues of the distributions $f(\gamma)$.

Our asymptotic formula is entirely trivial in the abelian case. It might nonetheless still be suggestive. Suppose that T is a variable point in $\mathfrak{a}_G$, a real vector space that in general stands for the Lie algebra of the noncompact part of the center of $G(\mathbb{R})$. Then $\gamma_T = \gamma \exp T$ is a variable point in the abelian group $G(\mathbb{R})$, and the limit of $f(\gamma_T)$ is 0 as T approaches infinity. However, we can also make the test function vary with T. Let f_T be the Schwartz function on $G(\mathbb{R})$ such that

$$\tau(f_T) = \tau(\exp T)\tau(f),$$

for any character τ on $G(\mathbb{R})$. Then $f_T(\gamma)$ equals $f\big(\gamma(\exp T)^{-1}\big)$, and $f_T(\gamma_T)$ reduces simply to $f(\gamma)$. We can therefore write

$$\lim_{T\to\infty} f_T(\gamma_T) = \int_{T_{\mathrm{temp}}(G)} \theta(\gamma,\tau)\tau(f)d\tau, \tag{1}$$

where $T_{\mathrm{temp}}(G)$ denotes the group of characters on $G(\mathbb{R})$, and

$$\theta(\gamma,\tau) = \tau^{-1}(\gamma) = \tau(\gamma^{-1}).$$

This is of course just the Fourier inversion formula for $G(\mathbb{R})$, since the function on the left hand side is independent of T. Written in this slightly extravagant way, it serves as a model for the general asymptotic formula we shall establish.

Suppose now that G is a general connected reductive algebraic group over $\mathbb{R}$. The most direct analogues of the distributions $f(\gamma)$ above are in some sense the invariant orbital integrals

$$f_G(\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x)dx$$

of Harish-Chandra. They are parametrized by strongly regular elements $\gamma \in G_{\mathrm{reg}}(\mathbb{R})$, and are defined for any function $f \in \mathcal{C}(G)$ in the Schwartz space on $G(\mathbb{R})$. The subscript G is meant to emphasize that invariant orbital integrals are part of a more general family of tempered distributions, indexed by Levi subgroups M of G. These are weighted orbital integrals

$$J_M(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x)v_M(x)dx,$$

in which γ is a strongly G-regular element in $M(\mathbb{R})$, and $v_M(x)dx$ represents a noninvariant measure on the conjugacy class of γ in $G(\mathbb{R})$. The weight factor $v_M(x)$ is the volume in $\mathfrak{a}_M/\mathfrak{a}_G$ of a certain convex hull, which depends on x in general, but reduces to 1 in the case $M = G$.

The distributions $J_M(\gamma, f)$ are the generic archimedean terms on the geometric side of the trace formula. In the case of a discrete subgroup with compact quotient, only the invariant distributions

$$f_G(\gamma) = I_G(\gamma, f)$$

occur. The more general distributions arise in the case of noncompact quotient. They represent terms that fail to be invariant under conjugation of f by $G(\mathbb{R})$. The invariant trace formula is a refined version of the basic trace formula, in which the weighted orbital integrals are replaced by invariant distributions $I_M(\gamma, f)$. We thus have two families of tempered distributions

$$J_M(\gamma, f),\ I_M(\gamma, f), \qquad\qquad \gamma \in M(\mathbb{R}) \cap G_{\mathrm{reg}},\ f \in \mathcal{C}(G),$$

on $G(\mathbb{R})$. They each contain subtle information, the analysis of which remains an important consideration for obtaining a deeper understanding of the trace formula [L3].

Suppose that M, γ and f are fixed, and that T varies over the real vector $\mathfrak{a}_M$. We can then form the variable point

$$\gamma_T = \gamma \exp T$$

in $M(\mathbb{R})$, as in the abelian case. In §1, we shall introduce a variable function $f_T \in \mathcal{C}(G)$. To define f_T, it suffices to characterize the operator valued Fourier transform $f_T \longrightarrow \pi(f_T)$ as π ranges over the irreducible tempered representations of $G(\mathbb{R})$. We do so by setting

$$\pi(f_T) = \Big(\sum_u e^{\nu_{\pi_1}(uT)}\Big)\pi(f),$$

where ν_{π_1} is the "imaginary part" of the infinitesimal character of π, and u ranges over a finite set of embeddings of $\mathfrak{a}_M$ into the space on which the linear form ν_{π_1} is defined. The distributions $J_M(\gamma_T, f_T)$ and $I_M(\gamma_T, f_T)$ can then be studied as functions of T. The problem is to calculate their limits, as T approaches infinity in a cone $\mathfrak{a}_P^r \subset \mathfrak{a}_M$ attached to a parabolic subgroup $P \in \mathcal{P}(M)$ and a small positive number r.

Our main results are in §6. They give a solution to the problem in the case that f belongs to the Hecke algebra $\mathcal{H}(G)$ on $G(\mathbb{R})$, and the element γ is elliptic in M. Theorem 6.1 treats the weighted orbital integrals, while Corollary 6.2 applies to their invariant counterparts. We shall describe the latter.

Corollary 6.2 asserts the existence of an asymptotic formula

$$\lim_{\substack{T \longrightarrow \infty \\ \mathfrak{a}_P^r}} I_M(\gamma_T, f_T) = \int_{T_\varepsilon(M)} \theta_M(\gamma, \tau) m_M(\tau, P) f_M(\tau) d\tau, \tag{2}$$

whose constituents are as follows. The subscript $\varepsilon = \varepsilon_P$ represents a small element in $\mathfrak{a}_M^*$ that lies in the chamber of P. The associated domain $T_\varepsilon(M)$ is obtained from a natural basis $T_{\text{temp}}(M)$ of virtual tempered characters on $M(\mathbb{R})$, essentially the "singular invariant distributions" of Harish-Chandra, by twisting with the real central character defined by ε. It comes with a natural measure $d\tau$. Since f belongs to the Hecke algebra, the class function f_M on $M(\mathbb{R})$ defined by descent is holomorphic in the spectral variables. It is therefore defined at any element τ in $T_\varepsilon(M)$. The function $\theta_M(\gamma, \tau)$ is the kernel (with M in place of G) in the expansion

$$f_G(\gamma) = \int_{T_{\text{temp}}(G)} \theta_G(\gamma, \tau) f_G(\tau) d\tau \tag{3}$$

of an invariant orbital integral in terms of virtual characters. Finally, $m_M(\tau, P)$ is a slowly increasing function of $\tau \in T_\varepsilon(M)$, built out of logarithmic derivatives of Plancherel densities. This function is the most distinctive term on the right hand side of (2). It can have singularities at $\varepsilon = 0$, reflecting the zeros of Plancherel densities. If we take $M = G$, $m_M(\tau, P)$ equals 1, and since

$$I_G(\gamma_T, f_T) = I_G(\gamma, f) = f_G(\gamma), \qquad T \in \mathfrak{a}_G,$$

the formula (2) reduces in this case simply to (3). It can thus be regarded as a generalization of the formula (1) for abelian G.

Corollary 6.2 is a straightforward consequence of Theorem 6.1 and a general estimate, which we establish as Corollary 5.2 in §5. The proof of Theorem 6.1 occupies most of the first six sections. In §1 we first review the relevant distributions. We then define the mapping $f \to f_T$ as a special case of a family of multipliers on the Schwartz space. In general, our methods will be based on Harish-Chandra's asymptotic theory of spherical functions, particularly his theory of the constant term. We review some of the salient points of this theory in §2. We also reformulate some of the Harish-Chandra's estimates (Lemma 2.2 and its corollaries) for use in §4.

Sections 3 and 4 contain the heart of the argument. As steps in the proof of Theorem 6.1, they pertain to the weighted orbital integrals $J_M(\gamma_T, f_T)$. To exploit Harish-Chandra's theory of the constant term, we replace f by a suitable τ-spherical Schwartz function $f_1 \in \mathcal{C}(G,\tau)$. The corresponding analogue of $J_M(\gamma_T, f_T)$ is a finite sum of functions $J_M(\gamma_T, f_1^S)$, where $f_1^S \in \mathcal{C}(G,\tau)$ is obtained in an obvious way from f_1 and a linear image $S \in \mathfrak{a}_1$ of T. In §3, we show that as T approaches infinity, the Eisenstein integrals that determine f_1 may be replaced by their constant terms. The relevant estimates are summarized in Lemma 3.4. In §4, we express the contribution of these constant terms in the form of a relatively simple integral. This is summarized in Lemma 4.3. Techniques in both sections include inequalities that relate the polar decomposition, the Iwasawa decomposition, and the conjugacy class decomposition, all relative to $G(\mathbb{R})$. We refer the reader to the text for further discussion.

Sections 5 and 6 are designed to allow us to interpret the integral of Lemma 4.3. In §5 we describe the domain $T_\varepsilon(M)$, and attach weighted characters to elements in this set. We then derive Corollary 5.2 from Lemma 4.4, an estimate obtained from some of the arguments used to prove Lemma 4.3. In §6 we establish Theorem 6.1 as a harder consequence of Lemma 4.3. The problem at this point is to relate the weighted characters (5.10) attached to spherical functions with weighted characters (5.8) for a Schwartz function f. Though somewhat complicated, the computations of §6 are straightforward. Like the techniques of §3 and §4, they generalize methods that were first applied to the group $SL(2)$ [AHS]. At the end of §6, we derive Corollary 6.2 from Theorem 6.1, Corollary 5.2, and the inductive definition (1.4) of $I_M(\gamma, f)$ in terms of the weighted orbital integral $J_M(\gamma, f)$ and the weighted characters (5.8).

In the interests of simplicity, we have limited the context of our results to that required for applications in [A13]. It would not have been difficult to work in greater generality. We conclude the paper at the end of §6 with some very brief remarks on how one might extend the results. We discuss in turn the possibilities of allowing γ to be any element in

$M(\mathbb{R}) \cap G_{\mathrm{reg}}$, of taking f to be a general Schwartz function on $G(\mathbb{R})$, and of replacing $\mathbb{R}$ by an arbitrary local field F of characteristic 0.

It has been a longstanding problem to compute the Fourier transforms of weighted orbital integrals [L1]. The case of groups of real rank 1 was solved by Hoffmann [Ho], following earlier papers [AHS], [W] on the topic. The rank 1 analogue of Theorem 6.1 was an essential part of the process. The asymptotic formula is considerably easier in this case, for the reasons that a proper Levi subgroup M is compact and that a corresponding chamber in $\mathfrak{a}_M$ is just a half line. In general, the problem of computing Fourier transforms is equivalent to that of writing $I_M(\gamma, f)$ explicitly as a distribution on $T_{\mathrm{temp}}(G)$. The formula (2) of Corollary 6.2 can be regarded as a step in this direction. For example, it reduces to the Fourier inversion formula (1) in the case of abelian G. Given its general form, and its possible extension to the Schwartz space mentioned at the end of §6, the formula (2) amounts to an asymptotic formula for the invariant Fourier transform of $I_M(\gamma, f)$.

We have been motivated by a different application. In the sequel [A13] to this paper, we shall solve a comparison problem for the invariant distributions attached to different groups. More precisely, we shall establish identities that relate the invariant distributions $I_M(\gamma, f)$ for a given G with corresponding stable distributions for endoscopic groups G' of G. The proof of such identities is part of the stabilization of the global trace formula, and in fact can be regarded as a local archimedean analogue of the global question. There is considerable common ground between the problem of computing Fourier transforms and that of comparison. The latter is undoubtedly simpler. It entails the proof of a given identity rather than the construction of what is likely to be a complicated function. However, the methods needed to attack either problem seem to be closely related. Be that as it may, the asymptotic formula (2) will be a key part of the comparison in [A13].

§1. Distributions and multipliers

Let G be a connected, reductive algebraic group over the real field $\mathbb{R}$. Our concern is the harmonic analysis of functions and distributions on the real Lie group $G(\mathbb{R})$. We begin with a brief review of the distributions of interest.

Suppose that T is a maximal torus in G that is defined over $\mathbb{R}$. We write $T_{\mathrm{reg}} = T_{G\text{-reg}}$ for the open subset of elements in T that are strongly G-regular, in the sense that their centralizer in G equals T. Harish-Chandra's invariant orbital integral is defined for any element γ in $T_{\mathrm{reg}}(\mathbb{R})$ and any function f in the Schwartz space $\mathcal{C}(G) = \mathcal{C}\big(G(\mathbb{R})\big)$ on $G(\mathbb{R})$. It is given by an absolutely convergent integral

$$f_G(\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{T(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x)dx, \tag{1.1}$$

where dx is a $G(\mathbb{R})$-invariant measure on $T(\mathbb{R})\backslash G(\mathbb{R})$, and

$$D(\gamma) = D^G(\gamma) = \det\big(1 - \mathrm{Ad}(\gamma)\big)_{\mathfrak{g}/\mathfrak{t}}$$

is the Weyl discriminant. (Following a general practice of denoting the Lie algebra of a given group by the appropriate lower case gothic letter, we have written $\mathfrak{g}$ and $\mathfrak{t}$ for the Lie algebras of G and T respectively.) Invariant orbital integrals play a central role in Harish-Chandra's proof of the Plancherel formula for $G(\mathbb{R})$.

Invariant orbital integrals are part of a broader family of distributions, known as weighted orbital integrals. These objects depend on a Levi subgroup M of G, by which we mean a Levi subgroup over $\mathbb{R}$ of some parabolic subgroup of G over $\mathbb{R}$, and a maximal compact subgroup K of G. We assume that the maximal torus $T = T_M$ is contained in M, and therefore that T_M contains the split component A_M of the center of M. We assume also that the Lie algebras of K and $A_M(\mathbb{R})$ are orthogonal with respect to the Killing form. Since we are working over the field $\mathbb{R}$, we can identify the Lie algebra of $A_M(\mathbb{R})$ with the real vector space

$$\mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}\big(X(M)_{\mathbb{R}}, \mathbb{R}\big).$$

We recall that there is a canonical homomorphism

$$H_M: \ M(\mathbb{R}) \longrightarrow \mathfrak{a}_M.$$

The weighted orbital integral attached to M is defined for any $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$ and $f \in \mathcal{C}(G)$ by a noninvariant integral

$$J_M(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{T(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x) v_M(x) dx. \tag{1.2}$$

The weight factor

$$v_M(x) = \lim_{\zeta \to 0} \sum_{P \in \mathcal{P}(M)} v_P(\zeta, x)\theta_P(\zeta)^{-1}$$

is obtained from the (G, M)-family of functions

$$v_P(\zeta, x) = e^{-\zeta(H_P(x))}, \qquad P \in \mathcal{P}(M),$$

of $\zeta \in i\mathfrak{a}_M^*$, according to the prescription of [A3, Lemma 6.2], and equals the volume of the convex hull in $\mathfrak{a}_M/\mathfrak{a}_G$ of the set

$$\big\{ - H_P(x) : \ P \in \mathcal{P}(M) \big\}.$$

We are following standard notation and terminology, as for example in [A10, §1]. Thus $\mathcal{P}(M) = \mathcal{P}^G(M)$ denotes the finite set of parabolic subgroups $P = MN_P$ of G with Levi component M. The function

$$H_P: \ G(\mathbb{R}) \longrightarrow \mathfrak{a}_M$$

is the mapping of Harish-Chandra, defined by

$$H_P(nmk) = H_M(m), \qquad n \in N_P(\mathbb{R}), \ m \in M(\mathbb{R}), \ k \in K.$$

The denominator $\theta_P(\zeta)$ is a homogeneous function of ζ, of degree equal to the dimension of $\mathfrak{a}_M/\mathfrak{a}_G$, which depends on a choice of metric $\|\cdot\|$ on $\mathfrak{a}_M$. The integral (1.2) converges absolutely, and defines a smooth function of γ on $T_{\mathrm{reg}}(\mathbb{R})$ [A2, §8]. It depends only on the conjugacy class of γ in $M(\mathbb{R})$, and in fact only on the orbit of the conjugacy class under the Weyl group

$$W(M) = W^G(M) = \mathrm{Norm}_G(M)/M = \mathrm{Norm}_{K\cap M}(M)/K \cap M$$

of M.

Weighted orbital integrals have two drawbacks. They are not invariant under conjugation of f by $G(\mathbb{R})$, and they depend on the choice of K. However, there is a natural construction that gives a parallel family of distributions with better properties. It is based on the dual family of distributions defined by weighted characters.

We write $\Pi(G) = \Pi\big(G(\mathbb{R})\big)$ for the set of equivalence classes of irreducible representations of $G(\mathbb{R})$, and $\Pi_{\mathrm{temp}}(G) = \Pi_{\mathrm{temp}}\big(G(\mathbb{R})\big)$ for the subset of irreducible tempered representations. The distributional character

$$f_G(\pi) = \mathrm{tr}\big(\pi(f)\big) = \int_{G(\mathbb{R})} f(x)\Theta(\pi, x)dx$$

attached to any $\pi \in \Pi_{\mathrm{temp}}(G)$ and $f \in \mathcal{C}(G)$ may be regarded as a spectral analogue of the invariant orbital integral (1.1). The resulting space of functions

$$\mathcal{I}(G) = I\mathcal{C}(G) = \big\{f_G : \ f \in \mathcal{C}(G)\big\}$$

on $\Pi_{\mathrm{temp}}(G)$ forms a natural Schwartz space [A11]. The weighted character is defined for any $\pi \in \Pi_{\mathrm{temp}}(M)$ and $f \in \mathcal{C}(G)$ by a "noninvariant trace"

$$J_M(\pi, f) = \mathrm{tr}\big(\mathcal{M}_M(\pi, P)\mathcal{I}_P(\pi, f)\big). \tag{1.3}$$

As usual, $\mathcal{I}_P(\pi)$ denotes the representation of $G(\mathbb{R})$ induced from the pullback of π to $P(\mathbb{R})$, acting on a Hilbert space $\mathcal{H}_P(\pi)$ of operator valued functions on K. The weight factor

$$\mathcal{M}_M(\pi, P) = \lim_{\zeta\to 0} \sum_{Q\in\mathcal{P}(M)} \mathcal{M}_Q(\zeta, \pi, P)\theta_Q(\zeta)^{-1}$$

is obtained from a (G, M)-family of operator valued functions

$$\mathcal{M}_Q(\zeta, \pi, P), \qquad Q \in \mathcal{P}(M),\ \zeta \in i\mathfrak{a}_M^*,$$

on $\mathcal{H}_P(\pi)$. It was defined in terms of unnormalized intertwining operators and Plancherel densities in [A12]. (We shall recall the construction in §5 in order to modify it slightly.) The correspondence that sends any $f \in \mathcal{C}(G)$ to the function

$$\phi_M(f):\ \pi \longrightarrow \phi_M(f, \pi) = J_M(\pi, f), \qquad \pi \in \Pi_{\mathrm{temp}}(M),$$

is then a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{I}(M)$.

Consider the family of mappings

$$\phi_L:\ \mathcal{C}(G) \longrightarrow \mathcal{I}(L), \qquad L \in \mathcal{L}(M),$$

parametrized by the finite set $\mathcal{L}(M) = \mathcal{L}^G(M)$ of Levi subgroups of G that contain M. Like the distributions (1.2), these mappings are noninvariant, and depend on the choice of K. We use them to construct invariant tempered distributions

$$I_M(\gamma, f) = I_M^G(\gamma, f), \qquad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),$$

inductively by setting

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big), \tag{1.4}$$

where $\widehat{I}_M^L(\gamma)$ denotes the continuous linear form on $\mathcal{I}(L)$ such that

$$\widehat{I}_M^L(\gamma, h_L) = I_M^L(\gamma, h), \qquad h \in \mathcal{C}(L).$$

Since

$$I_G(\gamma, f) = J_G(\gamma, f) = f_G(\gamma),$$

the families (1.2) and (1.4) both include the original invariant orbital integrals (1.1). However, the distributions $I_M(\gamma)$ in the second family have the advantage of being invariant.

They also turn out to be independent of the choice of K. (See [A12, §3].) For these reasons, they represent the more natural generalizations of invariant orbital integrals.

If $P \in \mathcal{P}(M)$, we write Δ_P as usual for the set of simple roots of (P, A_M). Elements $\alpha \in \Delta_P$ can be regarded either as quasicharacters $a \to a^\alpha$ on $A_M(\mathbb{R})$, or as linear forms $H \to \alpha(H)$ on $\mathfrak{a}_M$. Then

$$\mathfrak{a}_P^+ = \big\{ H \in \mathfrak{a}_M : \ \alpha(H) > 0, \ \alpha \in \Delta_P \big\}$$

is the open chamber in $\mathfrak{a}_P$ attached to P. (Given P, we often write $\mathfrak{a}_P = \mathfrak{a}_M$ and $A_P = A_M$.) The closure $\overline{\mathfrak{a}}_P^+$ of $\mathfrak{a}_P^+$ is a cone whose boundary components $\overline{\mathfrak{a}}_Q^+$ are parametrized by parabolic subgroups Q of G that contain P. We recall that any such Q has a Levi decomposition $Q = M_Q N_Q$, for a unique Levi component M_Q that contains M, and that the real vector space $\mathfrak{a}_Q = \mathfrak{a}_{M_Q}$ has a canonical embedding into $\mathfrak{a}_M$. We write Δ_P^Q as usual for the subset of roots in Δ_P that vanish on the subspace $\mathfrak{a}_Q$ of $\mathfrak{a}_M$. We shall also write

$$H = H_Q + H^Q, \qquad H_Q \in \mathfrak{a}_Q, \ H^Q \in \mathfrak{a}_M^Q, \tag{1.5}$$

for the decomposition of a point $H \in \mathfrak{a}_M$, in which $\mathfrak{a}_M^Q$ denotes the kernel of the canonical projection of $\mathfrak{a}_M$ onto $\mathfrak{a}_Q$.

The space $\mathfrak{a}_M$ attached to a given M comes with an implicitly chosen Euclidean metric $\| \cdot \|$. We assume that this metric is given by the restriction of a fixed $W(M_0)$-invariant Euclidean inner product $(\cdot, \cdot)$ on a space $\mathfrak{a}_{M_0}$ (which we also denote by $\| \cdot \|$), for some minimal Levi subgroup M_0 contained in M. The decomposition (1.5) is then orthogonal with respect to the underlying inner product. Actually, for much of the paper it will be convenient to fix M_0, as well as a minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$. We will then employ the usual abbreviated notation $N_0 = N_{P_0}$, $A_0 = A_{M_0}$, $W_0 = W(M_0)$, $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$, $H_0 = H_{P_0}$, $\Delta_0 = \Delta_{P_0}$, $\mathfrak{a}_0^+ = \mathfrak{a}_{P_0}^+$, $\Delta_0^Q = \Delta_{P_0}^Q$, etc., for objects attached to M_0 and P_0.

The Levi subgroup M will be fixed from now on. Our goal is to establish asymptotic formulas for the distributions $J_M(\gamma, f)$ and $I_M(\gamma, f)$. The formulas are to depend on a point $T \in \mathfrak{a}_P^+$ that approaches infinity. (We follow several precedents in denoting the point by T. It is for this reason that we sometimes fall back on the notation T_M for the maximal torus in the Levi subgroup M.) For any element γ as in (1.2), we set

$$\gamma_T = \gamma \exp T, \qquad T \in \mathfrak{a}_P^+. \tag{1.6}$$

If T is chosen to be far from the walls of $\mathfrak{a}_P^+$, γ_T will be strongly G-regular, and $J_M(\gamma_T, f)$ will be defined. However, for fixed $f \in \mathcal{C}(G)$, $J_M(\gamma_T, f)$ approaches 0 as T approaches infinity. We need to replace f with a function that also varies with T.

We shall transform f by a variable object, which is to be a special case of what we can call a Schwartz multiplier. The construction is quite natural. To put it into perspective, we pause briefly for a few general remarks about such objects.

We define a *Schwartz multiplier* for G to be a continuous endomorphism $f \to f_\alpha$ of $\mathcal{C}(G)$ that commutes with left and right translation. We write $\mathcal{M}(G)$ for the algebra of all Schwartz multipliers for G. Any $\alpha \in \mathcal{M}(G)$ is determined by its dual function

$$\widehat{\alpha} : \ \Pi_{\mathrm{temp}}(G) \longrightarrow \mathbb{C},$$

defined by the property

$$\pi(f_\alpha) = \widehat{\alpha}(\pi)\pi(f), \qquad \pi \in \Pi_{\mathrm{temp}}(G),\ f \in \mathcal{C}(G). \tag{1.7}$$

We write $\Pi_{\mathrm{temp,cusp}}(G)$ and $\Pi_{\mathrm{cusp}}(G)$ for the subsets of cuspidal representations in $\Pi_{\mathrm{temp}}(G)$ and $\Pi(G)$ respectively. The dual function $\widehat{\alpha}$ is characterized in turn by the family of functions

$$\widehat{\alpha}_L : \ \Pi_{\mathrm{temp,cusp}}(L) \longrightarrow \mathbb{C},$$

parametrized by (cuspidal) Levi subgroups L of G, such that

$$\widehat{\alpha}_L(\pi) = \widehat{\alpha}\big(\mathcal{I}_Q(\pi)\big), \qquad Q \in \mathcal{P}(L), \tag{1.8}$$

where π now denotes a representation in $\Pi_{\mathrm{temp,cusp}}(L)$. Since α is continuous, $\widehat{\alpha}_L(\pi)$ is determined by the values it takes at the open dense set of representations π for which the induced representation $\mathcal{I}_Q(\pi)$ is irreducible.

The family of functions $\{\widehat{\alpha}_L\}$ attached to any $\alpha \in \mathcal{M}(G)$ has two basic properties.

(i) For each L, $\widehat{\alpha}_L$ is a smooth function on $\Pi_{\mathrm{temp,cusp}}(L)$, of which any invariant derivative is tempered.

(ii) The family is symmetric, in the sense that

$$g\widehat{\alpha}_L = \widehat{\alpha}_{gL},$$

for any L and any $g \in G(\mathbb{R})$.

In the growth condition (i), an invariant derivative means the transfer of an invariant differential operator on $i\mathfrak{a}_L^*$, relative to the action

$$\pi \longrightarrow \pi_\lambda(x) = \pi(x)e^{\lambda(H_L(x))}, \qquad \lambda \in i\mathfrak{a}_L^*,\ x \in L(\mathbb{R}).$$

A tempered function on $\Pi_{\mathrm{temp,cusp}}(L)$ is understood to be one whose value at π is bounded by a polynomial in the norm of the infinitesimal character of π, or rather the norm of a linear form on a Cartan subalgebra that represents the infinitesimal character. In the symmetry condition (ii), it is understood that $gL = \mathrm{Int}(g)L$, and that

$$(g\widehat{\alpha}_L)(\pi_g) = \widehat{\alpha}_L(\pi_g \circ \mathrm{Int}\, g), \qquad \pi_g \in \Pi_{\mathrm{temp,cusp}}(gL).$$

Conversely, for any family $\{\widehat{\alpha}_L\}$ of functions that satisfy the conditions (i) and (ii), there is a unique multiplier $\alpha \in \mathcal{M}(G)$ such that (1.7) and (1.8) hold. This fact is a simple consequence of the main theorem of [A1], which describes the image of $\mathcal{C}(G)$ under noninvariant Fourier transform. We thus obtain a simple characterization of the algebra $\mathcal{M}(G)$.

The noninvariant Fourier transform of a function $f \in \mathcal{C}(G)$ is defined as the operator valued function

$$\widehat{f}_Q(\pi) = \mathcal{I}_Q(\pi, f^\vee), \qquad Q \in \mathcal{P}(L),\ \pi \in \Pi_{\mathrm{temp,cusp}}(L), \tag{1.9}$$

on $\mathcal{H}_Q(\pi)$, where

$$f^\vee(x) = f(x^{-1}),$$

and L ranges (cuspidal) Levi subgroups of G. The inverse Fourier transform can be defined for any operator valued function a_Q that is rapidly decreasing in the natural sense. It equals

$$a_Q^\vee(x) = |W(L)|^{-1} \int_{\Pi_{\mathrm{temp,cusp}}(L)} \mathrm{tr}\big(\mathcal{I}_Q(\pi, x) a_Q(\pi)\big)\varepsilon_Q(\pi) d\pi, \tag{1.10}$$

where $\varepsilon_Q(\pi)$ is the Plancherel density, and $d\pi$ is the measure on $\Pi_{\mathrm{temp,cusp}}(L)$ induced from the measure on $i\mathfrak{a}_L^*$ determined by a fixed Euclidean metric on $\mathfrak{a}_L$ and the free action $\pi \to \pi_\lambda$ of $i\mathfrak{a}_L^*$. The noninvariant Fourier inversion formula is the identity

$$f(x) = \sum_{\{L\}} (\widehat{f}_Q)^\vee(x), \tag{1.11}$$

where L ranges over conjugacy classes of cuspidal Levi subgroups, and Q represents a group in $\mathcal{P}(L)$. (See [A1], for example.) The main theorem of [A1] characterizes the image of $\mathcal{C}(G)$ under the noninvariant Fourier transform (1.9) as the appropriate Schwartz space of operator valued functions on $\Pi_{\mathrm{temp}}(G)$. If $\alpha \in \mathcal{M}(G)$ is a multiplier,

$$(f_\alpha)_Q^\wedge(\pi) = \widehat{\alpha}_L(\pi^\vee)\widehat{f}_Q(\pi), \qquad \pi \in \Pi_{\mathrm{temp,cusp}}(L),$$

where $\pi^\vee$ is the contragredient of π. The inversion formula (1.11) therefore gives rise to a natural expression for $f_\alpha(x)$.

Now suppose that M_1 is a fixed cuspidal Levi subgroup, and that S is a point in the space $\mathfrak{a}_{M_1}$. A representation π_1 in $\Pi_{\mathrm{temp,cusp}}(M_1)$ has an infinitesimal character, represented by a linear form on any Cartan subalgebra of $\mathfrak{m}_1(\mathbb{C})$. We write $\nu_{\pi_1} \in i\mathfrak{a}_{M_1}^*$ for the restriction of this linear form to $\mathfrak{a}_{M_1}$, and call it the *imaginary part* of the infinitesimal character of π_1. If f is any function in $\mathcal{C}(G)$, we set

$$f^S(x) = (\widehat{f}_{P_1}^S)^\vee(x), \tag{1.12}$$

for the operator valued function

$$\widehat{f}^{S}_{P_1}(\pi_1) = e^{-\nu_{\pi_1}(S)}\widehat{f}_{P_1}(\pi_1), \qquad \pi_1 \in \Pi_{\mathrm{temp,cusp}}(M_1),$$

on $\mathcal{H}_{P_1}(\pi_1)$. Then f^S is a function in $\mathcal{C}(G)$ whose noninvariant Fourier transform is supported on the conjugacy class of M_1. Its transform at M_1 and P_1 is given by

$$\begin{aligned}(f^S)^{\wedge}_{P_1}(\pi_1) &= |W(M_1)|^{-1} \sum_{w\in W(M_1)} e^{-(w\nu_{\pi_1})(S)}\widehat{f}_{P_1}(\pi_1)\\ &= \widehat{\alpha}^S(\pi_1^{\vee})\widehat{f}_{P_1}(\pi_1),\end{aligned}$$

where

$$\widehat{\alpha}^S(\pi_1) = |W(M_1)|^{-1} \sum_{w\in W(M_1)} e^{(w\nu_{\pi_1})(S)}.$$

In other words,

$$f^S = f_{\alpha^S},$$

where $\alpha^S \in \mathcal{M}(G)$ is the multiplier such that

$$\widehat{\alpha}^S_L(\pi) = \begin{cases} \widehat{\alpha}^S(\pi_1), & \text{if } (L,\pi) = (M_1,\pi_1),\\ \\ 0, & \text{if } L \text{ is not conjugate to } M_1.\end{cases}$$

We shall use the multipliers α^S of f to construct an element in $\mathcal{M}(G)$ that varies with our point $T \in \mathfrak{a}_P^+$. This necessitates a brief investigation of a family of linear injections attached to M and the Levi subgroup M_1. We define $U(M,M_1)$ to be the set of embeddings

$$u: \ \mathfrak{a}_M \ \hookrightarrow \ \mathfrak{a}_{M_1}$$

induced by elements $g \in G(\mathbb{R})$ such that gMg^{-1} contains M_1. The set could of course be empty. In general, the Weyl group $W(M_1)$ acts by left composition on $U(M,M_1)$, while $W(M)$ acts freely by right composition. For future reference, we shall describe $U(M,M_1)$ in terms of the Weyl sets introduced by Langlands [L2, §2].

Assume for the moment that M and M_1 are both standard with respect to M_0 and P_0. In other words, M_0 is a minimal Levi subgroup contained in both M and M_1, and

$P_0 \in \mathcal{P}(M_0)$ is contained in parabolic subgroups $P \in \mathcal{P}(M)$ and $P_1 \in \mathcal{P}(M_1)$. Let $W(P_1; P)$ be the finite set of isomorphisms

$$w: \ \mathfrak{a}_{P_1} \longrightarrow \mathfrak{a}_{P_1'}, \qquad P_0 \subset P_1' \subset P,$$

obtained by restriction to $\mathfrak{a}_{P_1} = \mathfrak{a}_{M_1}$ of elements in the Weyl group $W_0 = W(M_0)$ such that $w^{-1}\alpha$ is a root of (P_1, A_1) for every root α in the subset $\Delta_{P_1'}^P$ of $\Delta_{P_1'}$. (See [A1, §II.5], for example.) If $P = P_1$, $W(P_1; P)$ is just the group $W(M_1)$. In general, however, the parabolic subgroup P_1' varies with w.

Lemma 1.1. *For any $w \in W(P_1; P)$, let w_M^{-1} denote the restriction of w^{-1} to the subspace $\mathfrak{a}_M$ of $\mathfrak{a}_{P_1'}$. Then the mapping*

$$w \longrightarrow u = w_M^{-1}, \qquad w \in W(P_1; P), \tag{1.13}$$

is a bijection from $W(P_1; P)$ onto $U(M, M_1)$.

Proof. Suppose that u belong to $U(M, M_1)$. Let $Q = \widetilde{u}P\widetilde{u}^{-1}$, where $\widetilde{u}$ is an element in G that induces the mapping u on $\mathfrak{a}_M$. Then Q is a parabolic subgroup of G, with Levi component $M_Q = \widetilde{u}M\widetilde{u}^{-1}$ that contains M_1. Let Q_1 be the unique parabolic subgroup in $\mathcal{P}(M_1)$ that is contained in Q, and such that $Q_1 \cap M_Q = P_1 \cap M_Q$. Of course, Q_1 need not be standard, but its chamber $\mathfrak{a}_{Q_1}^+$ is an open subset of $\mathfrak{a}_{M_1}$. A general result of Langlands [L2, Lemma 2.13] implies that $\mathfrak{a}_{Q_1}^+$ equals $w^{-1}\mathfrak{a}_{P_1'}^+$, for a unique standard parabolic subgroup P_1' of G, and a unique linear isomorphism w from $\mathfrak{a}_{P_1}$ to $\mathfrak{a}_{P_1'}$ defined by restriction of some element in W_0. (See [A1, Lemma I.3.1].) If $\widetilde{w}$ is a representative of w in G, $\widetilde{w}\widetilde{u}$ conjugates P to a group that contains the standard parabolic subgroup $P_1' = \widetilde{w}Q_1\widetilde{w}^{-1}$. Since P is also standard, we conclude that

$$P = (\widetilde{w}\widetilde{u})P(\widetilde{w}\widetilde{u})^{-1}$$

In particular, P contains P_1'. If α belongs to $\Delta_{P_1'}^P$, $w^{-1}\alpha$ is a root of $(Q_1 \cap M_Q, A_{P_1})$, and hence also of (P_1, A_{P_1}), since $Q_1 \cap M_Q$ equals $(P_1 \cap M_Q)$. Therefore w belongs to

$W(P_1;P)$. The last identity implies also that $\widetilde{w}\widetilde{u}$ lies in P. It follows that the mapping $wu = w_M u$ is trivial on $\mathfrak{a}_M$. In other words, u equals w_M^{-1}.

We have now established that the mapping $w \to u$ is surjective. In proving it, we established also that the element w attached to u is unique. The mapping is therefore injective, and hence a bijection. $\square$

If u belongs to $U(M, M_1)$, we shall often write $P_1^u = P_1'$, where P_1' is the standard parabolic subgroup such that the preimage w of u in $W(P;P_1)$ maps $\mathfrak{a}_{P_1}$ onto $\mathfrak{a}_{P_1'}$. In other words, the Levi component M_1^u of P_1^u equals the group $w(M_1) = \widetilde{w}M_1\widetilde{w}^{-1}$. This definition is dependent upon the condition that both M and M_1 be standard. Without the condition, M_1^u is well defined only as an M-orbit of Levi subgroups of M.

Lemma 1.2 *The mapping*

$$u \longrightarrow M_1^u, \qquad\qquad u \in U(M, M_1),$$

is a surjection from $U(M, M_1)$ onto the set of W_0^M-orbits of Levi subgroups of M in the W_0-orbit of M_1. The Weyl group $W(M_1^u)$ acts transitively on the fibre of M_1^u in $U(M, M_1)$, and the stabilizer of u in $W(M_1^u)$ equals $W^M(M_1^u)$.

Proof. Suppose that M_1' is a Levi subgroup of M in the W_0-orbit of M_1. Replacing M_1' by a W_0^M-conjugate if necessary, we can assume that M_1' is standard relative to the minimal parabolic subgroup $P_0 \cap M$ of M. It follows that $M_1' = M_{P_1'}'$, for a parabolic subgroup P_1' of G with $P_0 \subset P_1' \subset P$. By assumption, M_1' equals $w_0(M_1)$, for an element $w_0 \in W_0$. Replacing M_1' again by a W_0^M-conjugate if necessary, we can assume that $w^{-1}\alpha$ is a root of (P_1, A_1) for every $\alpha \in \Delta_{P_1'}^P$. This implies that the restriction w of w_0 to $\mathfrak{a}_{P_1}$ lies in $W(P_1;P)$. It follows that $M_1' = w(M_1) = M_1^u$, where $u = w_M^{-1}$. The mapping is therefore surjective. The assertions about the fibre of M_1^u follow easily from the definitions. $\square$

We now introduce a function that varies with T. We set

$$f_T = \sum_{\{M_1\}} \sum_{u \in U(M,M_1)} f^{uT}, \qquad f \in \mathcal{C}(G), \tag{1.14}$$

where M_1 again ranges over conjugacy classes of cuspidal Levi subgroups. The summand f^{uT} of course stands for the function (1.12), for the point $S = uT$ in $\mathfrak{a}_{M_1}$. In other words,

$$f_T = f_{\alpha_T},$$

for the multiplier

$$\alpha_T = \sum_{\{M_1\}} \sum_{u \in U(M,M_1)} \alpha^{uT}$$

in $\mathcal{M}(G)$. Then f_T is a Schwartz function, whose noninvariant Fourier transform at any M_1 and P_1 is given by

$$\widehat{f}_{T,P_1}(\pi_1) = \sum_{u \in U(M,M_1)} e^{-\nu_{\pi_1}(uT)} \widehat{f}_{P_1}(\pi_1), \qquad \pi_1 \in \Pi_{\mathrm{temp,cusp}}(M_1).$$

Let r be a positive number that is small relative to the underlying norm $\|\cdot\|$ on $\mathfrak{a}_M$. We define

$$\mathfrak{a}_P^r = \big\{ H \in \mathfrak{a}_M : \ \alpha(H) > r\|H\|, \ \alpha \in \Delta_P \big\},$$

and we write

$$T \underset{P,r}{\longrightarrow} \infty$$

if T becomes large within the open cone $\mathfrak{a}_P^r$. The general problem is to obtain explicit formulas for the limits

$$\lim_{T \underset{P,r}{\longrightarrow} \infty} J_M(\gamma_T, f_T) \tag{1.15}$$

and

$$\lim_{T \underset{P,r}{\longrightarrow} \infty} I_M(\gamma_T, f_T), \tag{1.16}$$

for any function f in $\mathcal{C}(G)$. If the maximal torus $T = T_M$ is not elliptic in M, standard descent formulas lead to a reduction of the problem from G to a proper Levi subgroup. We therefore may as well assume that T_M is elliptic in M, and in particular, M is a cuspidal Levi subgroup of G. With this condition, we shall solve the problem in the special case that f lies in the Hecke algebra $\mathcal{H}(G)$.

§2. Spherical functions

We shall begin our proof of the asymptotic formulas in §3. The general argument will be based on Harish-Chandra's spectral decomposition of generalized spherical functions. We use this section to recall a few basic properties of the decomposition. After doing so, we shall formulate two lemmas from Harish-Chandra's proof of these properties. Lemma 2.1 will be at heart of the reduction we carry out in §3. Lemma 2.2 and its corollaries will be an essential part of our estimates of constant terms in §4.

The families of spherical functions that occur in the spectral decomposition are again parametrized by conjugacy classes of cuspidal Levi subgroups M_1 of G. For much of our discussion M_1 will be fixed, with the assumption that the Lie algebras of K and $A_{M_1}(\mathbb{R})$ are orthogonal. Then

$$K_1 = K_{M_1} = K \cap M_1(\mathbb{R})$$

is a maximal compact subgroup of $M_1(\mathbb{R})$. We let $P_1 = M_1 N_1$ be a parabolic subgroup in $\mathcal{P}(M_1)$. Extending notation we have adopted for minimal Levi subgroups M_0, we shall frequently replace a subscript M_1 or P_1 simply by 1.

Suppose that V is a finite dimensional Hilbert space, equipped with a unitary, two-sided representation τ of K. In other words, V comes with commuting left and right K-actions

$$(k^1, k^2):\ v \longrightarrow \tau(k^1)v\tau(k^2), \qquad v \in V,\ k^1, k^2 \in K.$$

In future discussions, we shall often write

$$\pi^K:\ v \longrightarrow v^K = \int_K \tau(k^{-1})v\tau(k)dk, \qquad v \in V,$$

for the projection of v onto the subspace V^K of diagonally K-invariant vectors in V. Harish-Chandra's results pertain to functions from $G(\mathbb{R})$ to V that are τ-spherical, in the sense that

$$f(k^1 x k^2) = \tau(k^1)f(x)\tau(k^2), \qquad x \in G(\mathbb{R}),\ k^1, k^2 \in K.$$

Let

$$\tau_1 = \tau_{M_1} = \tau_{P_1}$$

denote the restriction of τ from K to the subgroup K_1. Following Harish-Chandra, we write $\mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$ for the finite dimensional space of cuspidal τ_1-spherical functions from the group

$$M_1(\mathbb{R})^1 = \big\{m \in M_1(\mathbb{R}) :\ H_1(m) = H_{M_1}(m) = 0\big\}$$

to V. Harish-Chandra's Eisenstein integral is defined for any $P_1 \in \mathcal{P}(M_1)$, $x \in G(\mathbb{R})$, and $\psi \in \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$, and for λ_1 in the complex vector space $\mathfrak{a}^*_{1,\mathbb{C}} = \mathfrak{a}^*_{M_1,\mathbb{C}}$, by the formula

$$E_{P_1}(x, \psi, \lambda_1) = \int_K \tau(k^{-1})\psi_{P_1}(kx)e^{(\lambda_1+\rho_1)(H_1(kx))}dk,$$

where

$$\psi_{P_1}(nmak) = \psi(m)\tau(k), \qquad n \in N_1(\mathbb{R}),\ m \in M_1(\mathbb{R})^1,\ a \in A_1(\mathbb{R})^0,\ k \in K.$$

As usual $\rho_1 = \rho_{P_1}$ denotes the linear form on $\mathfrak{a}_1$ such that

$$e^{2\rho_1(H_1(m))} = \big|\det\big(\mathrm{Ad}(m)\big)_{\mathfrak{n}_1}\big|, \qquad m \in M_1(\mathbb{R}),$$

where $\mathfrak{n}_1 = \mathfrak{n}_{P_1}$ is the Lie algebra of N_1. The Eisenstein integral is a τ-spherical function of x.

Suppose that f belongs to the space $\mathcal{C}(G, \tau)$ of τ-spherical Schwartz functions. For any $P_1 \in \mathcal{P}(M_1)$, the spherical transform $\widehat{f}_1 = \widehat{f}_{P_1}$ of f is a function from $i\mathfrak{a}^*_1 = i\mathfrak{a}^*_{M_1}$ to $\mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$. It is defined by

$$(2.1) \qquad \big(\widehat{f}_1(\lambda_1), \psi\big) = \int_{G(\mathbb{R})} \big(f(x), E_{P_1}(x, \psi, \lambda_1)\big)dx, \qquad \lambda_1 \in i\mathfrak{a}^*_1,$$

for any $\psi \in \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$. It follows easily from Harish-Chandra's definition of the Schwartz space on $G(\mathbb{R})$ that the transform $F_1 = \widehat{f}_1$ belongs to the space

$\mathcal{C}\big(i\mathfrak{a}_1^*, \mathcal{A}_{\mathrm{cusp}}(M_1,\tau_1)\big)$ of Schwartz functions from $i\mathfrak{a}_1^*$ to $\mathcal{A}_{\mathrm{cusp}}(M_1,\tau_1)$. Moreover, F_1 satisfies the symmetry condition

$$F_1(w\lambda_1) = {}^0c_1(w,\lambda_1)F_1(\lambda_1), \qquad w \in W(M_1), \tag{2.2}$$

where

$${}^0c_1(w,\lambda_1) = {}^0c_{P_1|P_1}(w,\lambda_1) = c_{P_1|P_1}(1,w\lambda_1)^{-1}c_{P_1|P_1}(w,\lambda_1)$$

is defined in terms of Harish-Chandra's c-functions

$$c_{P_1}(w,\lambda_1) = c_{P_1|P_1}(w,\lambda_1).$$

We recall that the general c-function

$$c_{P_1'|P_1}(w,\lambda_1):\ \mathcal{A}_{\mathrm{cusp}}(M_1,\tau_1) \longrightarrow \mathcal{A}_{\mathrm{cusp}}(M_1',\tau_1'), \qquad \lambda_1 \in \mathfrak{a}_{1,\mathbb{C}}^*,$$

is attached to a second parabolic subgroup $P_1' = M_1'N_1'$, and an isomorphism

$$w = \mathrm{Int}(\widetilde{w}):\ \mathfrak{a}_1 \longrightarrow \mathfrak{a}_1' = \mathfrak{a}_{M_1'}, \qquad \widetilde{w} \in K,$$

that maps M_1 to a Levi component M_1' of P_1'. It is a meromorphic function from $\mathfrak{a}_{1,\mathbb{C}}^*$ to the finite dimensional space of linear transformations from $\mathcal{A}_{\mathrm{cusp}}(M_1,\tau_1)$ to $\mathcal{A}_{\mathrm{cusp}}(M_1',\tau_1')$, introduced by Harish-Chandra to describe the constant term of $E_{P_1}(x,\psi,\lambda_1)$ in the direction of P_1'.

Conversely, suppose that $F_1 = F_{P_1}$ belongs to $\mathcal{C}\big(i\mathfrak{a}_1^*, \mathcal{A}_{\mathrm{cusp}}(M_1,\tau_1)\big)$. We recall that there is a decomposition

$$\mathcal{A}_{\mathrm{cusp}}(M_1,\tau_1) = \bigoplus_{\pi_1} \mathcal{A}_{\pi_1}(M_1,\tau_1), \qquad \pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1,$$

where $\Pi_{\mathrm{cusp}}(M_1)^1 = \Pi_{\mathrm{temp,cusp}}(M_1)^1$ denotes the set of square integrable representations of $M_1(\mathbb{R})^1$. Following Harish-Chandra, we identify the group $M_1(\mathbb{R})^1$ with the quotient $M_1(\mathbb{R})/A_1(\mathbb{R})^0$, thereby allowing ourselves to write

$$\Pi_{\mathrm{temp,cusp}}(M_1) = \big\{\pi_{1,\lambda_1}:\ \pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1,\ \lambda_1 \in i\mathfrak{a}_1^*\big\}$$

and

$$\Pi_{\mathrm{cusp}}(M_1) = \{\pi_{1,\lambda_1} : \ \pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1, \ \lambda_1 \in \mathfrak{a}^*_{1,\mathbb{C}}\}.$$

Recall that the μ-function

$$\mu_1(\pi_{1,\lambda_1}) = \mu_{P_1}(\pi_{1,\lambda_1}) = \mu_{\bar{P}_1|P_1}(\pi_{1,\lambda_1}), \qquad \lambda_1 \in \mathfrak{a}^*_{1,\mathbb{C}},$$

attached by Harish-Chandra to any $\pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1$ is a meromorphic function, which is analytic and slowly increasing on a cylindrical neighbourhood of $i\mathfrak{a}^*_1$ in $\mathfrak{a}^*_{1,\mathbb{C}}$. The direct sum

$$\mu_1(\lambda_1) = \mu_{P_1}(\lambda_1) = \bigoplus_{\pi} \mu_{\bar{P}_1|P_1}(\pi_{1,\lambda_1}), \qquad \lambda_1 \in \mathfrak{a}^*_{1,\mathbb{C}}, \ \pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1,$$

can then be regarded as a meromorphic function from $\mathfrak{a}^*_{1,\mathbb{C}}$ to the space of endomorphisms of $\mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$, whose restriction to $i\mathfrak{a}^*_1$ is analytic and tempered. The inverse spherical transform of F_1 is the τ-spherical function

$$F_1^\vee(x) = |W(M_1)|^{-1} \int_{i\mathfrak{a}^*_1} E_{P_1}\big(x, \mu_1(\lambda_1)F_1(\lambda_1), \lambda_1\big) d\lambda_1, \qquad x \in G(\mathbb{R}). \tag{2.3}$$

One of Harish-Chandra's basic results is that $F_1^\vee$ is a Schwartz function from $G(\mathbb{R})$ to V. Furthermore, if F_1 satisfies the symmetry condition (2.2) and f equals $F_1^\vee$, then $\widehat{f}_1$ equals F_1.

The spectral decomposition for spherical functions is now easy to state. It is the assertion that any function $f \in \mathcal{C}(G, \tau)$ can be written as a sum of functions $F_1^\vee$. More precisely, f satisfies the Fourier inversion formula

$$f(x) = \sum_{\{M_1\}} (\widehat{f}_1)^\vee(x), \tag{2.4}$$

where $\{M_1\}$ ranges over conjugacy classes of cuspidal Levi subgroups, and $\widehat{f}_1 = \widehat{f}_{P_1}$ is defined with respect to any parabolic subgroup $P_1 \in \mathcal{P}(M_1)$. (See [Ha5].) This is of course closely related to noninvariant inversion formula (1.11). In particular, the Plancherel densities and μ-functions implicit in (1.11) and (2.4) respectively satisfy the identity

$$\varepsilon_{P_1}(\pi_{1,\lambda_1}) = d_{\pi_1} \mu_{P_1}(\pi_{1,\lambda_1}),$$

where d_{π_1} is the formal degree of π_1.

Harish-Chandra's theory of spherical functions is based on his asymptotic estimates for the function

$$\phi(x) = E_{P_1}(x, \psi, \lambda_1), \qquad\qquad \psi \in \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1),\ \lambda_1 \in i\mathfrak{a}_1^*,$$

in terms of their constant terms. If Q is any parabolic subgroup with Levi component M_Q, the constant term ϕ_Q of ϕ along Q is a τ_Q-spherical, $A_Q(\mathbb{R})$-finite function on M_Q. It has the property that for any $m \in M_Q(\mathbb{R})^1$,

$$\lim_{\substack{a \longrightarrow \infty \\ Q,r}} \left(e^{\rho_Q(\log\, a)}\phi(ma) - \phi_Q(ma)\right) = 0,$$

in the notation at the end of §1. The proof that $F_1^\vee$ is a Schwartz function depends on a sharper estimate for the difference between ϕ and its constant term, which we shall state in terms of a fixed minimal parabolic subgroup.

Let $P_0 = M_0N_0$ be a minimal parabolic subgroup for which M_1 is standard. Then M_1 contains M_0, and is a Levi component of a unique parabolic subgroup $P_1 \in \mathcal{P}(M_1)$ that contains P_0. We assume until further notice that P_0 is fixed, and that r_0 is a small positive number. Suppose that $Q = M_QN_Q$ is a standard parabolic subgroup relative to P_0. We define $\mathfrak{a}_{P_0,Q}^{r_0}$ to be the set of points H in the space $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$ such that

$$0 \leq \alpha(H) \leq r_0\|H\|$$

for every α in the subset $\Delta_0^Q = \Delta_{P_0}^Q$ of $\Delta_0 = \Delta_{P_0}$, and such that

$$\alpha(H) > r_0\|H\|$$

for every α in the complement of Δ_0^Q in Δ_0. The closure $\overline{\mathfrak{a}}_0^+$ of the chamber $\mathfrak{a}_0^+ = \mathfrak{a}_{P_0}^+$ is then a disjoint union over $Q \supset P_0$ of the sets $\mathfrak{a}_{0,Q}^{r_0}$. Observe that if $Q = P_0$, $\mathfrak{a}_{P_0,Q}^{r_0}$ is the analogue $\mathfrak{a}_{P_0}^{r_0}$ for P_0 of the cone $\mathfrak{a}_P^r$ defined in §1.

The following lemma is included in a general estimate [Ha4, Lemma 10.8] of Harish-Chandra. The reader can find some of Harish-Chandra's terms defined at the beginning of Sections 3, 8 and 10 of [Ha4].

Lemma 2.1. *Suppose that*

$$\phi(x) = E_{P_1}(x, \psi, \lambda_1), \qquad \psi \in \mathcal{A}_{\mathrm{cusp}}(M_1, T_1),\ \lambda_1 \in i\mathfrak{a}_1^*.$$

Then for any standard parabolic subgroup $Q \supset P_0$, we can find a positive number δ and a polynomial p on $i\mathfrak{a}_1^*$, with the property that

$$\|\phi(h) - e^{-\rho_Q(\log\, h)}\phi_Q(h)\| \leq |p(\lambda_1)|\|\psi\|e^{-(1+\delta)\rho_0(\log\, h)}, \tag{2.5}$$

for any point $h \in A_0(\mathbb{R})$ such that $\log h$ *lies in $\mathfrak{a}_{P_0,Q}^{r_0}$.* □

We need to establish a second lemma, which depends on the detailed structure of constant terms. Let us first recall how to express the constant term ϕ_Q of Lemma 2.1 in terms of c-functions. If

$$w:\ \mathfrak{a}_{P_1} \xrightarrow{\sim} \mathfrak{a}_{P_1'}$$

belongs to the set of transformations $W(P_1;Q)$ attached in §1 to the standard parabolic subgroups P_1 and Q of G, the subscript

$$P_1' = P_1^u, \qquad u = w_Q^{-1},\ w_Q = w_{M_Q},$$

represents another standard parabolic subgroup of G. The intersection

$$R = P_1' \cap M_Q$$

is then a parabolic subgroup of M_Q that is standard relative to the minimal parabolic subgroup $R_0 = P_0 \cap M_Q$. The constant term ϕ_Q is given by the sum

$$\phi_Q(ma) = \sum_{w \in W(P_1;Q)} \phi_{Q,w}(ma), \qquad m \in M_Q(m)^1,\ a \in A_Q(\mathbb{R}), \tag{2.6}$$

where

$$
\begin{aligned}
\phi_{Q,w}(ma) &= E_R\big(ma, c_R(1, w\lambda_1)^{-1} c_{P_1'|P_1}(w, \lambda_1)\psi, w\lambda_1\big) \\
&= E_R\big(m, c_R(1, w\lambda_1)^{-1} c_{P_1'|P_1}(w, \lambda_1)\psi, w\lambda_1\big) e^{(w\lambda_1)(\log a)}.
\end{aligned}
$$

The objects E_R and c_R are of course defined with respect to the parabolic subgroup R of M_Q. In case M_Q is conjugate to M_1, (2.6) is Theorem 18.1 of [Ha2]. For general Q, the formula is easily derived from this special case and [Ha2, Lemma 18.3]. (See for example [A1, §II.6].)

Suppose that F_1 is a function in $\mathcal{C}\big(i\mathfrak{a}_1^*, \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)\big)$, as in (2.3). If S is a point in $\mathfrak{a}_1$, the function

$$
F_1^S(\lambda_1) = F_1(\lambda_1) e^{-\lambda_1(S)}, \qquad \lambda_1 \in i\mathfrak{a}_1^*, \tag{2.7}
$$

also belongs to $\mathcal{C}\big(i\mathfrak{a}_1^*, \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)\big)$. We shall need an estimate for the Schwartz function $(F_1^S)^\vee$ that is uniform in S.

Lemma 2.2. *For any $n \geq 0$, there is a positive constant c_n such that the norm of*

$$
e^{\rho_0(H)} (F_1^S)^\vee(\exp H) \tag{2.8}
$$

is bounded by

$$
c_n \sup_{Q' \supset P_0} \sup_{w' \in W(P_1; Q')} \big(1 + \|H - (w'S)_{Q'}\|\big)^{-n}, \tag{2.9}
$$

for any $H \in \overline{\mathfrak{a}}_0^+$ and $S \in \mathfrak{a}_1$.

Proof. The argument has two stages. The first will be an application of Lemma 2.1 to the integrand in

$$
(F_1^S)^\vee(\exp H) = |W(M_1)|^{-1} \int_{i\mathfrak{a}_1^*} E_{P_1}\big(\exp H, \mu_1(\lambda_1) F_1^S(\lambda_1), \lambda_1\big) d\lambda_1.
$$

For this to provide a useful reduction of the problem, we assume that H belongs to the interior $\mathfrak{a}_0^+$ of $\overline{\mathfrak{a}}_0^+$. We are of course free to do so, since (2.8) and (2.9) are both continuous in H. We shall also need to write the integral over $i\mathfrak{a}_1^*$ as a double integral over the product of $i(\mathfrak{a}_1^G)^*$ with $i\mathfrak{a}_G^*$.

If ζ belongs to $i\mathfrak{a}_G^*$, $\mu_1(\lambda_1+\zeta)$ equals $\mu_1(\lambda_1)$. Moreover,

$$E_{P_1}\big(\exp H, \mu_1(\lambda_1)F_1^S(\lambda_1), \lambda_1+\zeta\big) = E_{P_1}\big(\exp H, \mu_1(\lambda_1)F_1^S(\lambda_1), \lambda_1\big)e^{\zeta(Z)},$$

where $Z = H_G - S_G$. It follows that

$$(F_1^S)^\vee(\exp H) = \int_{i(\mathfrak{a}_1^G)^*} E_{P_1}\big(\exp\eta, \mu_1(\lambda_1)F_{1,Z}^\sigma(\lambda_1), \lambda_1\big)d\lambda_1,$$

where $\eta = H^G = H - H_G$, $\sigma = S^G = S - S_G$, and

$$F_{1,Z}^\sigma(\lambda_1) = |W(M_1)|^{-1}\int_{i\mathfrak{a}_G^*} F_1^\sigma(\lambda_1+\zeta)e^{\zeta(Z)}d\zeta.$$

By assumption, the point η belongs to the projection $(\mathfrak{a}_0^G)^+$ of $\mathfrak{a}_0^+$ onto $\mathfrak{a}_0^G$. It therefore lies in a set $\mathfrak{a}_{P_0,Q}^{r_0}$, for a unique *proper* parabolic subgroup $Q \supset P_0$. We use Lemma 2.1 to approximate the last integrand by its constant term

$$E_{P_1,Q}\big(\exp\eta, \mu_1(\lambda_1)F_{1,Z}^\sigma(\lambda_1), \lambda_1\big)$$

at Q. We see that for any $\lambda_1 \in i(\mathfrak{a}_1^G)^*$ and $\eta \in \mathfrak{a}_{P_0,Q}^{r_0}$, the norm of the difference between

$$e^{\rho_0(\eta)}E_{P_1}\big(\exp\eta, \mu_1(\lambda_1)F_{1,Z}^\sigma(\lambda_1), \lambda_1\big)$$

and

$$e^{(\rho_0-\rho_Q)(\eta)}E_{P_1,Q}\big(\exp\eta, \mu_1(\lambda_1)F_{1,Z}^\sigma(\lambda_1), \lambda_1\big) \tag{2.10}$$

is bounded by

$$|p(\lambda_1)|\mu_1(\lambda_1)\|F_{1,Z}^\sigma(\lambda_1)\|e^{-\delta\rho_0(\eta)},$$

for $p(\lambda_1)$ and δ as in Lemma 2.1. The integral over $\lambda_1 \in i(\mathfrak{a}_1^G)^*$ of the last expression converges. This is because both $p(\lambda_1)$ and $\mu_1(\lambda_1)$ are slowly increasing in λ_1, while $F_{1,Z}^\sigma(\lambda_1)$ is rapidly decreasing. In fact, the norm

$$\|F_{1,Z}^\sigma(\lambda_1)\| = \|F_{1,Z}(\lambda_1)\|$$

is rapidly decreasing in both $\lambda_1 \in i(\mathfrak{a}_1^G)^*$ and $Z \in \mathfrak{a}_G$. Moreover, the linear form $\rho_0(\eta) = \rho_{P_0}(\eta)$ is bounded below by a positive multiple of $\|\eta\|$. It follows that for any n, we can choose c_n such that

$$e^{-\delta\rho_0(\eta)} \int_{i(\mathfrak{a}_1^G)^*} |p(\lambda_1)|\mu_1(\lambda_1)\|F_{1,Z}^{\sigma}(\lambda_1)\|d\lambda_1 \leq c_n(1 + \|\eta\| + \|Z\|)^{-n}.$$

Since

$$\|\eta\| + \|Z\| = \|\eta + Z\| = \|H - S_G\|,$$

this is bounded by (2.9). To complete the first stage of the argument, we observe that the integral over $\lambda_1 \in i(\mathfrak{a}_1^G)^*$ of (2.10) equals the expression

$$e^{(\rho_0-\rho_Q)(H)}|W(M_1)|^{-1} \int_{i\mathfrak{a}_1^*} E_{P_1,Q}\big(\exp H, \mu_1(\lambda_1)F_1^S(\lambda_1), \lambda_1\big)d\lambda_1. \tag{2.11}$$

We have established that the norm of the difference between (2.8) and (2.11) is bounded by (2.9), for any $S \in \mathfrak{a}_1$ and $H \in \mathfrak{a}_0^+$ such that $\eta = H^G$ belongs to $\mathfrak{a}_{P_0,Q}^{r_0}$. The second stage of the proof will be to show that the norm of (2.11) is bounded by (2.9), for any $S \in \mathfrak{a}_1$ and any $H \in \mathfrak{a}_0^+$ at all. We shall argue by induction.

The exponential factor in (2.11) equals

$$e^{(\rho_0-\rho_Q)(H)} = e^{\rho_{R_0}(H)},$$

where R_0 is the minimal parabolic subgroup $P_0 \cap M_Q$ of M_Q. The integrand in (2.11) is given by (2.6). It equals the sum over $w \in W(P_1; Q)$ of Eisenstein integrals

$$E_R\big(\exp H, c_R(1, w\lambda_1)^{-1}c_{P_1'|P_1}(w, \lambda_1)\mu_1(\lambda_1)F_1^S(\lambda_1), w\lambda_1\big),$$

where $P_1' = P_1^u$, $u = w_Q^{-1}$, and $R = P_1' \cap M_Q$. To deal with this last summand, we set $\Lambda = w\lambda_1$, $T = wS$, and

$$F_R(\Lambda) = \mu_R(\Lambda)^{-1}c_R(1, \Lambda)^{-1}c_{P_1'|P_1}(w, w^{-1}\Lambda)\mu_{P_1}(w^{-1}\Lambda)F_1(w^{-1}\Lambda),$$

for the given element $w \in W(P_1;Q)$. We claim that as a function of Λ in $i\mathfrak{a}_R^* = i(\mathfrak{a}_{P_1'}^*)$, $F_R(\Lambda)$ belongs to the space $\mathcal{C}\big(i\mathfrak{a}_R^*, \mathcal{A}_{\mathrm{cusp}}(M_R,\tau_R)\big)$. Granting this claim for the moment, we see that the integral over λ_1 in (2.11) may be taken inside the sum over w, and then changed to an integral over Λ. The expression (2.11) becomes

$$\sum_{w\in W(P_1;Q)} e^{\rho_{R_0}(H)}|W(M_1)|^{-1}\int_{i\mathfrak{a}_R^*} E_R\big(\exp H, \mu_R(\Lambda)F_R^T(\Lambda),\Lambda\big)d\Lambda,$$

with

$$F_R^T(\Lambda) = F_R(\Lambda)e^{-\Lambda(T)} = F_R(\Lambda)e^{-\lambda_1(S)}.$$

In other words, (2.11) equals the product of a quotient

$$|W^{M_Q}(M_R)||W(M_1)|^{-1}$$

of orders of Weyl groups with the sum

$$\sum_{w\in W(P_1;Q)} e^{\rho_{R_0}(H)}(F_R^T)^\vee(\exp H).$$

We are assuming that H is any point in $\mathfrak{a}_0^+$. Since the chamber $\mathfrak{a}_0^+ = \mathfrak{a}_{P_0}^+$ is contained in $\mathfrak{a}_{R_0}^+$, we can estimate the last summands by applying the lemma inductively to the proper Levi subgroup M_Q of G. We obtain a bound

$$e^{\rho_{R_0}(H)}\|(F_R^T)^\vee(\exp H)\| \le c_n' \sup_{Q',w_R}\big(1+\|H-(w_RT)_{Q'\cap M_Q}\|\big)^{-n},$$

where Q' ranges over parabolic subgroups with $P_0 \subset Q' \subset Q$, and w_R ranges over elements in $W(R;Q'\cap M_Q)$. For any such w_R, the product $w' = w_Rw$ belongs to $W(P_1;Q')$, and

$$(w_RT)_{Q'\cap M_Q} = (w'S)_{Q'}.$$

It thus follows that the norm of (2.11) satisfies a bound (2.9). The same is therefore true of the original function (2.8).

We have established the lemma, given our claim that F_R belongs to $\mathcal{C}\big(i\mathfrak{a}_R^*, \mathcal{A}_{\text{cusp}}(M_R, \tau_R)\big)$. To justify the claim, it would be enough to show that the coefficient

$$\mu_R(\Lambda)^{-1} c_R(1,\Lambda)^{-1} c_{P_1'|P_1}(w, w^{-1}\Lambda)\mu_{P_1}(w^{-1}\Lambda) \tag{2.12}$$

of $F_1(w^{-1}\Lambda)$ is a smooth function with slowly increasing derivatives. Results of this kind were part of Harish-Chandra's theory of spherical functions, and are now well known. For example, one can write

$$\mu_{P_1}(w^{-1}\Lambda) = \mu_{P_1'}(\Lambda) = \big(r_{P_1'}(\Lambda)\overline{r_{P_1'}(\Lambda)}\big)^{-1},$$

where

$$r_{P_1'}(\Lambda) = \prod_{\alpha'} r_{\alpha'}(\Lambda)$$

is a product over the reduced roots α' of $(P_1', A_{P_1'})$ of rank one normalizing factors attached to standard intertwining operators. Both the normalized c-function

$$r_{P_1'}(\Lambda)^{-1} c_{P_1'|P_1}(w, w^{-1}\Lambda),$$

and the inverse

$$\big(r_R(\Lambda)^{-1} c_R(1,\Lambda)\big)^{-1}$$

of its analogue for R, are rational functions of Λ whose singularities do not intersecct $i\mathfrak{a}_R^*$. (See [A6, Theorem 2.1 (R_6), (R_7)] and formulas [A5, (2.4)] for c-functions in terms of intertwining operators.) The remaining component of (2.12) is a product of factors $(\overline{r_{\alpha'}(\Lambda)})^{-1}$, taken over reduced roots of P_1' that are not roots of R. From the explicit formulas is the appendix of [A6] on sees easily that each of these factors is an analytic, slowly increasing function on some cylindrical neighbourhood of $i\mathfrak{a}_R^*$ in $\mathfrak{a}_{R,\mathbb{C}}^*$. It follows from the Cauchy integral formula that as a function of $\Lambda \in i\mathfrak{a}_R^*$, (2.12) has slowly increasing derivatives. The claim is therefore valid, and the estimate (2.9) of the lemma holds. $\square$

Remark. We did not keep track of the dependence of the estimate (2.9) on the function F_1. It would have been easy to do so. An inspection of the argument reveals that we could set the constant c_n in (2.9) equal to $\|F_1\|_n$, for a continuous seminorm $\|\cdot\|_n$ on the space $\mathcal{C}\big(i\mathfrak{a}_1^*, \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)\big)$ that contains F_1. Suppose that F_1 also satisfies the symmetry condition (2.2). Then F_1 is the image of the function $f_1 = F_1^\vee$ in $\mathcal{C}(G, \tau)$, under the continuous mapping (2.1). In this case, we can take

$$c_n = \|f_1\|_n,$$

where $\|\cdot\|_n$ is now a continuous seminorm on $\mathcal{C}(G, \tau)$. As constructed, the last seminorm still depends on M_1 and (τ, V). However, one can actually arrange that it is the seminorm on $\mathcal{C}(G, \tau)$ attached to a continuous seminorm on $\mathcal{C}(G)$ that is independent of M_1 and (τ, V), and the Hermitian seminorm on V whose value at (2.8) is part of the statement of the lemma. This is a straightforward consequence of the proof of the easy half of the main theorem in [A1].

For future reference, we formulate as a separate corollary the conclusion we drew at the end of the proof of the lemma.

Corollary 2.3. *Set*

$$F_R(\Lambda) = \big(\mu_R(\Lambda)^{-1} c_R(1, \Lambda)^{-1} c_{P_1'|P_1}(w, w^{-1}\Lambda)\mu_{P_1}(w^{-1}\Lambda)\big)F_{P_1}(w^{-1}\Lambda), \quad \Lambda \in i\mathfrak{a}_R^*, \tag{2.13}$$

for $Q \supset P_0$, $w \in W(P_1; Q)$, $P_1' = P_1^u$, $u = w_Q^{-1}$, and $R = P_1' \cap M_Q$, as in the proof of the lemma. Then the coefficient of $F_1(w^{-1}\Lambda)$ on the right extends to an analytic, slowly increasing function on a cylindrical neighbourhood of $i\mathfrak{a}_R^*$ in $\mathfrak{a}_{R,\mathbb{C}}^*$, and the function $F_R(\Lambda)$ itself belongs to $\mathcal{C}\big(i\mathfrak{a}_R^*, \mathcal{A}_{\mathrm{cusp}}(M_R, \tau_R)\big)$. □

The estimates of the lemma will be used primarily in the form taken by the next corollary.

Corollary 2.4. *Set*

$$\phi^S(\lambda_1, x) = E_{P_1}\big(x, \mu_1(\lambda_1)F_1^S(\lambda_1), \lambda_1\big), \qquad S \in \mathfrak{a}_1, \ \lambda_1 \in i\mathfrak{a}_1^*.$$

Then for any $Q \supset P_0$ and $w \in W(P_1; Q)$, and any point $h \in A_0(\mathbb{R})$ with $\log(h) \in \overline{\mathfrak{a}}_0^+$, the norm of

$$e^{-\rho_Q(\log h)}|W(M_1)|^{-1}\int_{i\mathfrak{a}_1^*}\phi^S_{Q,w}(\lambda_1, h)d\lambda_1 \tag{2.14}$$

is bounded by

$$c_n e^{-\rho_0(\log h)} \sup_{Q',w'} \big(1 + \|(\log h) - (w'S)_{Q'}\|\big)^{-n},$$

where Q' ranges over parabolic subgroups with $P_0 \subset Q' \subset Q$, w' ranges over elements in $W(P_1; Q')$ such that $w'w^{-1}$ leaves $\mathfrak{a}_\Delta$ pointwise fixed, and c_n depends only on n.

Proof. The corollary is a consequence of the lemma and its proof. We have seen that the expression (2.14) equals the product of $e^{-\rho_0(\log h)}$ with

$$|W^{M_Q}(M_R)||W(M_1)|^{-1}e^{\rho_{R_0}(\log h)}(F^T_R)^\vee(h), \qquad T = wS,$$

in the notation of the second stage of the proof. We have only to apply the lemma to the function $(F^T_R)^\vee$ in $\mathcal{C}(M_Q, \tau_Q)$, as we did inductively near the end of the proof. We see that (2.14) satisfies a bound of the required kind. $\square$

§3. Reduction to constant terms

We return to the problem posed at the end of §1. Then M is a fixed, cuspidal Levi subgroup of G with M-elliptic maximal torus T_M. We fix a compact subset C of $\mathfrak{a}_M$. We shall then let γ range over the relatively compact subset

$$\Gamma = T_{M,G\text{-reg}}(\mathbb{R})^C = \big\{\gamma \in T_{M,G\text{-reg}}(\mathbb{R}) :\ H_M(\gamma) \in C\big\}$$

of $T_M(\mathbb{R})$. As in §1, T is to range over a cone $\mathfrak{a}_P^r$ in $\mathfrak{a}_P^+$, for a fixed parabolic subgroup $P = MN_P$ in $\mathcal{P}(M)$. Our aim will be to study the limit (1.13) in terms of spherical functions.

The terms in (1.13) depend only on M and P (in addition of course to f and γ). However to prove the formula, we shall fix a minimal parabolic subgroup $P_0 = M_0N_0$ with $P \supset P_0$ and $M \supset M_0$. Having chosen P_0, we take M_1 to be a standard cuspidal Levi subgroup as in §2. Then M_1 contains M_0, and comes with a parabolic subgroup $P_1 = M_1N_1$ in $\mathcal{P}(M_1)$ that contains P_0. With this setting we will be able to apply the estimates of Harish-Chandra summarized in Lemma 2.1.

We fix a double representation τ of K on the finite dimensional Hilbert space V. We then fix a Schwartz function $F_1 = F_{P_1}$ on the space $i\mathfrak{a}_1^* = i\mathfrak{a}_{M_1}^*$, with values in the finite dimensional complex vector space $\mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1) = \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_{M_1})$. The inverse spherical transform $f_1 = F_1^\vee$ is a function in $\mathcal{C}(G, \tau)$. The transformation $f \to f_T$ of §1 can be applied to this (vector-valued) function. It yields another function $f_{1,T} = F_{1,T}^\vee$ in $\mathcal{C}(G, T)$. The weighted orbital integral $J_M(\gamma_T, f_{1,T})$ is then defined as a function of T with values in V. We assume that F_1 satisfies the symmetry condition (2.2). This implies that the spherical transform $\widehat{f}_{1,P_1}$ equals F_1. Moreover, the spherical transform $(f_{1,T})^\wedge_{P_1}$ of $f_{1,T}$ equals the function

$$F_{1,T}(\lambda_1) = \sum_{u \in W(M,M_1)} e^{-\lambda_1(uT)} F_1(\lambda_1), \qquad \lambda_1 \in i\mathfrak{a}_1^*.$$

Our interest is in the limit of $J_M(\gamma_T, f_{1,T})$ as T approaches infinity in $\mathfrak{a}_P^r$.

We shall actually have to consider a slightly more general problem. Let us write

$$f_1^S = (F_1^S)^\vee \tag{3.1}$$

for any point $S \in \mathfrak{a}_1$, where we recall that $F_1^S(\lambda_1)$ is defined in (2.7) as $F_1(\lambda_1)e^{-\lambda_1(S)}$. The function $f_{1,T}$ above is a sum of functions f_1^S, with S ranging over the $W(M, M_1)$-translates of T. For reasons of induction, it will eventually be necessary to study integrals $J_M(\gamma_T, f_1^S)$ in which S is a more general linear form in T. We shall make this precise later. In the meantime, we allow S to be any variable point in $\mathfrak{a}_1$. We shall establish some estimates for $J_M(\gamma_T, f_1^S)$ that are uniform in S.

We need to investigate the weighted orbital integral (1.2), with f replaced by the τ-spherical function f_1^S. Since the maximal torus $T = T_M$ in M is elliptic, its split component is A_M. We may therefore replace the domain of integration $T(\mathbb{R})\backslash G(\mathbb{R})$ in (1.2) by $A_M(\mathbb{R})\backslash G(\mathbb{R})$, with the understanding that the quotient $A_M(\mathbb{R})\backslash T_M(\mathbb{R})$ has Haar measure 1. Applying the decomposition $G(\mathbb{R}) = M(\mathbb{R})N_P(\mathbb{R})K$ to the integral, we obtain

$$\begin{aligned}
&J_M(\gamma_T, f_1^S)\\
&= |D(\gamma_T)|^{\frac{1}{2}} \int_{A_M(\mathbb{R})\backslash G(\mathbb{R})} f_1^S(x^{-1}\gamma_T x)v_M(x)dx\\
&= |D(\gamma_T)|^{\frac{1}{2}} \int_K \int_{A_M(\mathbb{R})\backslash M(\mathbb{R})} \int_{N_P(\mathbb{R})} f_1^S(k^{-1}n^{-1}m^{-1}\gamma_T mnk)v_M(n)dndmdk.
\end{aligned}$$

Before attempting any estimates, we shall make two structural changes in the last expression.

The first change will be in the integral over $N_P(\mathbb{R})$. For a given $m \in M(\mathbb{R})$, we shall sometimes write

$$\delta_T = \delta_T(m) = m^{-1}\gamma_T m.$$

This equals $\delta \exp T$, where

$$\delta = \delta(m) = m^{-1}\gamma m.$$

If n belongs to $N_P(\mathbb{R})$, we can also write

$$n^{-1}m^{-1}\gamma_T mn = n^{-1}\delta_T n = \delta_T \nu,$$

for the point

$$\nu = \delta_T^{-1} n^{-1} \delta_T n \tag{3.2}$$

in $N_P(\mathbb{R})$. For fixed m and T, the map $n \to \nu$ is an invertible morphism of N_P. Taking the inverse of this map, we consider the point

$$n = n(\nu, \delta_T) = n(\nu, m^{-1}\gamma_T m)$$

as a function of ν and δ_T. In fact, we shall change variables from n to ν in the last integral over $N_P(\mathbb{R})$. We can do so as long as we multiply the integrand by the corresponding Jacobian determinant

$$|D^G(\gamma_T)|^{-\frac{1}{2}} |D^M(\gamma_T)|^{\frac{1}{2}} e^{\rho_P(H_M(\gamma_T))} = |D^G(\gamma_T)|^{-\frac{1}{2}} |D^M(\gamma)|^{\frac{1}{2}} e^{\rho_P(H_M(\gamma)+T)}.$$

The first factor in this product cancels the normalizing factor $|D(\gamma_T)|^{\frac{1}{2}} = |D^G(\gamma_T)|^{\frac{1}{2}}$ in the original integral. We find that $J_M(\gamma_T, f_1^S)$ equals the product of

$$|D^M(\gamma)|^{\frac{1}{2}} e^{\rho_P(H_M(\gamma)+T)} \tag{3.3}$$

with

$$\int_K \int_{A_M(\mathbb{R})\backslash M(\mathbb{R})} \int_{N_P(\mathbb{R})} f_1^S(k^{-1}m^{-1}\gamma_T m \nu k) v_M\big(n(\nu, m^{-1}\gamma_T m)\big) d\nu dm dk. \tag{3.4}$$

The second change will be to express the variables of integration in terms of the polar decomposition of $G(\mathbb{R})$. For any $x \in G(\mathbb{R})$, we write $x_0^+ = x_{P_0}^+$ for the noncompact part of x in the polar decomposition. In other words, x_0^+ is the unique point in $A_0(\mathbb{R})^0 = \exp(\mathfrak{a}_0)$ whose logarithm lies in the closure of the chamber $\mathfrak{a}_0^+ = \mathfrak{a}_{P_0}^+$, and such that x lies in Kx_0^+K. For any m and ν in the integral (3.4), we set

$$h_T = h_T(m, \nu) = (m^{-1}\gamma_T m \nu)_0^+ = (\delta_T \nu)_0^+.$$

Thus

$$m^{-1}\gamma_T m \nu = k_T^1 h_T k_T^2,$$

for points

$$k_T^i = k_T^i(m, \nu), \qquad i = 1, 2,$$

in K. We shall also write

$$h'_T = h'_T(m) = (m^{-1}\gamma_T m)^+_{R_0} = (\delta_T)^+_{R_0}$$

for the noncompact part of $m^{-1}\gamma_T m = \delta_T$ in the polar decomposition with respect to the minimal parabolic subgroup $R_0 = P_0 \cap M$ of M. Then $h'_T = h' \exp T$, where $h' = h'(m)$ is the value of h'_T at $T = 0$, and

$$m^{-1}\gamma_T m = k_T^{\prime,1} h'_T k_T^{\prime,2},$$

for points

$$k_T^{\prime,i} = k_T^{\prime,i}(m), \qquad i = 1, 2,$$

in $K_M = K \cap M(\mathbb{R})$. Moreover,

$$h_T = (h'_T \nu')^+_0, \tag{3.5}$$

where

$$\nu' = (k_T^{\prime,2})\nu(k_T^{\prime,2})^{-1}. \tag{3.6}$$

We return to the discussion of $J_M(\gamma_T, f_1^S)$. The τ-spherical function f_1^S is the inverse spherical transform of the function $F_1^S(\lambda_1)$. It is thus the $|W(M_1)|^{-1}$-normalized average in λ_1 of the Eisenstein integral

$$\phi^S(\lambda_1, x) = E_{P_1}\big(x, \mu_{P_1}(\lambda_1)F_1^S(\lambda_1), \lambda_1\big) \tag{3.7}$$

attached to P_1. We can therefore write

$$\int_K f_{1,T}(k^{-1}xk)dk = |W(M_1)|^{-1} \int_{i\mathfrak{a}_1^*} \pi^K\big(\phi^S(\lambda_1, x)\big)d\lambda_1,$$

for the projection π^K defined at the beginning of §2. Moreover,

$$\phi^S(\lambda_1, m^{-1}\gamma_T m\nu) = \tau(k_T^1)\phi^S(\lambda_1, h_T)\tau(k_T^2),$$

in the notation above. The expression (3.4) can therefore be written as the integral over $m \in A_M(\mathbb{R})\backslash M(\mathbb{R})$ and $\nu \in N_P(\mathbb{R})$ of

$$|W(M_1)|^{-1} \int_{i\mathfrak{a}_1^*} \pi^K\big(\tau(k_T^1)\phi^S(\lambda_1, h_T)\tau(k_T^2)\big)d\lambda_1 \cdot v_M\big(n(\nu, m^{-1}\gamma_T m)\big). \tag{3.8}$$

Our intention is to apply Harish-Chandra's theory of the constant term to the function $\phi^S(\lambda_1, h_T)$ in (3.8). We will estimate the contribution of the error term to (3.8) in this section, and the contribution of the constant term itself in §4. In both cases, we shall need a bound for the exponential function

$$e^{-\rho_0(\log\, h_T)} = \big|\det\big(\mathrm{Ad}(h_T)\big)_{\mathfrak{n}_0}\big|^{-\frac{1}{2}},$$

where $\mathfrak{n}_0$ is the Lie algebra of $N_0 = N_{P_0}$. We may as well establish it now, with the definition of h_T still freshly in mind. The bound will be given in terms of familiar coordinates on $G(\mathbb{R})$, which we will need later in the context of a general standard parabolic subgroup $Q \supset P_0$. Recall that $\overline{Q}$ denotes the parabolic subgroup in $\mathcal{P}(M_Q)$ opposite to Q. For any such $\overline{Q}$, and any $x \in G(\mathbb{R})$, we write

$$x = n_{\bar{Q}}(x)m_{\bar{Q}}(x)a_{\bar{Q}}(x)k_{\bar{Q}}(x), \tag{3.9}$$

for points $n_{\bar{Q}}(x) \in N_{\bar{Q}}(\mathbb{R})$, $m_{\bar{Q}}(x) \in M_{\bar{Q}}(\mathbb{R})^1$, $a_{\bar{Q}}(x) \in A_Q(\mathbb{R})^0$ and $k_{\bar{Q}}(x) \in K$. In the case $Q = P_0$ that we will use here, we denote the four points by $\overline{n}_0(x)$, $\overline{m}_0(x)$, $\overline{a}_0(x)$, and $\overline{k}_0(x)$ respectively. The point $\overline{m}_0(x)$ actually belongs to K in this case, and can therefore be ignored.

In estimating the exponential function, we shall derive a geometric property of the vector $\log(h_T) - T$ in $\mathfrak{a}_0$. Define

$${}^+\mathfrak{a}_0 = \big\{X \in \mathfrak{a}_0 :\ (X, H) > 0,\ H \in \mathfrak{a}_0^+\big\},$$

the chamber in $\mathfrak{a}_0$ that is dual to $\mathfrak{a}_0^+$ relative to our fixed Euclidean inner product $(\cdot,\cdot)$ on $\mathfrak{a}_0$. Notice that ${}^+\mathfrak{a}_0$ is a closed cone in the subspace $\mathfrak{a}_0^G$ of $\mathfrak{a}_0$.

Lemma 3.1. (a) *We can choose positive constants C_0 and c_0 depending only on G such that*

$$e^{-\rho_0(\log\, h_T)} \leq C_0 e^{-\rho_0(T)} e^{-\rho_0(\log\, h')} e^{-\rho_0(\log\, \overline{a}_0(\nu'))} \tag{3.10}$$

and

$$\rho_0(\log\, h_T) + c_0 \geq \rho_0(T) + \rho_0(\log\, h') + \rho_0\big(\log\, \overline{a}_0(\nu')\big). \tag{3.11}$$

(b) *There is a point $T_0^G \in \mathfrak{a}_0^G$ that depends only on G such that the vector*

$$\big(\log(h_T) - T\big) - \big(H_M(\gamma) + T_0^G\big)$$

belongs to the chamber ${}^+\mathfrak{a}_0$.

Proof. The elements h_T, h' and ν' in (a) are of course defined as above. The two inequalities in (a) become equivalent under exponentiation, so it suffices to prove (3.11). We shall do so in the course of establishing (b).

Let $\mu \in \mathfrak{a}_0^*$ be a highest weight relative to (P_0, A_0). Then $\mu' = (-\mu)$ is a lowest weight, in the sense that there is an irreducible, finite dimensional representation (r', V') of G over $\mathbb{R}$ with lowest weight μ'. We choose a K-invariant Hermitian inner product on $V'(\mathbb{C})$ that has an orthonormal basis of weight vectors. Then

$$\|r'(h_T)^{-1}\|_2 = \Big(\sum_{\eta} e^{-2\eta(\log\, h_T)}\Big)^{\frac{1}{2}},$$

where $\|\cdot\|_2$ is the corresponding Hilbert-Schmidt norm on $\mathrm{End}\big(V'(\mathbb{C})\big)$, and η is summed over the A_0-weights (with multiplicity) of r'. For any weight η, $\mu' - \eta$ lies in the negative chamber $(-{}^+\mathfrak{a}_0)$. Since $(\log\, h_T)$ lies in the closure of the positive chamber $\mathfrak{a}_0^+$,

$$(\mu' - \eta)(\log\, h_T) \leq 0.$$

The inequality

$$\begin{aligned}\|r'(h_T)^{-1}\|_2 &= e^{\mu(\log\, h_T)}\Big(1+\sum_{\eta\neq\mu'} e^{2(\mu'-\eta)(\log\, h_T)}\Big)^{\frac{1}{2}}\\ &\leq (\dim\, V')^{\frac{1}{2}} e^{\mu(\log\, h_T)})\end{aligned}$$

follows. To obtain a second inequality, we evaluate the operator $r'(h_T)^{-1}$ at a lowest weight vector v' of norm 1. It follows from (3.5) and (3.9) that

$$\begin{aligned}\|r'(h_T)^{-1}\|_2 &= \|r'(h'_T\nu')^{-1}\|_2\\ &\geq \|r'(h'_T\nu')^{-1}v'\|\\ &= \|r'(\nu')^{-1}r'(h'_T)^{-1}v'\|\\ &= e^{\mu(\log\, h'_T)}\|r'(\nu')^{-1}v'\|\\ &= e^{\mu(\log\, h'_T)}\Big\|r'\big(\overline{k}_0(\nu')\big)^{-1}r'\big(\overline{a}_0(\nu')\big)^{-1}r'\big(\overline{n}_0(\nu')^{-1}\big)v'\Big\|\\ &= e^{\mu(\log\, h'_T)}e^{\mu(\log\, \overline{a}_0(\nu'))}.\end{aligned}$$

The two inequalities together become

$$e^{\mu(\log\, h'_T)}e^{\mu(\log\, \bar{a}_0(\nu'))} \leq (\dim\, V')^{\frac{1}{2}} e^{\mu(\log\, h_T)}.$$

It follows from the definitions that $\log(h'_T)$ equals $\log\, h' + T$. Taking logarithms of the last inequality, we see that

$$\mu(\log\, h_T) - \mu(T) + c' \geq \mu(\log\, h') + \mu\big(\log\, \overline{a}_0(\nu')\big), \tag{3.12}$$

where $c' = \frac{1}{2}\log(\dim\, V')$. The required inequality (3.11) then follows with $\mu = 2\rho_0$.

To establish (b), we recall that $\log\, h'$ lies in the closure of the chamber $\mathfrak{a}^+_{R_0}$ for (R_0, A_0). Moreover, the projection of this point onto $\mathfrak{a}_M$ equals the vector

$$H_M(h') = H_M(\gamma).$$

The subset of points in $\mathfrak{a}^+_{R_0}$ that project to 0 in $\mathfrak{a}_M$ is contained in the dual chamber ${}^+\mathfrak{a}_{R_0}$, which is in turn contained in ${}^+\mathfrak{a}_0$. The difference $\big(\log\, h' - H_M(\gamma)\big)$ therefore lies in ${}^+\mathfrak{a}_0$.

The second vector $\big(\log \overline{a}_0(\nu')\big)$ on the right hand side of (3.12) is well known also to lie in ${}^+\mathfrak{a}_0$. In fact, one sees directly from the argument above that $\mu\big(\log \overline{a}_0(\nu')\big)$ is nonnegative for any μ. It thus follows from (3.12) that for any μ, there is a constant c' such that

$$\mu\big(\log h_T - T - H_M(\gamma)\big) + c' \geq 0.$$

Now, it is a consequence of the definitions that the vector

$$\log h_T - T - H_M(\gamma)$$

lies in the subspace $\mathfrak{a}_0^G$ of $\mathfrak{a}_0$. Let $\{\mu\}$ be a finite set of highest weights, which all lie in $(\mathfrak{a}_0^G)^*$ and for which the intersections of the half spaces

$$\{H \in \mathfrak{a}_0^G :\ \mu(H) \leq 0\}$$

equals ${}^+\mathfrak{a}_0$. We take $T_0^G \in \mathfrak{a}_0^G$ to be a vector such that for any of these μ, the sum of $\mu(T_0^G)$ with the corresponding constant c' is negative. The assertion of (b) follows. $\square$

Returning again to the discussion of $J_M(\gamma_T, f_1^S)$, we shall apply the estimate (2.5) of Lemma 2.1 to the integrand in (3.8). If Δ is any subset of the simple roots Δ_0 of (P_0, A_0), let $P_\Delta \supset P_0$ be the corresponding subgroup of G. We shall generally replace any subscript P_Δ simply by Δ. For example, Δ determines the subspace

$$\mathfrak{a}_\Delta = \mathfrak{a}_{P_\Delta} = \{H \in \mathfrak{a}_0 :\ \alpha(H) = 0,\ \alpha \in \Delta\}$$

of $\mathfrak{a}_0$, which we have agreed to identify with the Lie algebra of $A_\Delta(\mathbb{R})$. We fix, for once and for all, a small positive number r with respect to which we will ultimately take the limits (1.15) and (1.16). Having chosen r, we then fix the positive number r_0 of §2 so that it is small relative to r. In particular, we assume that $0 < r_0 < r$. The closed chamber $\bar{\mathfrak{a}}_0^+$ is then a disjoint union over Δ of the sets

$$\mathfrak{a}_{0,\Delta} = \mathfrak{a}_{P_0,P_\Delta}^{r_0},$$

defined in §2. Given Δ and T, we write

$$P(\mathbb{R})_{\Delta,T} = \bigl\{(m,\nu) \in A_M(\mathbb{R})\backslash M(\mathbb{R}) \times N_P(\mathbb{R}) :\ \log\bigl(h_T(m,\nu)\bigr) \in \mathfrak{a}_{0,\Delta}\bigr\}.$$

We can then apply Lemma 2.1 (with P_Δ in place of Q) to the points $h_T = h_T(m,\nu)$ with (m,ν) in $P(\mathbb{R})_{\Delta,T}$. We see that the norm

$$\bigl\|\pi^K\bigl(\tau(k_T^1)\phi^S(\lambda_1,h_T)\tau(k_T^2)\bigr) - e^{-\rho_\Delta(\log\, h_T)}\pi^K\bigl(\tau(k_T^1)\phi_\Delta^S(\lambda_1,h_T)\tau(k_T^2)\bigr)\bigr\| \tag{3.13}$$

has a bound

$$|p_1(\lambda_1)|\cdot\|\mu_1(\lambda_1)F_1^S(\lambda_1)\|e^{-(1+\delta_1)\rho_0(\log\, h_T)},$$

for a polynomial p_1 and a positive number δ_1, which is valid for any $T \in \mathfrak{a}_P^+$ and any $(m,\nu) \in P(\mathbb{R})_{\Delta,T}$.

Consider the contribution of $P(\mathbb{R})_{\Delta,T}$ to the formula for $J_M(\gamma_T, f_1^S)$ if the integrand in (3.8) is replaced by the absolute value (3.13). The contribution is bounded by the product of (3.3) with the two integrals

$$|W(M_1)|^{-1}\int_{i\mathfrak{a}_1^*} |p_1(\lambda_1)|\cdot\|\mu_1(\lambda_1)F_1^S(\lambda_1)\|d\lambda_1$$

and

$$\int_{P(\mathbb{R})_{\Delta,T}} e^{-(1+\delta_1)\rho_0(\log\, h_T)}\bigl|v_M\bigl(n(\nu,m^{-1}\gamma_T m)\bigr)\bigr|d\nu dm. \tag{3.14}$$

Since $\mu_1(\lambda_1)$ is slowly increasing and $F_1^S(\lambda_1)$ is rapidly decreasing, the first integral is finite, and is bounded independently of S. To treat the second integral, we require a couple of simple lemmas.

Following Harish-Chandra, we write

$$\sigma(x) = \log\|r_G(x)\|_2, \qquad x \in G(\mathbb{R}),$$

where (r_G, V_G) is a suitably fixed finite dimensional representation of G, and $\|\cdot\|_2$ is the Hilbert-Schmidt norm attached to a suitable Hermitian inner product on $V_G(\mathbb{C})$.

Lemma 3.2. *There are positive constants c and p such that*

$$\left|v_M\big(n(\nu, m^{-1}\gamma_T m)\big)\right| \leq c\big(1+\sigma(\nu)\big)^p\big(1+\sigma(m^{-1}\gamma m)\big)^p,$$

for all ν and m, all $\gamma \in \Gamma$, and all $T \in \mathfrak{a}_P^r$ sufficiently large relative to Γ.

Proof. The estimate in a consequence of the proof of [A2, Lemma 7.2]. According to [Ha2, Lemma 10], the correspondence

$$n \longrightarrow \nu = \delta_T^{-1} n^{-1} \delta_T n, \qquad \delta_T = m^{-1}\gamma_T m,$$

is the exponential transfer of an invertible polynomial mapping of $\mathfrak{n}_P$ onto itself. The matrix coefficients of this mapping are linear combinations of monomials, with coefficients that are in turn polynomials in the matrix coefficients of the map

$$A_{\delta_T} = \big(1 - \mathrm{Ad}(\delta_T)^{-1}\big) : \ \mathfrak{n}_P \longrightarrow \mathfrak{n}_P.$$

The same goes for the inverse correspondence

$$\nu \longrightarrow n = n(\nu, \delta_T),$$

except that the monomial coefficients are polynomials in the matrix coefficients of A_{δ_T} divided by the determinant of A_{δ_T}. The determinant equals

$$\det(A_{\delta_T}) = 1 - e^{-2\rho_1(H_M(\delta_T))} = 1 - e^{-2\rho_1(H_M(\gamma)+T)}.$$

It is bounded away from 0 whenever $T \in \mathfrak{a}_P^r$ is large relative to $\gamma \in \Gamma$. The matrix coefficients of A_{δ_T} can be bounded independently of T, or in other words, in terms of the matrix coefficients of $A_\delta = A_{m^{-1}\gamma m}$. The required estimate follows easily from the usual formula [A2, (4.1)] for $v_M\big(n(\nu, m^{-1}\gamma_T m)\big)$, and standard properties of the Harish-Chandra function $\sigma(\cdot)$. (See [A2, (7.2) and Lemma 7.1].) $\square$

We apply Lemma 3.2 and (3.10) to the integral (3.14). We see that (3.14) is bounded by a constant multiple of the product of two expressions

$$e^{-(1+\delta_1)\rho_0(T)} \int_{A_M(\mathbb{R})\backslash M(\mathbb{R})} e^{-(1+\delta_1)\rho_0(\log\, h')}\big(1+\sigma(m^{-1}\gamma m)\big)^p dm$$

and

$$\int_{N_P(\mathbb{R})} e^{-(1+\delta_1)\rho_0(\log(\overline{a}_0(\nu')))}\big(1+\sigma(\nu)\big)^p d\nu.$$

In dealing with the first expression, we can substitute

$$\rho_0(\log h') = \rho_{R_0}(\log h') + \rho_P\big(H_M(\gamma)\big), \qquad\qquad R_0 = P_0 \cap M,$$

into the exponential part of the integrand, since the projection of $\log h'$ onto $\mathfrak{a}_M$ equals $H_M(\gamma)$. Recall that h' equals $(m^{-1}\gamma m)^+_{R_0}$, so in particular $\log h'$ lies in the closure of the chamber $\mathfrak{a}^+_{R_0}$, and that γ lies in the relatively compact subset Γ of $T_M(\mathbb{R})$. It follows that the product

$$e^{-\delta_1\rho_{R_0}(\log\, h')}\big(1+\sigma(h')\big)^{n+p}$$

is bounded for any n. Moreover, an elementary estimate [Ha1, Lemma 36] of Harish-Chandra tells us that

$$e^{-\rho_{R_0}(\log\, h')} \leq \Xi_M(h'),$$

where Ξ_M is the function used to define the Schwartz space on $M(\mathbb{R})$ [Ha3]. We can thus find a constant c_n for any given n such that

$$\begin{aligned} & e^{-(1+\delta_1)\rho_0(\log\, h')} \\ & \leq c_n e^{-(1+\delta_1)\rho_P(H_M(\gamma))}\Xi_M(h')\big(1+\sigma(h')\big)^{-(n+p)} \\ & = c_n e^{-(1+\delta_1)\rho_P(H_M(\gamma))}\Xi_M(m^{-1}\gamma m)\big(1+\sigma(m^{-1}\gamma m)\big)^{-(n+p)}, \end{aligned}$$

since the functions Ξ_M and σ are both biinvariant under K_M. It follows that the product of the first expression above with the earlier factor (3.3) is bounded by the product of

$$c_n e^{-\delta_1\rho_P(T+H_M(\gamma))}$$

with

$$|D^M(\gamma)|^{\frac{1}{2}} \int_{A_M(\mathbb{R})\backslash G(\mathbb{R})} \Xi_M(m^{-1}\gamma m)\big(1+\sigma(m^{-1}\gamma m)\big)^{-n} dm.$$

We have of course used the fact that $\rho_0(T) = \rho_P(T)$. According to a basic estimate [Ha3, Theorem 5] of Harish-Chandra, the last factor is finite for n sufficiently large, and is bounded independently of γ. We conclude that the product of the first expression with (3.3) approaches 0 at T approaches infinity in $\mathfrak{a}_P^r$, uniformly in $\gamma \in \Gamma$.

The second expression equals

$$\int_{N_P(\mathbb{R})} e^{-(1+\delta_1)\rho_0(\log(\bar{a}_0(\nu)))} \big(1+\sigma(\nu)\big)^p d\nu,$$

since

$$\sigma(\nu) = \sigma\big(\mathrm{Int}(k_T'^{,2})\nu\big) = \sigma(\nu'),$$

and $d\nu = d\nu'$. Moreover, for any n, this expression is bounded by a constant multiple of

$$\int_{N_P(\mathbb{R})} e^{-\rho_0(\log(\bar{a}_0(\nu)))} \big(1+\rho_0\big(\log(\bar{a}_0(\nu))\big)\big)^{-n} \big(1+\sigma(\nu)\big)^p d\nu.$$

The finiteness of this integral for large n is the case that $Q = \overline{P}$ in the following lemma.

Lemma 3.3. *For any $p > 0$ and $Q \in \mathcal{P}(M)$, we can choose n so that the integral*

$$\int_{N_P(\mathbb{R})\cap N_{\bar{Q}}(\mathbb{R})} e^{-\rho_0(\log\,\bar{a}_0(\nu)))} \big(1+\rho_0(\log\,\bar{a}_0(\nu))\big)^{-n} \big(1+\sigma(\nu)\big)^p d\nu \tag{3.15}$$

is finite.

Proof. The lemma is a variant of a classical estimate of Harish-Chandra ([Ha1, Lemma 45] or [Ha3, Lemma 89]). In the case at hand, we shall argue by induction on $d(P,Q)$, the minimal number of singular hyperplanes in $\mathfrak{a}_M$ that separate the chambers $\mathfrak{a}_P^+$ and $\mathfrak{a}_Q^+$. This is a familiar technique, so we shall be brief. If $p = 0$, for example, the integrand in (3.15) is essentially the function (I.4.7) in [A1], and the finiteness of the corresponding integral is established in [A1, p. I.4.22–I.4.24, p. I.4.7–I.4.9].

One can choose a quasisplit inner twist ψ: $G \to G^*$ of G, and parabolic subgroups P^*, $Q^* \in \mathcal{P}(M')$ of G^*, such that ψ restricts to an $\mathbb{R}$-isomorphism from $N_P \cap N_{\bar{Q}}$ onto $N_{P^*} \cap N_{\overline{Q^*}}$. Moreover, this isomorphism maps the functions $\log \bar{a}_0(\nu)$ and $\sigma(\nu)$ on $N_P(\mathbb{R}) \cap N_{\bar{Q}}(\mathbb{R})$ to corresponding functions on $N_{P^*}(\mathbb{R}) \cap N_{\overline{Q^*}}(\mathbb{R})$. We may therefore assume that G is quasisplit. The intersection $N_P \cap N_{\bar{Q}}$ equals $N_{P_0} \cap N_{\bar{Q}_0}$, for a minimal parabolic subgroup $Q_0 \in \mathcal{P}(M_0)$. We can therefore also assume that $M = M_0$ and $P = P_0$. In particular, we shall write

$$\rho_0\big(\log \overline{a}_0(\nu)\big) = \rho_P\big(H_{\bar{P}}(\nu)\big)$$

in the integrand of (3.15).

Given Q, we choose a group $P_1 \in \mathcal{P}(M)$, with $d(P_1, Q) = 1$, such that $d(P, P_1)$ is less than $d(P, Q)$. We then have a decomposition of integrals

$$\int_{N_P(\mathbb{R})\cap N_{\bar{Q}}(\mathbb{R})} \phi(\nu)d\nu = \int_{N_{P_1}(\mathbb{R})\cap N_{\bar{Q}}(\mathbb{R})} \int_{N_P(\mathbb{R})\cap N_{\bar{P}_1}(\mathbb{R})} \phi(\nu_1 x)d\nu_1 dx,$$

for any nonnegative measurable function ϕ on $N_P(\mathbb{R}) \cap N_{\bar{Q}}(\mathbb{R})$. To deal with the two variables on the right, we write

$$\begin{aligned} H_{\bar{P}}(\nu_1 x) &= H_{\bar{P}}\big(\nu_1 n_{\bar{P}}(x) a_{\bar{P}}(x) k_{\bar{P}}(x)\big) \\ &= H_{\bar{P}}\big(\nu'_{1,x} a_{\bar{P}}(x)\big) \\ &= H_{\bar{P}}(\nu_{1,x}) + H_{\bar{P}}(x), \end{aligned}$$

where $\nu'_{1,x}$ is the point in $N_P(\mathbb{R})\cap N_{\bar{P}_1}(\mathbb{R})$ such that $\nu_1 n_{\bar{P}}(x)$ belongs to $N_{\bar{P}}(\mathbb{R})\nu'_{1,x}$, and $\nu_{1,x}$ is the conjugate of $\nu'_{1,x}$ by $a_{\bar{P}}(x)^{-1}$. For any x, the mapping $\nu_1 \to \nu_{1,x}$ is a diffeomorphism of $N_P(\mathbb{R}) \cap N_{\bar{P}_1}(\mathbb{R})$, which is easily seen to transform the Haar measure according to the formula

$$e^{-\rho_P(H_{\bar{P}}(x))} d\nu_1 = e^{-\rho_{P_1}(H_{\bar{P}}(x))} d\nu_{1,x}.$$

Moreover, it is not difficult to establish a bound

$$\sigma(\nu_1 x) \leq c\big(\sigma(\nu_{1,x}) + \sigma(x)\big)$$

from the subadditive property [Ha3, Lemma 10] of σ. The integrand in (3.15) is therefore bounded by a constant multiple of the function

$$e^{-\rho_P(H_{\bar{P}}(\nu_{1,x})+H_{\bar{P}}(x))}\big(1 + \rho_P\big(H_{\bar{P}}(\nu_{1,x}) + H_{\bar{P}}(x)\big)\big)^{-n}\big(1 + \sigma(\nu_{1,x}) + \sigma(x)\big)^p.$$

Changing the inner variable of integration from ν_1 to $\nu_{1,x}$ (which we then write again as ν_1), we see that (3.15) itself is bounded by a constant multiple of the product of the two integrals

$$\int_{N_P(\mathbb{R})\cap N_{\bar{P}_1}(\mathbb{R})} e^{-\rho_P(H_{\bar{P}}(\nu_1))}\big(1+\rho_P\big(H_{\bar{P}}(\nu_1)\big)\big)^{-n'}\big(1+\sigma(\nu_1)\big)^p d\nu_1$$

and

$$\int_{N_{P_1}(\mathbb{R})\cap N_{\bar{Q}}(\mathbb{R})} e^{-\rho_{P_1}(H_{\bar{P}}(x))}\big(1+\rho_P\big(H_{\bar{P}}(x)\big)\big)^{-n'}\big(1+\sigma(x)\big)^p dx,$$

for the positive multiple $n' = \frac{1}{2}n$ of n. The first integral is the analogue of (3.15) with Q and n replaced by P_1 and n'. It converges for large n, by our induction assumption. The terms in the second integrand satisfy $H_{\bar{P}}(x) = H_{\bar{P}_1}(x)$, and

$$\rho_P\big(H_{\bar{P}}(x)\big) = \rho_P\big(H_{\bar{P}_1}(x)\big) = r_1\rho_{P_1}\big(H_{\bar{P}_1}(x)\big),$$

where r_1 is a positive number. The second integral is therefore bounded by a constant multiple of the analogue of (3.15) with P and n replaced by P_1 and n'.

We have reduced the proof to the case that G is quasisplit, $P = P_0$, and P and Q are adjacent. Let G' be the Levi subgroup of G such that $\mathfrak{a}_{G'}$ is the subspace of $\mathfrak{a}_M = \mathfrak{a}_0$ spanned by the common wall of the adjacent chambers $\mathfrak{a}_P^+$ and $\mathfrak{a}_Q^+$. The various terms in (3.15) readily reduce to their analogues for the minimal parabolic subgroups $P' = G' \cap P$ and $Q' = G' \cap Q = \overline{P}'$ of G. From direct computations on the groups $SL(2)$ and $SU(2,1)$, one knows that the function

$$e^{-\rho_0(\log(\bar{a}_0(\nu)))}, \qquad \nu \in N_P(\mathbb{R}),$$

is bounded below by a positive definite quadratic form in the coordinates of ν, and hence that

$$\big(1+\sigma(\nu)\big) \leq c\big(1+\rho_0\big(\log \overline{a}_0(\nu)\big)\big),$$

for some constant c. We may therefore assume that $p = 0$. The assertion of the lemma then follows in this case either by direct computation, or an appeal to Harish-Chandra's original estimate. $\square$

We have now dealt with each of the two expressions whose product was used to bound the integral (3.14). Our conclusion is that the product of (3.14) with (3.3) approaches 0 uniformly for $\gamma \in \Gamma$ as T approaches infinity in $\mathfrak{a}_P^r$. It follows that for any Δ, the contribution of (3.13) to the formula for $J_M(\gamma_T, f_1^S)$ approaches 0 uniformly in γ and S as T approaches infinity in $\mathfrak{a}_P^r$. In calculating the required limit, we can thus replace the integral over m and ν of (3.8) with a sum of integrals over the sets $P(\mathbb{R})_{\Delta,T}$ of expressions obtained from (3.8) by replacing $\phi^S(\lambda_1, h_T)$ with $e^{-\rho_\Delta(\log h_T)}\phi_\Delta^S(\lambda_1, h_T)$. The constant term $\phi_\Delta^S(\lambda_1, h_T)$ is given by (2.6). It equals

$$\phi_\Delta^S(\lambda_1, h_T) = \sum_{w \in W(P_1;P_\Delta)} \phi_{\Delta,w}^S(\lambda_1, h_T),$$

where if w maps $\mathfrak{a}_{P_1}$ to $\mathfrak{a}_{P_1'}$ and $R_\Delta = P_1' \cap M_\Delta$, $\phi_{\Delta,w}^S(\lambda_1, h_T)$ equals

$$E_{R_\Delta}\big(h_T, c_{R_\Delta}(1, w\lambda_1)^{-1} c_{P_1'|P_1}(w, \lambda_1)\mu_{P_1}(\lambda_1)F_1^S(\lambda_1), w\lambda_1\big).$$

We shall write

$$\Phi_{\Delta,w}^S(m, \nu; \gamma, T), \qquad\qquad w \in W(P_1;P_\Delta),\ (m,\nu) \in A_M(\mathbb{R})\backslash M(\mathbb{R}) \times N_P(\mathbb{R}),$$

for the product of

$$e^{-\rho_\Delta(\log h_T)}|W(M_1)|^{-1}\int_{i\mathfrak{a}_1^*} \pi^K\big(\tau(k_T^1)\phi_{\Delta,w}^S(\lambda_1, h_T)\tau(k_T^2)\big)d\lambda_1 \tag{3.16}$$

with

$$|D^M(\gamma)|^{\frac{1}{2}} e^{\rho_P(H_M(\gamma)+T)} v_M\big(n(\nu, m^{-1}\gamma_T m)\big). \tag{3.17}$$

The second factor (3.17) here is just the product of the original normalizing factor (3.3) for $J_M(\gamma_T, f_1^S)$ with the weight factor in the integrand of (3.8). We have established

Lemma 3.4. *The difference*

$$J_M(\gamma_T, f_1^S) - \sum_\Delta \int_{P(\mathbb{R})_{\Delta,T}} \sum_{w \in W(P_1;P_\Delta)} \Phi_{\Delta,w}^S(m, \nu; \gamma, T)d\nu dm \tag{3.18}$$

approaches 0 *uniformly in* γ *and* S *as* T *approaches infinity in* $\mathfrak{a}_P^r$. □

§4. Estimation of the constant terms

We continue with the discussion of the last section. We have reduced the asymptotic study of $J_M(\gamma_T, f_1^S)$ to that of integrals of constant terms attached to f_1^S. Our task is now to estimate these integrals. We shall show that if S is a $U(M, M_1)$-transform of T, all but at most one of the integrals converge to 0 at T approaches infinity. We shall then examine the limit of the remaining integral.

For the reasons of induction mentioned in §3, we shall have to be prepared also to let S be a more general image of T. In fact, at the end of the section, we shall establish some of the estimates for independent parameters T and S, subject only to a weak condition on their relative position. However, our main task is still to carry out a finer analytic argument under the restrictive conditions on T and S. We therefore assume for the time being that $S = uT$, for some $u \in U(M, M_1)$.

The integrals are the terms on the right hand side of (3.18). For any indices of summation Δ and w in (3.18), we shall estimate the integral over $P(\mathbb{R})_{\Delta,T}$ of the corresponding summand $\Phi^S_{\Delta,w}(m,\nu;\gamma,T)$. Among other things, this will allow us to interchange the integral in (3.18) with the sum over w. We recall that the two factors (3.16) and (3.17) of $\Phi^S_{\Delta,w}(m,\nu,\gamma,T)$ both depend on the variables (m,ν) of integration, even though this is explicit in the notation only in the case of (3.17).

The first step is to apply Corollary 2.4, with (P_Δ, h_T) in place of (Q, h), to the factor (3.16). We see that for any n, the norm of (3.16) is bounded by an expression

$$c_n e^{-\rho_0(\log\, h_T)} \sup_{\Delta', w'} \big(1 + \|\log\, h_T - (w'S)_{\Delta'}\|\big)^{-n},$$

for a constant c_n that depends only on n. The supremum is taken over subsets Δ' of Δ and elements $w' \in W(P_1; P_{\Delta'})$ such that $w'w^{-1}$ leaves a_Δ pointwise fixed, while $(w'S)_{\Delta'}$ denotes the projection of $w'S$ onto $\mathfrak{a}_{\Delta'}$. To put this estimate into tractible form, we have to make use of the various assertions of Lemma 3.1.

We can certainly decompose the vector

$$(\log h_T) - (w'S)_{\Delta'}$$

in $\mathfrak{a}_0$ into a sum

$$\big(\log h_T - T - H_M(\gamma) - T_0^G\big) + \big(T - (w'S)_{\Delta'}\big) + \big(H_M(\gamma) + T_0^G\big)$$

of three vectors. According to Lemma 3.1(b), the first of these vectors belongs to the dual chamber ${}^+\mathfrak{a}_0$ of $\mathfrak{a}_0^+$. We are assuming that $S = uT$ is a $U(M, M_1)$ transform of T. It then follows from standard properties of the convex hull of W_0T in $\mathfrak{a}_0$ ([A2, Lemma 3.2(iii)], [A4, Lemma 3.1]) that the second vector

$$T - (w'S)_{\Delta'} = T - (w'uT)_{\Delta'}$$

also lies in the dual chamber ${}^+\mathfrak{a}_0$. Since ${}^+\mathfrak{a}_0$ is the cone spanned by a linearly independent set of vectors in $\mathfrak{a}_0$, we can bound the sum of the first two vectors below by a positive constant multiple of the sum of their norms. We can therefore write

$$\|\log h_T - (w'S)_{\Delta'}\| \geq \delta\big(\|\log h_T - T\| + \|T - (w'S)_{\Delta'}\| - (1+\delta)\|H_M(\gamma) + T_0^G\|\big),$$

for a constant $\delta > 0$. We can therefore write the third vector is the sum of a point T_0^G that depends only on G, and a vector $H_M(\gamma)$ whose norm remains bounded as γ ranges over the bounded set Γ. It then follows that

$$\big(1 + \|\log h_T - (w'S)_{\Delta'}\|\big) \geq \delta_\Gamma\big(1 + \|\log h_T - T\| + \|T - (w'S)_{\Delta'}\|\big), \tag{4.1}$$

for a positive constant δ_Γ that depends only on Γ.

We now use the inequalities of Lemma 3.1(a) to complete the first step of the argument. Combining (3.11) with the lower bound (4.1), we obtain inequalities

$$\begin{aligned}
&\big(1 + \|\log h_T - (w'S)_{\Delta'}\|\big)^{-n} \\
&\leq c_1\big(1 + \|T - (w'S)_{\Delta'}\big)^{-n'}\big(1 + \|\log h_T - T\|\big)^{-n'} \\
&\leq c_2\big(1 + \|T - (w'S)_{\Delta'}\|\big)^{-n'}\big(1 + \rho_0(\log h_T - T)\big)^{-n'} \\
&\leq c_3\big(1 + \|T - (w'S)_{\Delta'}\|\big)^{-n'}\big(1 + \rho_0(\log h')\big)^{-n''}\big(1 + \rho_0(\log \overline{a}_0(\nu'))\big)^{-n''},
\end{aligned}$$

for positive constants c_1, c_2 and c_3, and positive multiples $n' = \frac{1}{2}n$ and $n'' = \frac{1}{4}n$ of n. We then apply the other inequality (3.10) of Lemma 3.1(a) to the remaining factor $e^{-\rho_0(\log\, h_T)}$ in the estimate above for (3.16). We conclude that for any n, the norm of (3.16) is bounded by the product of

$$e^{-\rho_0(T)} \sup_{\Delta', w'} \big(1 + \|T - (w'S)_{\Delta'}\|\big)^{-n},$$
$$e^{-\rho_0(\log\, h')} \big(1 + \rho_0(\log\, h')\big)^{-n},$$

and

$$e^{-\rho_0(\log\, \bar{a}_0(\nu'))} \big(1 + \rho_0(\log\, \overline{a}_0(\nu'))\big)^{-n},$$

with a constant that depends only on n.

The second step is simply to apply the inequality of Lemma 3.2 to the other factor (3.17). Using the fact that $\rho_P(T) = \rho_0(T)$, we see that (3.17) is bounded by a constant multiple of the product of

$$|D^M(\gamma)|^{\frac{1}{2}} e^{\rho_P(H_M(\gamma))} e^{\rho_0(T)}$$

with

$$\big(1 + \sigma(\nu)\big)^p \big(1 + \sigma(m^{-1}\gamma m)\big)^p.$$

We now have an estimate for any summand in (3.18) in terms of a product of five factors. The element h' in the second of the three earlier factors was defined in terms of γ and m. It follows from Harish-Chandra's inequality [Ha1, Lemma 36], as it was applied in the discussion between Lemmas 3.2 and 3.3, that

$$e^{-\rho_0(\log\, h')} \big(1 + \rho_0(\log\, h')\big)^{-n} \big(1 + \sigma(m^{-1}\gamma m)\big)^p$$

is bounded by a constant multiple

$$e^{-\rho_P(H_M(\gamma))} \Xi_M(m^{-1}\gamma m) \big(1 + \sigma(m^{-1}\gamma m)\big)^{-n+p_1},$$

for a constant p_1 that is independent of n. The element ν' in the last of the earlier factors depends on ν and T, as well as γ and m. However, it follows immediately from its definition that $\sigma(\nu)$ equals $\sigma(\nu')$. Taking into account the cancellation of terms from the product of the five factors, we see that for any n, the norm

$$\|\Phi^S_{\Delta,w}(m,\nu;\gamma,T)\|$$

of the summand is bounded by the product of

$$\sup_{\Delta',w'} \big(1 + \|T - (w'S)_{\Delta'}\|\big)^{-n}, \tag{4.2}$$

$$|D_M(\gamma)|^{\frac{1}{2}}\Xi_M(m^{-1}\gamma m)\big(1+\sigma(m^{-1}\gamma m)\big)^{-n+p_1}, \tag{4.3}$$

and

$$e^{-\rho_0(\log(\bar{a}_0(\nu')))}\big(1+\rho_0(\overline{a}_0(\nu'))\big)^{-n}\big(1+\sigma(\nu')\big)^p, \tag{4.4}$$

with a constant that depends only on n. The exponents p and p_1 are constants that depend only on G.

We can of course assume that n is as large as we want. It then follows from [Ha3, Theorem 5] that the integral of (4.3) over m in $A_M(\mathbb{R})\backslash M(\mathbb{R})$ is bounded uniformly for γ in Γ. In dealing with the expression (4.4), we recall from the definition of §3 that $\nu \to \nu'$ is a measure preserving diffeomorphism of $N_P(\mathbb{R})$. It follows from Lemma 3.3 that the integral of (4.4) over ν in $N_P(\mathbb{R})$ converges. We conclude that for any Δ and w, and for any $\gamma \in \Gamma$ and $T \in \mathfrak{a}^r_P$, the integral

$$\int_{A_M(\mathbb{R})\backslash M(\mathbb{R})}\int_{N_P(\mathbb{R})} \|\Phi^S_{\Delta,w}(m,\nu;\gamma,T)\| d\nu dm, \qquad S = uT,$$

is bounded by the product of (4.2) with a constant that depends only on n. In particular, the integral over $P(\mathbb{R})_{\Delta,T}$ in (3.18) can be taken inside the sum over w. In other words, the difference

$$J_M(\gamma_T, f_1^S) - \sum_{\Delta}\sum_{w}\int_{P(\mathbb{R})_{\Delta,T}} \Phi^S_{\Delta,w}(m,\nu;\gamma,T) d\nu dm, \qquad S = uT, \tag{4.5}$$

converges to 0 uniformly in $\gamma \in \Gamma$ as T approaches infinity in $\mathfrak{a}_P^r$.

The bounds (4.2) we have obtained for the summands in (4.5) would be easy to analyze at this point. However, we may as well first establish that most of the summands can be eliminated. We shall show that any summand attached to Δ converges to 0 unless $\Delta = \Delta_0^P$. To simplify the notation in the remaining case, we write

$$P(\mathbb{R})_T = P(\mathbb{R})_{\Delta_0^P, T} = \big\{(m, \nu) :\ \log\big(h_T(m, \nu)\big) \in \mathfrak{a}_{P_0, P}^{r_0}\big\}.$$

If Ω is any subset of $P(\mathbb{R})$, we shall also write Ω_γ for the set of points (m, ν) in $A_M(\mathbb{R})\backslash M(\mathbb{R}) \times N_P(\mathbb{R})$ such that $m^{-1}\gamma m \nu$ belongs to Ω. We then let Ω_γ^c denote the complement of Ω_γ in $A_M(\mathbb{R})\backslash M(\mathbb{R}) \times N_P(\mathbb{R})$.

Lemma 4.1. (a) *For any $\varepsilon > 0$, we can choose a compact subset Ω of $P(\mathbb{R})$ such that*

$$\int_{\Omega_\gamma^c} \|\Phi_{\Delta, w}^S(m, \nu; \gamma, T)\| d\nu dm < \varepsilon, \qquad S = uT,$$

for all Δ, w and γ, and all T in $\mathfrak{a}_P^r$.
(b) *For any compact subset Ω of $P(\mathbb{R})$, we can choose a relatively compact subset C_Ω of a_P^r such that Ω_γ is contained in $P(\mathbb{R})_T$, for all γ and all T in the complement of C_Ω in $\mathfrak{a}_P^r$.*

Proof. (a) This is a recapitulation of the remarks above. The integrand is bounded by a constant multiple of the product of (4.2), (4.3) and (4.4). We want to choose Ω so that the integral of this product over the complement of Ω_γ is small. Observe that if $\Omega = \Omega' \times \Omega''$, for compact sets $\Omega' \subset M(\mathbb{R})$ and $\Omega'' \subset N_P(\mathbb{R})$, then Ω_γ equals $\Omega'_\gamma \times \Omega''$. The term (4.2) is bounded independently of T. The integral of (4.4) over ν in $N_P(\mathbb{R})$ is finite if n is sufficiently large, and can be approximated by the integral over a compact set Ω''. The integral of (4.3) over m in $A_M(\mathbb{R})\backslash M(\mathbb{R})$ is bounded independently of γ, so long as n is sufficiently large. Moreover, the integral over the complement $(\Omega'_\gamma)^c$ of Ω'_γ in $A_M(\mathbb{R})\backslash M(\mathbb{R})$ is bounded by the product of

$$\sup_{m \in (\Omega'_\gamma)^c} \big(1 + \sigma(m^{-1}\gamma m)\big)^{-1}$$

and the integral of the corresponding expression with $(n-1)$ in place of n. Since we can increase the size of n by 1, the latter integral is bounded independently of γ. Given the nature of the function σ, we can make the supremum as small as we wish simply by choosing the compact set $\Omega' \subset M(\mathbb{R})$ to be sufficiently large. The assertion (a) follows.

(b) Recall that for any Δ, $P(\mathbb{R})_{\Delta,T}$ is the set of (m,ν) in $A_M(\mathbb{R})\backslash M(\mathbb{R}) \times N_P(\mathbb{R})$ such that the point $H_T = \log h_T$ belongs to the subset $\mathfrak{a}_{0,\Delta}$ of $\overline{\mathfrak{a}}_0^+$. Recall also that

$$h_T = (m^{-1}\gamma_T m\nu)_0^+ = (\exp T \cdot m^{-1}\gamma m\nu)_0^+.$$

Suppose that (m,ν) belongs to the intersection of $P(\mathbb{R})_{\Delta,T}$ with Ω_γ. We need to show that if $T \in \mathfrak{a}_P^r$ is sufficiently large, Δ must equal Δ_0^P.

Since (m,ν) belongs to Ω_γ, the point $\omega = m^{-1}\gamma m\nu$ lies in the compact set Ω. Let (r,V) be an irreducible finite dimensional representation of G over $\mathbb{R}$, with highest weight $\mu \in \mathfrak{a}_0^*$ relative to (P_0, A_0). As in the proof of Lemma 3.1, we fix a K-invariant Hermitian inner product on $V(\mathbb{C})$ that has an orthonormal basis of weight vectors. We immediately obtain two estimates

$$\begin{aligned} \|r(h_T)\|_2 = \|r(m^{-1}\gamma_T m\nu)\|_2 &= \|r(\exp T\cdot\omega)\|_2 \\ &\leq \|r(\exp T)\|_2\, \|r(\omega)\|_2 \\ &\leq C_1 \|r(\exp T)\|_2 \end{aligned}$$

and

$$\begin{aligned} \|r(\exp T)\|_2 = \|r(m^{-1}\gamma_T m\nu\omega^{-1})\|_2 &\leq \|r(m^{-1}\gamma_T m\nu)\|_2\, \|r(\omega^{-1})\|_2 \\ &= \|r(h_T)\|_2\, \|r(\omega^{-1})\|_2 \\ &\leq C_1 \|r(h_T)\|_2, \end{aligned}$$

where

$$C_1 = \sup_{\omega\in\Omega} \big\{ \|r(\omega)\|_2, \|r(\omega^{-1})\|_2 \big\}.$$

Taking logarithms, we see that

$$\big|\log\|r(h_T)\|_2 - \log\|r(\exp T)\|_2\big| \leq c_1,$$

where $c_1 = \log C_1$. But

$$\|r(h_T)\|_2 = e^{(\log h_T)}\Big(1 + \sum_{\eta \neq \mu} e^{(2\eta - 2\mu)(\log h_T)}\Big)^{\frac{1}{2}}$$

and

$$\|r(\exp T)\|_2 = e^{\mu(T)}\Big(1 + \sum_{\eta \neq \mu} e^{(2\eta - 2\mu)(T)}\Big)^{\frac{1}{2}},$$

where η is summed over the A_0-weights (with multiplicity) of r. It follows that

$$\big|\big(\log\|r(h_T)\|_2 - \log\|r(\exp T)\|_2\big) - \mu(\log h_T - T)\big| \leq \log(\dim V).$$

Therefore

$$|\mu(\log h_T - T)| \leq c_1 + \log(\dim V).$$

Letting μ vary over a basis of $\mathfrak{a}_0^*$, we conclude that $\|\log h_T - T\|$ is bounded independently of T.

The point T belongs to the subset $\mathfrak{a}_P^r$ of $\overline{\mathfrak{a}}_0^+$, where we recall that $r > 0$ was fixed in §3. This set is in turn contained in the subset

$$\mathfrak{a}_{0,P} = \mathfrak{a}_{0,\Delta_0^P} = \mathfrak{a}_{P_0,P}^{r_0}$$

of $\overline{\mathfrak{a}}_0^+$ corresponding to Δ_0^P and r_0. In fact, the distance from T to the complement of $\mathfrak{a}_{0,P}$ in $\mathfrak{a}_0^+$ grows linearly with the norm of T. This is a consequence of the definitions and the fact that r_0 is strictly less than r. The assertion of (b) follows. □

It follows from the lemma that any summand in (4.5) with $\Delta \neq \Delta_0^P$ converges to 0 uniformly in $\gamma \in \Gamma$ as T approaches infinity in $\mathfrak{a}_P^r$. To the remaining case that $\Delta = \Delta_0^P$ we apply the bounds (4.2) for the corresponding summands, which were obtained prior to the lemma. In this case, we have

$$\begin{aligned}\|T - (w'S)_{\Delta'}\| &\geq \|T - (w'S)_{\Delta}\| \\ &= \|T - (wS)_{\Delta}\| = \|T - (wuT)_P\|,\end{aligned}$$

for any of the elements Δ' and w' that index the supremum (4.2) attached to Δ and w. The corresponding function in (4.2) therefore satisfies

$$\big(1 + \|T - (w'S)_{\Delta'}\|\big)^{-n} \leq \big(1 + \|T - (wuT)_P\big)^{-n}.$$

The restriction $\widetilde{w}$ of wu to $\mathfrak{a}_M$ coincides with that of some element in W_0. It maps $\mathfrak{a}_P^+$ to a cone in $\mathfrak{a}_0$ that is disjoint from $\mathfrak{a}_P^+$ unless it is actually trivial on $\mathfrak{a}_M$. Combining this with the fact that

$$\|(wuT)_P\| \leq \|T\|,$$

we deduce that (4.2) approaches 0 as T approaches infinity in $\mathfrak{a}_P^r$ unless $\widetilde{w}$ is the identity transformation of $\mathfrak{a}_M$. The element w is an index of summation in (4.5) that runs over the set $W(P_1; P_\Delta) = W(P_1; P)$. The restriction to $\mathfrak{a}_M$ of its inverse gives the bijection from $W(P_1; P_\Delta)$ to $U(M, M_1)$ of Lemma 1.1. We conclude that a summand in (4.5) corresponding to (Δ, w) converges to 0 uniformly in $\gamma \in \Gamma$ as T approaches infinity in $\mathfrak{a}_P^r$ unless Δ equals Δ_0^P, and u is the image of w under the bijection from $W(P_1; P)$ to the subset $U(M, M_1)$.

Let us write

$$\Phi_P^u(m, \nu; \gamma, T) = \Phi_{\Delta,w}^{uT}(m, \nu; \gamma, T), \qquad u \in U(M, M_1),$$

where $\Delta = \Delta_0^P$ and w is the preimage of u in $W(P_1; P)$. Lemma 4.1 implies that the integral of this function over (m, ν) in the complement of $P(\mathbb{R})_T$ converges to 0 uniformly in $\gamma \in \Gamma$ as T approaches infinity is $\mathfrak{a}_P^r$. We have obtained the following refinement of Lemma 3.4.

Lemma 4.2. *The difference*

$$J_M(\gamma_T, f_1^{uT}) - \int_{A_M(\mathbb{R})\backslash M(\mathbb{R})} \int_{N_P(\mathbb{R})} \Phi_P^u(m, \nu; \gamma, T) dn dm$$

converges to 0 *uniformly in* $\gamma \in \Gamma$ *as* T *approaches infinity in* $\mathfrak{a}_P^r$. $\square$

We have established that the limit of $J_M(\gamma_T, f_1^{uT})$ equals

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \int_{A_M(\mathbb{R})\backslash M(\mathbb{R})} \int_{N_P(\mathbb{R})} \Phi_P^u(m,\nu;\gamma,T)d\nu dm, \tag{4.6}$$

provided that the latter limit exists. To investigate (4.6), we need to be able to replace the domain of integration by a compact set. We first choose a positive number ε that is to be fixed until the end of the argument. It then follows from Lemma 4.1(a) that there is a compact subset Ω of $P(\mathbb{R})$ such that

$$\int_{\Omega_\gamma^c} |\Phi_P^u(m,\nu;\gamma,T)| d\nu dm < \frac{\varepsilon}{3}, \tag{4.7}$$

for any $\gamma \in \Gamma$ and $T \in \mathfrak{a}_P^r$. It will suffice to consider the limit of the integral

$$\int_{\Omega_\gamma} \Phi_P^u(m,\nu;\gamma,T) dm d\nu.$$

Our task will be to calculate the pointwise limit of the integrand.

Recall that $\Phi_P^u(m,\nu;\gamma,T)$ is the product of the function (3.17) with the value taken by (3.16) when $\Delta = \Delta_0^P$, $S = uT$, and w is the preimage of u in $W(P_1;P)$. The corresponding value of the constant term $\phi_{\Delta,w}^S(\lambda_1, h_T)$ in (3.16) is given by the general formula (2.6). It equals

$$E_R\big(h_T, \mu_R(\Lambda)F_R^T(\Lambda), \Lambda\big), \qquad\qquad R = P_1' \cap M, \Lambda = w\lambda_1,$$

in the notation of §2. In particular,

$$F_R^T(\Lambda) = F_R(\Lambda)e^{-\Lambda(T)} = F_R(\Lambda)e^{-\lambda_1(S)},$$

where $F_R(\Lambda)$ is defined as in the statement of Corollary 2.3. Let us write

$$E_R\big(h_T, \mu_R(\Lambda)F_R^T(\Lambda), \Lambda\big) = E_R\big(h_T^1, \mu_R(\Lambda)F_R(\Lambda), \Lambda\big)e^{\Lambda(H_M(h_T)-T)},$$

where

$$m^1 = m \exp\big(H_M(m)\big)^{-1}$$

denotes the projection of a point $m \in M(\mathbb{R})$ onto $M(\mathbb{R})^1$. Changing variables from λ_1 to Λ in the integral in (3.16), we can then express $\Phi_P^u(m,\nu;\gamma,T)$ as the product of

$$|D^M(\gamma)|^{\frac{1}{2}} e^{\rho_P(H_M(\gamma))}, \tag{4.8}$$

$$|W(M_1)|^{-1} \int_{i\mathfrak{a}_R^*} \pi^K\big(\tau(k_T^1)E_R(h_T^1,\mu_R(\Lambda)F_R(\Lambda),\Lambda)\tau(k_T^2)\big)e^{(\Lambda-\rho_P)(H_M(h_T)-T)}d\Lambda, \tag{4.9}$$

and

$$v_M\big(n(\nu, m^{-1}\gamma_T m)\big), \tag{4.10}$$

since $\rho_\Delta = \rho_P$. As functions of m and ν, (4.9) and (4.10) are both quite complicated. However, we shall see that both functions converge to simpler expressions as T approaches infinity.

We write

$$m^{-1}\gamma_T m\nu = m^{-1}\gamma_T m n_{\bar{P}}(\nu) m_{\bar{P}}(\nu) a_{\bar{P}}(\nu) k_{\bar{P}}(\nu),$$

in the notation of (3.9). This in turn equals

$$\kappa_P m^{-1}\gamma_T m m_{\bar{P}}(\nu) a_{\bar{P}}(\nu) k_{\bar{P}}(\nu),$$

where

$$\kappa_P = \mathrm{Int}(m^{-1}\gamma_T m) n_{\bar{P}}(\nu) = \mathrm{Int}(\exp T)\big(\mathrm{Int}(m^{-1}\gamma m) n_{\bar{P}}(\nu)\big).$$

We are assuming that (m,ν) lies in the set Ω_γ, so that $m^{-1}\gamma m\nu$ lies in the compact set Ω. Therefore

$$\mathrm{Int}(m^{-1}\gamma m) n_{\bar{P}}(\nu)$$

lies in a compact subset of $N_{\bar{P}}(\mathbb{R})$ that is independent of γ. It follows that κ_P converges to 1 uniformly in m, ν and γ as T approaches infinity in $\mathfrak{a}_P^r$. We have shown that the two points

$$m^{-1}\gamma_T m\nu = k_T^1 h_T k_T^2$$

and

$$m^{-1}\gamma_T m m_{\bar{P}}(\nu)a_{\bar{P}}(\nu)k_{\bar{P}}(\nu) = m^{-1}\gamma m m_{\bar{P}}(\nu)(\exp T)a_{\bar{P}}(\nu)k_{\bar{P}}(\nu)$$

differ only up to a left translate by some element that converges uniformly to 1. It is then easily seen, by [A8, Lemma 5.2] for example, that the respective components of the polar decompositions of the two points also differ only by translation by elements that converge to 1. But the polar decomposition of the second point can obviously be expressed in terms of the polar decomposition in $M(\mathbb{R})$ of the point $m^{-1}\gamma m m_{\bar{P}}(\nu)$. This gives us an asymptotic approximation to the polar decomposition of the first point.

We apply these remarks to the limit of the integrand in (4.9). We see that

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \big(\tau(k_T^1)E_R(h_T^1,\psi_R,\Lambda)\tau(k_T^2)\big) = E_R\big(m^{-1}\gamma^1 m m_{\bar{P}}(\nu),\psi_R,\Lambda\big)\tau\big(k_{\bar{P}}(\nu)\big),$$

for any vector $\psi_R \in \mathcal{A}_{\mathrm{cusp}}(M_R,\tau_R)$, and that

$$\lim_{\substack{T \longrightarrow \infty \\ \bar{P},r}} \big(H_M(h_T) - T\big) = H_M\big(\gamma a_{\bar{P}}(\nu)\big) = H_{\bar{P}}(m^{-1}\gamma m\nu).$$

The first limit is uniform in Λ, and both limits are uniform in m, ν and $\gamma \in \Gamma$. Let us write

$$E_{R,\bar{P}}(x,\psi_R,\Lambda) = E_R\big(m_{\bar{P}}(x),\psi_R,\Lambda\big)\tau\big(k_{\bar{P}}(x)\big)e^{(\Lambda-\rho_P)(H_{\bar{P}}(x))}, \tag{4.11}$$

for any $x \in G(\mathbb{R})$. The limit

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \big(\tau(k_T^1)E_R(h_T^1,\psi_R,\Lambda)\tau(k_T^2)\big)e^{(\Lambda-\rho_P)(H_M(h_T)-T)}$$

is then equal to

$$E_{R,\bar{P}}(m^{-1}\gamma m\nu,\psi_R,\Lambda),$$

uniformly in Λ, m, ν and $\gamma \in \Gamma$. We shall apply this limit formula to (4.9) in a moment.

Let us first recall that

$$\nu = (m^{-1}\gamma_T m)^{-1}n^{-1}(m^{-1}\gamma_T m)n = \big(\mathrm{Int}(\exp T)^{-1}\mathrm{Int}(m^{-1}\gamma m)^{-1}n^{-1}\big)n.$$

Since $m^{-1}\gamma m$ and ν remain in fixed compact sets, so does n. The point ν therefore converges to n uniformly in γ as T approaches infinity in $\mathfrak{a}_P^r$. Applying this to the inverse mapping $\nu \to n$, we see that

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} n(\nu, m^{-1}\gamma_T m) = \nu.$$

We obtain the limit formula

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} v_M\big(n(\nu, m^{-1}\gamma_T m)\big) = v_M(\nu)$$

for the function (4.10), which is again uniform in m, ν and γ.

We have now established that the limit in T of the integrand in (4.9) equals

$$\pi^K\big(E_{R,\bar{P}}(m^{-1}\gamma m\nu, \mu_R(\Lambda)F_R(\Lambda), \Lambda)\big).$$

According to Corollary 2.3, the function $F_R(\Lambda)$ belongs to the space $\mathcal{C}\big(i\mathfrak{a}_R^*, \mathcal{A}_{\mathrm{cusp}}(M_R, \tau_R)\big)$. Since the μ-function $\mu_R(\Lambda)$ is slowly increasing, the product $\mu_R(\Lambda)F_R(\Lambda)$ is rapidly decreasing in Λ. It follows that the limit of the product of the integrand in (4.9) with any polynomial in Λ is uniform in Λ. This implies that the limit in T of the integral (4.9) itself exists, and can be taken inside the integral over $i\mathfrak{a}_R^*$. Combining this with the limit we have obtained for (4.10), we conclude that

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \Phi_P^u(m, \nu; \gamma, T) = \Phi_P^u(m, \nu; \gamma),$$

where $\Phi_P^u(m, \nu; \gamma)$ equals the product of (4.8) with

$$(4.12) \qquad |W(M_1)|^{-1} \int_{i\mathfrak{a}_R^*} \pi^K\big(E_{R,\bar{P}}(m^{-1}\gamma m\nu, \mu_R(\Lambda)F_R(\Lambda), \Lambda)\big)v_M(\nu)d\Lambda.$$

The limit of $\Phi_P^u(m, \nu; \gamma, T)$ is uniform in $(m, \nu) \in \Omega$, as well as $\gamma \in \Gamma$. The limit

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \int_{\Omega_\gamma} \Phi_P^u(m, \nu; \gamma, T)d\nu dn$$

therefore exists, and equals

$$\int_{\Omega_\gamma} \Phi_P^u(m,\nu;\gamma)d\nu dm.$$

Moreover, we can use the Weyl discriminant $|D^M(\gamma)|^{\frac{1}{2}}$ in the factor (4.8) to control the behaviour of the integrals over Ω_γ as γ approaches the singular set. It is a consequence of [Ha3, Theorem 5] and the definition of Ω_γ that the last limit is uniform in γ. This discussion applies of course to any compact subset Ω of $P(\mathbb{R})$. It implies that for any such Ω, and $\varepsilon > 0$ as above, there is a relatively compact subset $C_{\Omega,\varepsilon}$ of $\mathfrak{a}_P^r$ such that

$$\Big|\int_{\Omega_\gamma} \Phi_P^u(m,\nu;\gamma,T)d\nu dm - \int_{\Omega_\gamma} \Phi_P^u(m,\nu;\gamma)d\nu dm\Big| < \frac{\varepsilon}{3}, \tag{4.13}$$

for all $\gamma \in \Gamma$ and all T in the complement of $C_{\Omega,\varepsilon}$ in $\mathfrak{a}_P^r$.

The function (4.12) is actually integrable over (m,ν) in the entire set $A_M(\mathbb{R})\backslash M(\mathbb{R}) \times N_P(\mathbb{R})$. In fact, the product of (4.8) with the integral of the absolute value of (4.12) is bounded independently of $\gamma \in \Gamma$. This follows from Lemmas 3.2 and 3.3, Theorem 5 of [Ha3], and the fact that the product $\mu_R(\Lambda)F_R(\Lambda)$ in the integrand of (4.12) is a Schwartz function of $\Lambda \in i\mathfrak{a}_R^*$. We may therefore choose the compact set Ω so that

$$\int_{\Omega_\gamma^c} |\Phi_P^u(m,\nu;\gamma)|d\nu dm < \frac{\varepsilon}{3}, \tag{4.14}$$

for all $\gamma \in \Gamma$.

It remains only to combine the estimates (4.7), (4.13) and (4.14) in the obvious way. Our conclusion is that the limit (4.6) exists, and equals the integral of $\Phi_P^u(m,\nu;\gamma)$ over all m and ν, uniformly in $\gamma \in \Gamma$. Recalling Lemma 4.2, we obtain the following lemma, which can be regarded as a summary of the work we have done in the last two sections.

Lemma 4.3. *The τ-spherical function*

$$f_1^{uT} = (F_1^{uT})^\vee, \qquad\qquad F_1 \in \mathcal{C}\big(i\mathfrak{a}_1^*, \mathcal{A}_{\mathrm{cusp}}(M_1,\tau_1), u \in U(M,M_1)\big),$$

satisfies the uniform limit formula

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \big(J_M(\gamma_T, f_1^{uT})\big) = \Phi_P^u(\gamma), \qquad \gamma \in \Gamma, \tag{4.15}$$

where

$$\Phi_P^u(\gamma) = \int_{A_M(\mathbb{R})\backslash M(\mathbb{R})} \int_{N_P(\mathbb{R})} \Phi_P^u(m,\nu;\gamma)d\nu dm. \tag{4.16}$$ $\square$

We now pause to see how the early part of the discussion above applies to more general points S. The restrictive condition that $S = uT$ was used only in deriving the inequality (4.1). To describe a weaker substitute for this condition, we let $c_0 = c_0^+$ be any open cone in $\mathfrak{a}_0$ that is contained in $\mathfrak{a}_0^+$. The dual cone

$$^+c_0 = \big\{X \in \mathfrak{a}_0 : \ (X,H) > 0, \ H \in c_0\big\}$$

then contains $^+\mathfrak{a}_0$. Recall that M and M_1 are both standard with respect to P_0, so in particular, the spaces $\mathfrak{a}_M = \mathfrak{a}_P$ and $\mathfrak{a}_1 = \mathfrak{a}_{M_1} = \mathfrak{a}_{P_1}$ are both contained in $\mathfrak{a}_0$. We assume that the closure of c_0 intersects $\mathfrak{a}_P^+$ in a subset with nonempty interior. We shall say that $T \in \mathfrak{a}_P^+$ is (c_0, S)-*dominant*, for a point $S \in \mathfrak{a}_1$, if $T - wS$ belongs to the closure $^+\bar{c}_0$ of $^+c_0$ for every $w \in W_0$. If this is so, we set

$$d_{c_0}(T,S) = \inf_{w\in W_0} \|T - wS\|.$$

Notice that these definitions do not actually require that M_1 be standard. They remain valid if M_1 is a general Levi subgroup, so long as we replace W_0 by the set $U(M_1, M_0)$.

For example, suppose that $\widetilde{W}_0$ is a Coxeter group for some root system for $\mathfrak{a}_0$ that contains the roots of (G, A_0). We take c_0 to be the chamber $\widetilde{\mathfrak{a}}_0^+$ defined by a system of positive roots that contains Δ_0, such that the closure of $\widetilde{\mathfrak{a}}_0^+$ intersects $\mathfrak{a}_P^+$ in a set with nonempty interior $\widetilde{\mathfrak{a}}_P^+$. Suppose that $S = uT$, where T is restricted to the subset $\widetilde{\mathfrak{a}}_P^+$ of $\mathfrak{a}_P^+$, and u belongs to the set $\widetilde{U}(M, M_1)$ of linear injections from $\mathfrak{a}_M$ to $\mathfrak{a}_{M_1}$ induced by elements in the larger group $\widetilde{W}_0$. It then follows from the properties of the convex hull of $\widetilde{W}_0 T$ in $\mathfrak{a}_0$ that $T - wS$ lies in the closure of $^+\widetilde{\mathfrak{a}}_0$ for every $w \in \widetilde{W}_0$. Since $\widetilde{W}_0$ contains W_0, T is thus (c_0, S)-dominant in this case.

The next lemma gives a a general estimate in terms of the chamber $c_0 \subset \mathfrak{a}_0^+$ and the relatively compact set $\Gamma \subset T_{G\text{-reg}}(\mathbb{R})$. We formulate it in a way that is independent of the choice of two-sided representation (τ, V) of K, even though this is stronger than we need for the present application. As a uniform estimate in τ, it may be regarded as a step towards generalizing our results from $\mathcal{H}(G)$ to $\mathcal{C}(G)$, even though it is stronger than will be needed for present applications.

Lemma 4.4. *For any $n \geq 0$, there is a continuous seminorm $\|\cdot\|_n$ on $\mathcal{C}(G)$ such that*

$$\|J_M(\gamma_T, f_1^S)\| \leq \|f_1\|_n\big(1 + d_{c_0}(T,S)\big)^{-n}, \qquad f_1^S = (F_1^S)^\vee, \tag{4.17}$$

for any (τ, V) and $\gamma \in \Gamma$, any $F_1 \in \mathcal{C}\big(i\mathfrak{a}_1, \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)\big)$ that satisfies the symmetry condition (2.2), *and any $T \in \mathfrak{a}_P^r$ and $S \in \mathfrak{a}_1$ such that T is (c_0, S)-dominant.*

Proof. On the left hand side of (4.17), $\|\cdot\|$ is the norm attached to any inner product on the finite-dimensional space V for which τ is unitary. On the right hand side, $\|f_1\|_n$ represents the value at the function $f_1 = F_1^\vee$ of the seminorm on $\mathcal{C}(G,V)$ defined by $\|\cdot\|_n$ and $\|\cdot\|$. It therefore makes sense that the seminorm on $\mathcal{C}(G)$ be independent of (τ, V).

We shall use the estimates derived prior to Lemma 4.1. We could apply these again to the summands in (3.18). However, since we are aiming only for an upper bound rather than an asymptotic formula, we may as well combine them directly with the estimate of Lemma 2.2.

Recall the discussion at the beginning of §3. It allows us to write $J_M(\gamma_T, f_1^S)$ as the integral over $A_M(\mathbb{R})\backslash M(\mathbb{R})$ and $\nu \in N_P(\mathbb{R})$ of the product of (3.3) and (3.8). This product is the same as the product of the expression

$$\pi^K\big(\tau(k_T^1)(F_1^S)^\vee(h_T)\tau(k_T^2)\big) \tag{4.18}$$

with (3.17). It follows from Lemma 2.2, together with the remark after its proof, that there is a continuous seminorm $\|\cdot\|_n$ on $\mathcal{C}(G)$ such that the norm of (4.18) is bounded by

$$\|f_1\|_n e^{-\rho_0(\log h_T)} \sup_{\Delta', w'} \big(1 + \|\log h_T - (w'S)_{\Delta'}\|\big)^{-n}.$$

The last expression is similar to the bound with which we began the discussion of this section, apart from the fact that the supremum is now taken over *all* subsets Δ' of Δ_0

and *all* elements $w' \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_{\Delta'}})$, a consequence of our having used Lemma 2.2 in place of Corollary 2.4. To apply the earlier discussion here, we need to establish the inequality (4.1) under the weaker condition on T and S.

The cone c_0 contains a smaller open cone

$$\widetilde{c}_0 = \big\{H \in \mathfrak{a}_0 : \ \widetilde{\alpha}(H) > 0, \ \widetilde{\alpha} \in \widetilde{\Delta}_0\big\},$$

defined by a linearly independent subset $\widetilde{\Delta}_0$ of $\mathfrak{a}_0^*$. The corresponding dual cone

$${}^+\widetilde{c}_0 = \big\{X \in \mathfrak{a}_0 : \ (X, H) > 0, \ H \in \widetilde{c}_0\big\}$$

then contains ${}^+c_0$. It therefore contains the vector

$$\log h_T - T - H_M(\gamma) - T_0^G,$$

which lies in ${}^+\mathfrak{a}_0$ by Lemma 3.1(b). Since T is assumed to be (c_0, S)-dominant, ${}^+\widetilde{c}_0$ contains the set $T - W_0 S$, and hence also the convex hull of this set. But any point $(w'S)_{\Delta'}$ represented in bound above lies in the convex hull of $W_0 S$, since it is the projection of a vertex onto to subspace $\mathfrak{a}_{\Delta'}$ of $\mathfrak{a}_0$. It follows that $T - (w'S)_{\Delta'}$ belongs to the closure of ${}^+\widetilde{c}_0$. On the other hand, $\widetilde{c}_0$ is the open cone generated by a linearly independent subset of $\mathfrak{a}_0$. It follows that there is a positive constant δ such that

$$\|X_1 + X_2\| \geq \delta(\|X_1\| + \|X_2\|), \tag{4.19}$$

for any two vectors X_1 and X_2 in the closure of ${}^+\widetilde{\mathfrak{a}}_0$. We then deduce the inequality (4.1) for T and S as before.

We can now follow the discussion prior to Lemma 4.1. We see that $\|J_M(\gamma_T, f_1^S)\|$ is bounded by the integral over m and ν of the product of $\|f\|_n$, (4.2), (4.3) and (4.4), and a constant that depends only on n. We can assume that n is large. Then as we noted earlier, the integral of (4.3) over m is bounded uniformly for γ in Γ, and the integral of (4.4) over ν is finite. To deal with the points $(w'S)_{\Delta'}$ represented in (4.2), we write

$$T - (w'S)_{\Delta'} = (T - S_1) + \big(S_1 - (w'S)_{\Delta'}\big),$$

where $S_1 = w_1 S$ is the W_0-translate of S that lies in the closure of ${}^+\mathfrak{a}_0$. The two vectors on the right both belong to the closure ${}^+\overline{c}_0$ of ${}^+c_0$. Moreover, the norm of $T - S_1$ equals the infimum $d_{c_0}(T,S)$, since T and S_1 both lie in $\bar{\mathfrak{a}}_0^+$. It follows from (4.19) that

$$\begin{aligned}\|T - (w'S)_{\Delta'}\| &\geq \delta\big(\|T - S_1\| + \|S_1 - (w'S)_{\Delta'}\|\big)\\ &\geq \delta d_{c_0}(T,S).\end{aligned}$$

The function (4.2) is therefore bounded by a constant multiple of

$$\big(1 + d_{c_0}(T,S)\big)^{-n}.$$

Adjusting $\|\cdot\|_n$ by a multiplicative constant, we obtain the required estimate (4.17). $\square$

§5. Weighted virtual characters and spherical transforms

It remains for us to apply Lemma 4.3 to the desired limit (1.15). We shall do so in the next section. More precisely, we shall compute the limit (4.15) of Lemma 4.3 in terms of distributions that closely resemble the weighted characters (1.3). In this section we shall collect various results and definitions needed for the application of Lemma 4.3. Among other things, we shall discuss variants of weighted characters, and some relations of these objects with spherical functions.

We begin with some remarks on the invariant Schwartz space $\mathcal{I}(G)$. This space (or rather its analogue for L) is an essential part of the construction of the invariant distributions (1.4). However, the presence of reducible induced representations complicates the description of $\mathcal{I}(G)$ as a Schwartz space of functions in $\Pi_{\mathrm{temp}}(G)$. The set $\Pi_{\mathrm{temp}}(G)$ of course parametrizes the irreducible tempered characters $\{\Theta(\pi)\}$ on $G(\mathbb{R})$. It is often preferable to work with a second basis $\{\Theta(\tau)\}$ of virtual characters, parametrized by the set $T(G)$ defined in [A9, §3]. We shall pause to review the properties of this basis, and the corresponding modification of the mappings $\phi_L(f)$ in (1.4).

In this paper, we shall write $T_{\mathrm{temp}}(G)$ for the set denoted $T(G)$ in [A9], since the latter set indexes tempered virtual characters. We reserve the symbol $T(G)$ for a larger set that indexes a basis of general virtual characters, and which may be described as follows. We first recall that $T_{\mathrm{temp}}(G)$ is a disjoint union

$$T_{\mathrm{temp}}(G) = \coprod_{\{M_1\}} \big(T_{\mathrm{temp,ell}}(M_1)/W(M_1)\big). \tag{5.1}$$

As earlier, $\{M_1\}$ is a set of representatives of $G(\mathbb{R})$-conjugacy classes of cuspidal Levi subgroups of G, while $T_{\mathrm{temp,ell}}(M_1)$ denotes the set of elements $\tau \in T_{\mathrm{temp}}(M_1)$ that are elliptic, in the sense that $\Theta(\tau)$ does not vanish on the regular elliptic set in $M_1(\mathbb{R})$. If ε belongs to $\mathfrak{a}_G^*$, we write $T_\varepsilon(G)$ for the set $\{\tau_\varepsilon\}$ that indexes the (nontempered) characters

$$\Theta(\tau_\varepsilon, x) = \Theta(\tau, x)e^{\varepsilon(H_G(x))}, \qquad \tau \in T_{\mathrm{temp}}(G),\ x \in G_{\mathrm{reg}}(\mathbb{R}).$$

We also write

$$T_{\mathrm{ell}}(G) = \coprod_{\varepsilon \in \mathfrak{a}_G^*} T_{\varepsilon,\mathrm{ell}}(G),$$

where $T_{\varepsilon,\mathrm{ell}}(G)$ denotes the set of elliptic elements in $T_\varepsilon(G)$. Then $T(G)$ is a disjoint union

$$T(G) = \coprod_{\{M_1\}} \big(T_{\mathrm{ell}}(M_1)/W(M_1)\big). \tag{5.2}$$

Each set $T_{\mathrm{temp,ell}}(M_1)$ on the right hand side of (5.1) may be interpreted as a disjoint union of real affine spaces, the union of whose complexifications is the corresponding set $T_{\mathrm{ell}}(M)$ on the right hand side of (5.2). We can identify $\mathcal{I}(G)$ with a space of functions

$$f_G(\tau) = \Theta(\tau, f) = \int_{G(\mathbb{R})} \Theta(\tau, x) f(x) dx, \qquad f \in \mathcal{C}(G),\ \tau \in T_{\mathrm{temp}}(G),$$

on $T_{\mathrm{temp}}(G)$. With this interpretation, $\mathcal{I}(G)$ becomes a direct sum of Schwartz spaces [A11]. (More precisely, $\mathcal{I}(G) = I\mathcal{C}(G)$ is a "rapidly decreasing" direct sum of Schwartz spaces, with symmetry conditions defined by the action of the finite groups $W(M_1)$.)

We shall presently have reason to consider the Hecke algebra $\mathcal{H}(G)$ on G, in addition to the Schwartz space $\mathcal{C}(G)$. We recall that $\mathcal{H}(G) = \mathcal{H}(G, K)$ is the space of smooth, K-finite functions of compact support on $G(\mathbb{R})$. Clozel and Delorme [CD] have characterized the image

$$I\mathcal{H}(G) = \big\{f_G :\ f \in \mathcal{H}(G)\big\}$$

of $\mathcal{H}(G)$ in $\mathcal{I}(G)$. As a space of functions on $T_{\mathrm{temp}}(G)$, $I\mathcal{H}(G)$ is a direct sum of Paley-Wiener spaces, defined by restriction to $T_{\mathrm{temp}}(G)$ of functions of Paley-Wiener type [A9, p. 93] on the complexification $T(G)$. (More precisely, $I\mathcal{H}(G)$ is an algebraic direct sum of Paley-Wiener spaces, with symmetry conditions defined by the action of the finite groups $W(M_1)$.)

The invariant orbital integral (1.1) can be described as a function on $T_{\mathrm{temp}}(G)$. From the results of [He] or [A10], one obtains an expansion

$$f_G(\gamma) = \int_{T_{\mathrm{temp}}(G)} \theta_G(\gamma, \tau) f_G(\tau) d\tau, \qquad f \in \mathcal{C}(G), \tag{5.3}$$

for a smooth tempered function

$$\theta_G(\gamma, \tau), \qquad \tau \in T_{\text{temp}}(G),$$

on $T_{\text{temp}}(G)$, and a natural measure $d\tau$ on $T_{\text{temp}}(G)$. (In [A10], (5.3) is the special case of Theorem 4.1 in which $M = G$. In this case, the function

$$I_G(\gamma, \tau) = I_G^G(\gamma, \tau), \qquad \tau \in T_{\text{disc}}(L),$$

in the expansion [A10, (4.1)] is supported on the subset $T_{\text{ell}}(L)$, and (5.3) follows from the decomposition of $T_{\text{temp}}(G)$ given by (5.1).) The function $\theta_G(\gamma, \tau)$ in (5.3) satisfies the formula

$$\theta_G(\gamma, \tau_\lambda) = e^{-\lambda(H_G(\gamma))}\theta_G(\gamma, \tau), \qquad \lambda \in i\mathfrak{a}_G^*.$$

It therefore continues analytically to a tempered function on the space $T_\varepsilon(G)$, for any $\varepsilon \in \mathfrak{a}_G^*$.

If M is our fixed Levi subgroup, and $\mathcal{I}(M)$ is to be regarded as a space of functions on $T_{\text{temp}}(M)$, we shall have to formulate the weighted character $J_M(\cdot, f) = \phi_M(f, \cdot)$ as a Schwartz function on $T_{\text{temp}}(M)$. Let us first review the weighted character $J_M(\pi, f)$ of (1.3), defined for a representation $\pi \in \Pi_{\text{temp}}(M)$ as in [A12].

The (G, M)-family $\mathcal{M}_G(\zeta, \pi, P)$ that goes into the construction (1.3) is defined by a product

$$\mathcal{J}_Q(\zeta, \pi, P) = m_Q(\zeta, \pi, P)\mathcal{M}_Q(\zeta, \pi, P), \qquad \zeta \in i\mathfrak{a}_M^*, \ Q \in \mathcal{P}(M). \tag{5.4}$$

The left hand side is a (G, M)-family of operator valued functions

$$\mathcal{J}_Q(\zeta, \pi, P) = J_{Q|P}(\pi)^{-1}J_{Q|P}(\pi_\zeta)$$

on $\mathcal{H}_P(\pi)$, in which

$$J_{Q|P}(\pi) : \ \mathcal{H}_P(\pi) \longrightarrow \mathcal{H}_Q(\pi)$$

is the standard (unnormalized) intertwining operator, defined for almost all π. The first factor on the right hand side of (5.4) represents a (G, M)-family of scalar valued functions

$$m_Q(\zeta, \pi, P) = m_{Q|P}(\pi)^{-1} m_{Q|P}(\pi_{\frac{1}{2}\zeta}),$$

where

$$m_{Q|P}(\pi) = \mu_{Q|P}(\pi)^{-1}$$

is the inverse of the partial μ-function

$$\mu_{Q|P}(\pi) = \big(J_{P|Q}(\pi) J_{Q|P}(\pi)\big)^{-1}, \tag{5.5}$$

defined again for almost all π. The elements in both (G, M)-families extend to meromorphic functions of $\pi \in \Pi(M)$. The number $\mathcal{M}_M(\pi, P)$ attached to the quotient (G, M)-family $\{\mathcal{M}_Q(\zeta, \pi, P)\}$ is an analytic function of $\pi \in \Pi_{\text{temp}}(M)$, any derivative of which is slowly increasing. It follows that for any $f \in \mathcal{C}(G)$, $J_M(\pi, f)$ is a Schwartz function of $\pi \in \Pi_{\text{temp}}(M)$. (See [A12, §2].)

Suppose now that τ belongs to $T(M)$. According to the definitions [A9], τ is an $M(\mathbb{R})$-orbit of triplets

$$(M_\tau, \rho_\tau, r_\tau),$$

where M_τ is a cuspidal Levi subgroup of M, ρ_τ belongs to the subset $\Pi_{\text{cusp}}(M_\tau)$ of representations in $\Pi(M_\tau)$ that are square integrable modulo $A_\tau = A_{M_\tau}$, and r_τ lies in the R-group $R^M(\rho_\tau)$ of τ relative to M_τ. For any parabolic subgroup $R_\tau \in \mathcal{P}^M(M_\tau)$, we form the induced representation

$$\pi = \mathcal{I}_{R_\tau}(\rho_\tau)$$

of $M(\mathbb{R})$. By normalizing the associated standard intertwining operators, we obtain a representation

$$w \longrightarrow \widetilde{R}(w, \rho_\tau) = A(\rho_\tau^w) R_{w^{-1}R_\tau w|R_\tau}(\rho_\tau), \qquad w \in R^M(\rho_\tau), \tag{5.6}$$

of $R^M(\rho_\tau)$ on $\mathcal{H}_{R_\tau}(\rho_\tau)$ that commutes with $\mathcal{I}_{R_\tau}(\rho_\tau)$. The operator

$$A(\rho_\tau^w):\ \mathcal{H}_{w^{-1}R_\tau w}(\rho_\tau) \longrightarrow \mathcal{H}_{\rho_\tau}(\rho_\tau)$$

here is given by an extension ρ_τ^w of ρ_τ to the group $M_\tau^w(\mathbb{R})$ generated by $M_\tau(\mathbb{R})$ and (a representative in K of) w, while $R_{w^{-1}R_\tau w|R_\tau}(\rho_\tau)$ is the product of $J_{w^{-1}R_\tau w|R_\tau}(\rho_\tau)$ with a scalar normalizing factor. (See [A9, §2]. In retaining conventions from [A9], we are asking the reader to tolerate some overlapping notation. In particular, we are using the symbol R for the finite group $R^M(\rho_\tau)$, a parabolic subgroup R_τ, and the operator $\widetilde{R}(w,\rho_\tau)$.)

Assume now that τ belongs to the subset $T_{\mathrm{temp}}(M)$ of $T(M)$. Setting $w = r_\tau$, we define the virtual character

$$f_M(\tau) = \mathrm{tr}\big(\widetilde{R}(r_\tau,\rho_\tau)\mathcal{I}_P(\pi,f)\big), \qquad f \in \mathcal{C}(G),$$

where $\widetilde{R}(r_\tau,\rho_\tau)$ acts on the space $\mathcal{H}_P(\pi)$ through its action on $\mathcal{H}_{R_\tau}(\rho_\tau)$. More generally, the weighted character attached to τ is defined by

$$J_M(\tau,f) = \mathrm{tr}\big(\widetilde{R}(r_\tau,\rho_\tau)\mathcal{M}_M(\pi,P)\mathcal{I}_P(\pi,f)\big). \tag{5.7}$$

We can then use the mappings

$$\phi_M(f):\ \tau \longrightarrow \phi_M(f,\tau) = J_M(\tau,f)$$

(with L in place of M) in the definition (1.4).

It is actually a variant of (5.7) that will be the main term in our asymptotic formula for $J_M(\gamma_T, f_T)$. We modify the right hand side of (5.7) by replacing the operator $\mathcal{M}_M(\pi,P)$ with a second operator

$$\mathcal{J}_M(\pi,P) = \lim_{\zeta\to 0} \sum_{Q\in\mathcal{P}(M)} \mathcal{J}_Q(\zeta,\pi,P)\theta_Q(\zeta)^{-1},$$

obtained from the (G,M)-family (5.4). The corresponding weighted character

$$J_M^P(\tau,f) = \mathrm{tr}\big(\widetilde{R}(r_\tau,\rho_\tau)\mathcal{J}_M(\pi,P)\mathcal{I}_P(\pi,f)\big) \tag{5.8}$$

is defined for any $f \in \mathcal{H}(G)$ as a meromorphic function of $\tau \in T(M)$. This will be our variant of (5.7). Unlike its better behaved counterpart $J_M(\tau, f)$, $J_M^P(\tau, f)$ depends on the choice of P, and could also have poles that lie in the tempered subspace $T_{\mathrm{temp}}(M)$ of $T(M)$. (See [A12, §2].) However, if $\varepsilon \in \mathfrak{a}_M^*$ is in general position, $J_M^P(\tau, f)$ is analytic for any τ in the translate $T_\varepsilon(M)$ of $T_{\mathrm{temp}}(M)$. In the next section we shall take $\varepsilon = \varepsilon_P$ to be any small point in the chamber $(\mathfrak{a}_P^*)^+$ of P in $\mathfrak{a}_M^*$.

The weighted characters (5.8) are linear forms on the Hecke algebra $\mathcal{H}(G)$. Our goal is to express the limit (1.15) in terms of these objects. However, the formula (4.13) we have established applies to the spherical function $f_1^{uT} = (F_1^{uT})^\vee$. We have therefore to introduce spherical analogues of the distributions (5.8), in preparation for the calculations of the next section. We shall then review the relations between Eisenstein integrals and induced representations.

Let (τ, V) be a unitary, finite dimensional, two-sided representation of K. For any cuspidal Levi subgroup M_1 of G, the usual (unnormalized) intertwining operators give rise to linear operators

$$J_{Q_1|P_1}^\ell(\lambda_1), J_{Q_1|P_1}^r(\lambda_1):\ \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1) \longrightarrow \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1), \qquad\qquad P_1, Q_1 \in \mathcal{P}(M_1),$$

that are meromorphic functions of $\lambda_1 \in \mathfrak{a}_{1,\mathbb{C}}^*$. (See [A5, p. 13]. The superscripts ℓ and r stand for "left" and "right".) These operators have decompositions

$$J_{Q_1|P_1}^\iota(\lambda_1) = \bigoplus_{\pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1} J_{Q_1|P_1}^\iota(\pi_{1,\lambda_1}), \qquad\qquad \iota = \ell, r,$$

into operators on the spaces $\mathcal{A}_{\pi_1}(M_1, \tau_1)$. Our interest will be in the case that the superscript ι equals r and that M_1 is replaced by a subgroup M_1^u of our fixed Levi subgroup M. We can then write $M_1^u = M_R$, where R is a fixed parabolic subgroup in $\mathcal{P}^M(M_1^u)$. We also consider only those parabolic subgroups $Q_1^u \in \mathcal{P}(M_1^u)$ of the form $Q(R)$, for some $Q \in \mathcal{P}(M)$. By definition, $Q(R)$ is the unique parabolic subgroup in $\mathcal{P}(M_R)$ that is contained in Q, and whose intersection with M equals R. If $P \in \mathcal{P}(M)$ is fixed and Λ belongs to $\mathfrak{a}^*_{R,\mathbb{C}}$, the product

$$\mathcal{J}^{r,P}_Q(\zeta,\Lambda) = J^r_{P(R)|Q(R)}(-\zeta+\Lambda)J^r_{P(R)|Q(R)}(\Lambda)^{-1}, \qquad Q \in \mathcal{P}(M),$$

is a (G,M)-family of functions of $\zeta \in i\mathfrak{a}^*_M$. The limit

$$\mathcal{J}^{r,P}_M(\Lambda) = \lim_{\zeta\to 0}\Big(\sum_{Q\in\mathcal{P}(M)} \mathcal{J}^{r,P}_Q(\zeta,\Lambda)\theta_Q(\zeta)^{-1}\Big)$$

therefore exists, and defines a meromorphic function of Λ with values in the space of linear operators on $\mathcal{A}_{\mathrm{cusp}}(M_R,\tau_R)$. We shall compose $\mathcal{J}^{r,P}_M(\Lambda)$ with the spherical analogue of an operator (5.6).

Suppose that $\tau \in T(M)$ is represented by a triplet $(M_\tau,\rho_\tau,r_\tau)$, as above. (We are now using the symbol τ to denote both a two-sided representation of K and an element in $T(M)$.) We take the Levi subgroup of M above to be equal to M_τ. Then $M_\tau = M_R$, for a fixed parabolic subgroup $R = R_\tau$ of $\mathcal{P}^M(M_\tau)$. We shall write

$$\rho_\tau = \rho_{\Lambda_\tau}, \qquad \rho \in \Pi_{\mathrm{cusp}}(M_R)^1,\ \Lambda_\tau \in \mathfrak{a}^*_{R,\mathbb{C}}. \tag{5.9}$$

The spherical analogues of (5.6) require operators

$$A^\iota(\rho^w_\tau), \qquad \iota = \ell, r,\ w \in R^M(\rho_\tau),$$

on $\mathcal{A}_\rho(M_R,\tau_R)$. Our chosen extension ρ^w_τ of the representation ρ_τ to the group $M^w_\tau(\mathbb{R}) = M^w_R(\mathbb{R})$ determines an extension of any matrix coefficient of ρ_τ to a function on $M^w_\tau(\mathbb{R})$. This allows us to identify elements in $\mathcal{A}_\rho(M_R,\tau_R)$ with $A_R(\mathbb{R})^0$-invariant functions from $M^w_\tau(\mathbb{R})$ to V. We define

$$\big(A^\ell(\rho^w_\tau)\psi\big)(m) = \tau(\widetilde{w})^{-1}\psi(\widetilde{w}m)$$

and

$$\big(A^r(\rho^w_\tau)\psi\big)(m) = \psi(m\widetilde{w})\tau(\widetilde{w})^{-1},$$

for any $\psi \in \mathcal{A}_\rho(M_R, \tau_R)$ and $m \in M_R(\mathbb{R})$, where $\widetilde{w}$ is a representative of w in K. We then define the spherical analogues of (5.6) as products

$$\widetilde{R}^\iota(w, \rho_\tau) = A^\iota(\rho_\tau^w) R^\iota_{w^{-1}Rw|R}(\rho_\tau), \qquad \iota = \ell, r, \ w \in R^M(\rho_\tau),$$

where the operators on the right are normalized by the same scalar factor used to define the corresponding operator in (5.6).

The spherical analogue of (5.8) we shall use requires a preliminary word about the invariant transform $f \to f_M$. If f belongs to $\mathcal{C}(G, \tau)$, we define f_M to be the Schwartz function from $T_{\text{temp}}(M)$ to V obtained from the familiar scalar valued mapping from $\mathcal{C}(G)$ to $\mathcal{I}(M)$. An obvious variant of this transform applies to cuspidal spherical functions on a Levi subgroup M_1. It is a mapping $\psi \to \psi_{M_1}$ that sends $\psi \in \mathcal{A}_{\text{cusp}}(M_1, \tau_1)$ to the function ψ_{M_1} of finite support from $\Pi_{\text{cusp}}(M_1)^1$ to V. In general, we shall write $\pi^\vee$ for the *contragredient* of a given representation π. If π_1 belongs to $\Pi_{\text{cusp}}(M_1)^1$, $\psi_{M_1}(\pi_1^\vee)$ is a vector in V that depends only on the image of ψ in $\mathcal{A}_{\pi_1}(M_1, \tau_1)$. If ψ_1 actually belongs to $\mathcal{A}_{\pi_1}(M_1, \tau_1)$, the orthogonality relations for matrix coefficients of discrete series tell us that the (V-valued) character $\psi_{M_1}(\pi_1^\vee)$ equals $d_{\pi_1}^{-1}\psi(1)$. We can therefore write the projection $\psi^K_{M_1}(\pi_1^\vee)$ of $\psi_{M_1}(\pi_1^\vee)$ onto V^K as

$$\psi^K_{M_1}(\pi_1) = d_{\pi_1}^{-1} \int_K \tau(k^{-1})\psi(1)\tau(k)dk = d_{\pi_1}^{-1} e^K(\psi),$$

for the Eisenstein integral

$$e^K(\psi) = E_{P_1}(1, \psi, 1), \qquad P_1 \in \mathcal{P}(M_1).$$

If $\tau \in T(M)$ is represented by the triplet $(M_\tau, \rho_\tau, r_\tau)$ as above, we write $\tau^\vee$ for the element in $T(M)$ represented by the triplet $(M_\tau, \rho_\tau^\vee, r_\tau)$.

Lemma 5.1. *Assume that τ lies in $T_{\text{temp}}(M)$, and that f belongs to $\mathcal{C}(G, \tau)$. Then*

$$f_M(\tau^\vee) = d_\rho^{-1} e^K\big(\widetilde{R}^\iota(r_\tau, \rho_\tau)\widehat{f}_{P(R)}(\Lambda_\tau)\big), \qquad \iota = \ell, r,$$

in the notation above.

Proof. Results of this kind are quite familiar, so we shall just sketch the proof. In fact, we shall treat only the special case that $M_\tau = M$, leaving the reader to check the general case. The representation $\pi = \rho$ then belongs to $\Pi_{\mathrm{cusp}}(M)^1$, the point $\lambda = \Lambda_\tau$ belongs to $i\mathfrak{a}_M^*$, and r_τ is trivial. According to the definition (2.1),

$$\begin{aligned}
\big(\widehat{f}_P(\lambda), \psi\big) &= \int_{G(\mathbb{R})} \big(f(x), E_P(x, \psi, \lambda)\big)dx \\
&= \int_{G(\mathbb{R})}\int_K \big(f(x), \tau(k^{-1})\psi_P(kx)\big)e^{(-\lambda+\rho_P)(H_P(kx))}dkdx \\
&= \int_{G(\mathbb{R})}\int_K \big(\tau(k)f(k^{-1}x), \psi_P(x)\big)e^{(-\lambda+\rho_P)(H_P(x))}dkdx \\
&= \int_{M(\mathbb{R})^1}\int_{\mathfrak{a}_M}\int_{N_P(\mathbb{R})} \big(f(m\exp Hn), \psi(m)\big)e^{(-\lambda+\rho_P)(H)}dndHdm,
\end{aligned}$$

for any $\psi \in \mathcal{A}_\pi(M, \tau_M)$. We have used the fact that f is τ-spherical to remove the integral over K in the third expression, and the implicit integral over K in the last expression. It then follows from the orthogonality relations for the square integrable representation $\pi^\vee$ that the value of $\big(\widehat{f}_P(\lambda)\big)_M^K$ at $\pi^\vee$ equals

$$\int_K \int_{M(\mathbb{R})^1}\int_{\mathfrak{a}_M}\int_{N_P(\mathbb{R})} f(k^{-1}m\exp Hnk)\Theta(\pi^\vee, m)e^{(-\lambda+\rho_P)(H)}dndHdmdk,$$

where $\Theta(\pi^\vee, \cdot)$ is the character of $\pi^\vee$. This is just the value of the function $f_M \in \mathcal{C}(M)\otimes V$ at the character of the representation $(\pi_\lambda)^\vee$ of $M(\mathbb{R})$. The required formula, in the special case that $M_\tau = M$, then follows from the remark preceding the lemma (with $M_1 = M$). $\square$

It is clear that the formula of Lemma 5.1 remains valid for nontempered $\tau \in T(M)$, if f belongs to the Hecke algebra. The spherical weighted characters we shall use in the argument of the next section apply to this setting. They pertain to an analytic function $F_{P(R)}(\Lambda)$ from $\mathfrak{a}_{R,\mathbb{C}}^*$ to the space

$$\mathcal{A}_{\mathrm{cusp}}(M_R, \tau_R) = \bigoplus_{\rho\in\Pi_{\mathrm{cusp}}(M_R)^1} \mathcal{A}_\rho(M_R, \tau_R).$$

The operator $\mathcal{J}_M^{r,P}(\Lambda)$ acts on this space, and the product $\mathcal{J}_M^{r,P}(\Lambda)F_{P(R)}(\Lambda)$ becomes a meromorphic function of Λ. Suppose that $\tau \in T(M)$ is represented by the triplet $(M_\tau, \rho_\tau, r_\tau)$, with $M_\tau = M_R$ as above. The operator $\widetilde{R}^r(r_\tau, \rho_\tau)$ acts on the space $\mathcal{A}_\rho(M_R, \tau_R)$, where ρ is the restriction of ρ_τ to $M_R(\mathbb{R})^1$. We extend it to $\mathcal{A}_{\mathrm{cusp}}(M_R, \tau_R)$ by defining it to be zero on the orthogonal complement of $\mathcal{A}_\rho(M_R, \tau_R)$. We assume that the triplet τ has the property that the image in $\mathfrak{a}_M^*$ of the real part of $\Lambda_\tau \in \mathfrak{a}_{R,\mathbb{C}}^*$ is in general position. By the nature of the (G, M)-family from which $\mathcal{J}_M^{r,P}(\Lambda)$ was constructed, this function is analytic at $\Lambda = \Lambda_\tau$. We define

$$(5.10) \qquad (\mathcal{J}_M^{r,P} F_{P(R)})_M^{\vee,K}(\tau^\vee) = d_\rho^{-1} e^K \big(\widetilde{R}^r(r_\tau, \rho_\tau)\mathcal{J}_M^{r,P}(\Lambda_\tau)F_{P(R)}(\Lambda_\tau)\big).$$

The notation we have chosen for the left hand side of this definition will be clearer when we encounter the objects on the right hand side in §6.

The composition of an Eisenstein integral with a linear form on V is a linear combination of matrix coefficients of induced representations. This is implicit in the original definition. The relations between Eisenstein integrals and induced representations become especially transparent for a particular choice of the two-sided representation τ.

Suppose that (σ_1, V_1) and (σ_2, V_2) are irreducible unitary representations of K. Let σ be the two-sided representation of K on the space

$$U_\sigma = \mathrm{Hom}_{\mathbb{C}}(V_2, V_1)$$

given by

$$\sigma(k_1)u\sigma(k_2) = \sigma_1(k_1) \circ u \circ \sigma_2(k_2), \qquad k_1, k_2 \in K,\ u \in U_\sigma.$$

We take $\tau = \tau_\sigma$ to be the two-sided representation of K on the space

$$V = V_\sigma = \mathrm{End}(U_\sigma) = \mathrm{Hom}_{\mathbb{C}}(U_\sigma, U_\sigma),$$

defined by

$$\big(\tau(k_1)v\tau(k_2)\big)(u) = \sigma(k_1)v(u)\sigma(k_2), \qquad v \in V,\ u \in U_\sigma.$$

Suppose that M_1 is a cuspidal Levi subgroup, and that $\tau_1 = \tau_{M_1}$ denotes as usual the restriction of the two-sided representation $\tau = \tau_\sigma$ to K_{M_1}. Suppose that $P_1 \in \mathcal{P}(M_1)$, and that π_1 is a representation in $\Pi_{\mathrm{cusp}}(M_1)^1$. The induced representation $\mathcal{I}_{P_1}(\pi_1)$ acts on the Hilbert space $\mathcal{H}_{P_1}(\pi_1)$ of vector valued functions on K. For any irreducible representation σ_* of K, let $\mathcal{H}_{P_1}(\pi_1)_{\sigma_*}$ be the finite dimensional subspace of functions in $\mathcal{H}_{P_1}(\pi_1)$ that transform under translation according to σ_*. Frobenius reciprocity can be used to describe this space in terms of the restriction of π_1 to K_{M_1}. By combining the cases that σ_* equals σ_1 and σ_2, one constructs a canonical isomorphism from the vector space

$$\mathrm{End}_\sigma\big(\mathcal{H}_{P_1}(\pi_1)\big) = \mathrm{Hom}_{\mathbb{C}}\big(\mathcal{H}_{P_1}(\pi_1)_{\sigma_1}, \mathcal{H}_{P_1}(\pi_1)_{\sigma_2}\big)$$

onto the space $\mathcal{A}_{\pi_1}(M_1, \tau_1)$, which we denote by

$$S \longrightarrow \psi(S), \qquad S \in \mathrm{End}_\sigma\big(\mathcal{H}_{P_1}(\pi_1)\big). \tag{5.11}$$

(See [A5, §3]. This isomorphism is essentially the one defined by Harish-Chandra in [Ha5, §7].)

The isomorphism satisfies the basic identity

$$\mathrm{tr}\big(\mathcal{I}_{P_1}(\pi_1, \pi_{1,\lambda_1}, x)S\big) = \mathrm{tr}\big(E_{P_1}(x, \psi(S), \lambda_1)\big), \qquad x \in G(\mathbb{R}),\ \lambda_1 \in \mathfrak{a}^*_{1,\mathbb{C}}, \tag{5.12}$$

where S is any operator in $\mathrm{End}_\sigma\big(\mathcal{H}_{P_1}(\pi_1)\big)$. (See [A5, p. 21], [A1, Lemma I.5.2].) This expresses the basic relation between Eisenstein integrals and induced representations. Other identities apply to the intertwining operators discussed above. It follows from the various definitions that

$$J^\ell_{Q_1|P_1}(\lambda_1)\psi(S) = \psi\big(SJ_{P_1|Q_1}(\pi_1, \pi_{1,\lambda_1})\big)$$

and

$$J^r_{Q_1|P_1}(\lambda_1)\psi(S) = \psi\big(J_{Q_1|P_1}(\pi_1, \pi_{1,\lambda_1})S\big), \tag{5.13}$$

for $P_1, Q_1 \in \mathcal{P}(M_1)$ and $\lambda_1 \in \mathfrak{a}^*_{1,\mathbb{C}}$. If M_1 and P_1 are specialized to the groups M_R and $P(R)$ above, one uses these relations to derive further identities

$$\widetilde{R}^\ell(w, \rho_\tau)\psi(S) = \psi\big(S\widetilde{R}(w, \rho_\tau)\big)$$

and

$$\widetilde{R}^r(w, \rho_\tau)\psi(S) = \psi\big(\widetilde{R}(w, \rho_\tau)S\big), \qquad w \in R^M(\rho_\tau), \tag{5.14}$$

for the operators in (5.6).

We conclude this section with a corollary of Lemma 4.4, which pertains to the chamber $c_0 \subset \mathfrak{a}_0^+$ of that lemma, and hence also parabolic subgroups $P_0 \in \mathcal{P}(M_0)$ and $P \in \mathcal{P}(M)$. We shall use (5.12) to convert the uniform estimate (4.17) in τ to a pair of uniform estimates in $f \in \mathcal{C}(G)$. Formulated in this generality, the corollary can be regarded as a first step towards extending our results from $\mathcal{H}(G)$ to $\mathcal{C}(G)$.

Corollary 5.2. *For any $n \geq 0$, there is a continuous seminorm $\|\cdot\|_n$ on $\mathcal{C}(G)$ such that*

$$|J_M(\gamma_T, f^S)| \leq \|f\|_n\big(1 + d_{c_0}(T, S)\big)^{-n} \tag{5.15}$$

and

$$|I_M(\gamma_T, f^S)| \leq \|f\|_n\big(1 + d_{c_0}(T, S)\big)^{-n}, \tag{5.16}$$

for any $\gamma \in \Gamma$ and $f \in \mathcal{C}(G)$, and any points $T \in \mathfrak{a}_P^+$ and $S \in \mathfrak{a}_1$ such that T is (c_0, S)-dominant.

Proof. We can express f as a convergent sum

$$f = \sum_{\sigma=(\sigma_1,\sigma_2)} f_\sigma, \qquad \sigma_i \in \Pi(K),$$

where f_σ transforms under right and left translation by K according to σ_1 and σ_2 respectively. Since the sum is convergent in $\mathcal{C}(G)$, it would be enough to establish (5.15) for each of the components $f_\sigma \in \mathcal{C}(G)$. We may therefore assume that $f = f_\sigma$, for a fixed pair $\sigma = (\sigma_1, \sigma_2)$ of irreducible representations of K. We do so, writing $\tau = \tau_\sigma$ in the notation above.

By definition (1.12), the function f^S depends only on the Fourier transform

$$\widehat{f}_{P_1}(\pi_{1,\lambda_1}) = \mathcal{I}_{P_1}(\pi_{1,\lambda_1}, f^\vee), \qquad \pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1,\ \lambda_1 \in i\mathfrak{a}_1^*,$$

of f. Since $\widehat{f}_{P_1}(\pi_{1,\lambda_1})$ lies in the space $\mathrm{End}_\sigma\big(\mathcal{H}_{P_1}(\pi_1)\big)$, we can define

$$F_1(\lambda_1) = \bigoplus_{\pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1} d_{\pi_1} \psi\big(\widehat{f}_{P_1}(\pi_{1,\lambda_1})\big).$$

Then F_1 is a Schwartz function from $i\mathfrak{a}_1^*$ to the finite dimensional vector space $\mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$. According to the definition (2.7) and its analogue for $\widehat{f}^S_{P_1}$ in §1, the functions F_1^S and $\widehat{f}^S_{P_1}$ depend on S in the same way. It follows from (1.12) and (5.12) that

$$\begin{aligned} f^S(x) &= |W(M_1)|^{-1} \int_{i\mathfrak{a}_1^*} \sum_{\pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1} \mathrm{tr}\big(\mathcal{I}_{P_1}(\pi_{1,\lambda_1}, x)\widehat{f}^S_{P_1}(\pi_{1,\lambda_1})\big)\varepsilon_{P_1}(\pi_{1,\lambda_1}) d\lambda_1 \\ &= |W(M_1)|^{-1} \int_{i\mathfrak{a}_1^*} \mathrm{tr}\big(E_{P_1}(x, \mu_{P_1}(\lambda_1) F_1^S(\lambda_1), \lambda_1)\big) d\lambda_1 \\ &= \mathrm{tr}\big((F_1^S)^\vee(x)\big), \end{aligned}$$

since the restriction of the operator $d_{\pi_1}\mu_{P_1}(\lambda_1)$ to the space $\mathcal{A}_{\pi_1}(M_1, \tau_1)$ equals the scalar $\varepsilon_{P_1}(\pi_{1,\lambda_1})$. The first estimate (5.15) then follows from Lemma 4.4.

To derive the second estimate (5.16) from the first, we shall need to be able to express the function $\phi_L(f^S)$ in terms of $\phi_L(f)$, for any $L \in \mathcal{L}(M)$. We note that any element in $\mathcal{M}(G)$ factors to an endomorphism of $\mathcal{I}(G)$, since it acts through the infinitesimal character. The multiplier $f \to f^S$ for $\mathcal{C}(G)$ therefore factors to an endomorphism $a \to a^S$ of $\mathcal{I}(G)$, with the property that $(f^S)_G = (f_G)^S$. However, it is not generally true that $\phi_L(f^S)$ equals $\phi_L(f)^S$ (or even that $\phi_L(f)^S$ is defined). The Fourier transform of f^S is supported on the G-conjugacy class of M_1, while the cuspidal support of the function $\phi_L(f^S)$ is a finite family (possibly empty) of classes of Levi subgroups of L. To be precise, let

$$M_{1,i} = \widetilde{w}_i M_1 \widetilde{w}_i^{-1}, \qquad w_i \in W_0 = W(M_0),\ 1 \le i \le k,$$

be a set of representatives of L-conjugacy classes of Levi subgroups of L that lie in the G-conjugacy class of M_1. (Keep in mind that the assertions of the corollary are framed in terms of a minimal Levi subgroup M_0 that is contained in both M_1 and M, and hence L as well. It is perhaps also helpful to observe that

$$M_i = M_1^{u_i}, \qquad u_i = w_{i,L}^{-1},$$

in the notation of Corollary 1.2.) It then follows from the definition of the mappings $f \to \phi_L(f)$ and $f \to f^S$, and the symmetry condition

$$f^S = |W(M_1)|^{-1} \sum_{w \in W(M_1)} f^{wS},$$

that

$$\phi_L(f^S) = |W(M_1)|^{-1} \sum_{w} \sum_{i=1}^{k} \phi_L(f)^{w_i wS}. \tag{5.17}$$

(The sum over w insures that the symmetry condition is reflected in the functions

$$\phi_L(f)^{S_i}, \qquad S_i \in \mathfrak{a}_{M_1,i}.)$$

We assume inductively that (5.16) holds if G is replaced by a proper Levi subgroup $L \in \mathcal{L}(M)$. It follows from the definitions prior to Lemma 4.4 that if T is (c_0, S)-dominant (relative to G), it is also $(c_0, w_i wS)$-dominant (relative to L), for any elements w and w_i in (5.16). Moreover, the distance function

$$d_{c_0}(T, S) = d_{c_0}^G(T, S)$$

for G satisfies

$$d_{c_0}^G(T, S) \leq d_{c_0}^L(T, ww_iS).$$

The estimate (5.16) for G then follows from the definition (1.4), the first estimate (5.15), and the fact that the mapping ϕ_L is continuous. □

§6. The asymptotic formula

We are now ready to solve the problem posed at the end of §1. We shall convert what we have already established into an asymptotic formula for the weighted orbital integrals (1.2). This is our main task. Once we have completed it, we shall derive an asymptotic formula for the invariant distributions (1.4).

The discussion of Sections 3 and 4 culminated in Lemma 4.3. This result applies to a spherical function f_1^{uT} in $\mathcal{C}(G,\tau)$, and gives an integral formula (4.16) for the limit of $J_M(\gamma_T, f_1^{uT})$. There are three further operations to be performed on the formula. We need to evaluate the integral (4.16) in more explicit terms, we must sum u over $U(M, M_1)$ to obtain the function f_T, and we must then convert the resulting expression to a linear form on $\mathcal{C}(G)$. We shall actually carry out the third operation only for functions f in the Hecke algebra $\mathcal{H}(G)$. For any such f, the function f_T is a K-finite Schwartz function, which does not generally lie in $\mathcal{H}(G)$. However, its operator valued Fourier transform is entire, a property that will be more than sufficient for our needs. As a matter of fact, the mapping $f \to f_T$ does lift to an endomorphism of the image $I\mathcal{H}(G)$ of $\mathcal{H}(G)$ in $\mathcal{I}(G)$, a fact that follows from the characterization [CD] of $I\mathcal{H}(G)$.

We recall again that M, $T_M \subset M$, $P \in \mathcal{P}(M)$, and $r > 0$ were fixed in §2 and §3. We shall evaluate the limit (1.15) in terms of the function $\theta_M(\gamma,\tau)$ in (5.3) (with M in place of G) and the linear form $J_M^P(\tau, f)$ given by (5.8).

Theorem 6.1. *Asume that $f \in \mathcal{H}(G)$, and that $\varepsilon = \varepsilon_P$ is a small point in the chamber $(\mathfrak{a}_P^*)^+$. Then*

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} J_M(\gamma_T, f_T) = \int_{T_\varepsilon(M)} \theta_M(\gamma,\tau) J_M^P(\tau, f)\, d\tau, \tag{6.1}$$

uniformly for γ in the relatively compact subset $\Gamma = T_{M,G\text{-reg}}(\mathbb{R})^C$ of $T_M(\mathbb{R})$.

Proof. There are two stages to the proof. The first is to convert the integral $\Phi_P^u(\gamma)$ in Lemma 4.3 to an analogue for spherical functions of the right hand side of (6.1). The second is to derive (6.1) itself from the relations between Eisenstein integrals and induced representations.

Lemma 4.3 applies to any two-sided representation (τ, V) of K, and any Schwartz function $f_1 \in \mathcal{C}(G, \tau)$ whose spherical transform $\{F_1\}$ is supported on the conjugacy class of M_1. We now make the assumption that the Schwartz function

$$F_1 : \; i\mathfrak{a}_1^* \longrightarrow \mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$$

extends to a holomorphic function of rapid decrease on a cylindrical neighbourhood of $i\mathfrak{a}_1^*$ in $i\mathfrak{a}_{1,\mathbb{C}}^*$. Recall that M_1 is a cuspidal Levi subgroup of G, equipped with a parabolic subgroup $P_1 \in \mathcal{P}(M_1)$ such that both P and P_1 are standard with respect to a minimal parabolic subgroup P_0. The element u in (4.15) belongs to the set $U(M, M_1)$, and equals the image w_M^{-1} of an element $w \in W(P_1; P)$ under the mapping of Lemma 1.1.

Lemma 4.3 asserts that the limit of $J_M(\gamma_T, f^{uT})$ as T approaches infinity in $\mathfrak{a}_P^r$ equals the integral $\Phi_P^u(\gamma)$ given by (4.16). Recall that the integrand $\Phi_P^u(m, \nu; \gamma)$ in (4.16) is the product of (4.8) and an integral (4.12) over $\Lambda \in i\mathfrak{a}_R^*$. With our assumptions on F_1, Corollary 2.3 implies that the mapping

$$F_R : \; i\mathfrak{a}_R^* \longrightarrow \mathcal{A}_{\mathrm{cusp}}(M_R, \tau_R),$$

defined for w as in (2.13), extends to a rapidly decreasing analytic function on a cylindrical neighbourhood of $i\mathfrak{a}_R^*$ in $i\mathfrak{a}_{R,\mathbb{C}}^*$. The same is therefore true of the product $\mu_R(\Lambda)F_R(\Lambda)$ in (4.12), since $\mu_R(\Lambda)$ is analytic and slowly increasing. It follows from the definition (4.11) that the integrand in (4.12) itself extends to a rapidly decreasing analytic function on a cylindrical neighbourhood of $i\mathfrak{a}_R^*$. This circumstance allows us to deform the contour of integration over Λ in (4.12) from the space $i\mathfrak{a}_R^*$ to its translate $(-\varepsilon_P + i\mathfrak{a}_R^*)$ by the point $(-\varepsilon_P)$.

The purpose of this change of contour is to allow an interchange of the integrals over Λ and ν. Indeed, for any Λ in $(-\varepsilon_P + i\mathfrak{a}_R^*)$, the exponential factor in the value of (4.11) at $x = m^{-1}\gamma m\nu$ is bounded by

$$e^{(-\varepsilon_P - \rho_P)(H_M(\gamma) + H_{\bar{P}}(\nu))} = c_\gamma e^{-(\varepsilon_P + \rho_P)(H_{\bar{P}}(\nu))},$$

where c_γ is independent of ν. It follows from the general estimate [Ha3, Lemma 89] of Harish-Chandra that the integral over ν of $\Phi_P^u(m, \nu; \gamma)$ can be taken inside the integral over Λ in the deformed contour. The factor $v_M(\nu)$ in (4.12) is defined as in §1 by the simple formula

$$\lim_{\zeta \to 0} \Big(\sum_{Q \in \mathcal{P}(M)} e^{-\zeta(H_Q(\nu))} \theta_Q(\zeta)^{-1} \Big), \qquad \zeta \in i\mathfrak{a}_M^*.$$

The other factor in (4.12) can obviously be taken inside the limit in ζ and the finite sum over Q. A second appeal to the estimate [Ha3, Lemma 89] tells us that the integral over ν can also be taken inside these operations. It follows that the limit $\Phi_P^u(\gamma)$ of $J_M(\gamma_T, f_1^{uT})$ can be expressed as the integral over $m \in A_M(\mathbb{R})\backslash M(\mathbb{R})$, the integral over $\Lambda \in (-\varepsilon_P + i\mathfrak{a}_R^*)$, the limit in ζ, and the sum over Q of the product of (4.8), $|W(M_1)|^{-1}$, $\theta_Q(\zeta)^{-1}$, and

$$\int_{N_P(\mathbb{R})} E_{R,\bar{P}}^K\big(m^{-1}\gamma m\nu, \mu_R(\Lambda)F_R(\Lambda), \Lambda\big) e^{-\zeta(H_Q(\nu))} d\nu. \tag{6.2}$$

We need to express (6.2) in terms of intertwining operators.

We claim that for any elements $y \in M(\mathbb{R})^1$, $\psi \in \mathcal{A}_{\mathrm{cusp}}(M_R, \tau_R)$, and $\Lambda \in (-\varepsilon_P + i\mathfrak{a}_R^*)$, the integral

$$\int_{N_P(\mathbb{R})} E_{R,\bar{P}}(y\nu, \psi, \Lambda) e^{-\zeta(H_Q(\nu))} d\nu \tag{6.3}$$

equals

$$E_R\big(y, J_{P(R)|Q(R)}^r(-\zeta + \Lambda) J_{Q(R)|\bar{P}(R)}^r(\Lambda)\psi, \Lambda\big). \tag{6.4}$$

According to the definition (4.11), the integral (6.3) is a weighted average of the function E_R under right translation by $N_P(\mathbb{R})$. The Eisenstein integral E_R is in turn a weighted average of the function ψ under left translation by K_M. These two operations commute.

We can therefore study (6.3) in terms of the weighted average of ψ under right translation by $N_P(\mathbb{R})$. One sees easily from this that it is enough to justify the claim in the special case that $M_R = M$. We shall do so, noting that $Q(R)$ equals Q, and that the point $\lambda = \Lambda$ lies in $(-\varepsilon_P + i\mathfrak{a}_M^*)$.

The required identity is a simple variant of the usual multiplicative property of intertwining operators, stated for example in [A5, (2.2)]. By definition [A5, §2],

$$\big(J_{Q|\bar{P}}(\lambda)\psi\big)(y) = \int_{N_Q(\mathbb{R})\cap N_P(\mathbb{R})} \psi_{\bar{P},\lambda}(yv)dv, \qquad Q \in \mathcal{P}(M),$$

where

$$\psi_{\bar{P},\lambda}(x) = \psi\big(m_{\bar{P}}(x)\big)\tau\big(k_{\bar{P}}(x)\big)e^{(\lambda+\rho_{\bar{P}})(H_{\bar{P}}(x))}, \qquad x \in G(\mathbb{R}).$$

The covergence of the integral is assured by the fact that the real part of λ equals $(-\varepsilon_P)$. We can therefore write

$$\begin{aligned}
&\big(J^r_{P|Q}(\lambda-\zeta)J^r_{Q|\bar{P}}(\lambda)\psi\big)(y) \\
&= \int \big(J^r_{Q|\bar{P}}(\lambda)\psi\big)_{Q,\lambda-\zeta}(yu)du \\
&= \int \big(J^r_{Q|\bar{P}}(\lambda)\psi\big)\big(ym_Q(u)\big)\tau\big(k_Q(u)\big)e^{(\lambda-\zeta+\rho_Q)(H_Q(u))}du \\
&= \int\int \psi_{\bar{P},\lambda}\big(ym_Q(u)v\big)\tau\big(k_Q(u)\big)e^{(\lambda-\zeta+\rho_Q)(H_Q(u))}dvdu \\
&= \int\int \psi_{\bar{P},\lambda}\big(yvm_Q(u)\big)\tau\big(k_Q(u)\big)e^{(\lambda-\zeta+\rho_Q)(H_Q(u))}dvdu,
\end{aligned}$$

where u and v are integrated over $N_P(\mathbb{R}) \cap N_{\bar{Q}}(\mathbb{R})$ and $N_Q(\mathbb{R}) \cap N_P(\mathbb{R})$ respectively. Appealing to the definitions and the appropriate changes of variable in the integrals, we see that this in turn can be written as

$$\begin{aligned}
&\int\int \psi_{\bar{P},\lambda}\big(ya_Q(u)vm_Q(u)k_Q(u)\big)e^{(\rho_Q-\rho_{\bar{P}}-\zeta)(H_Q(u))}dvdu \\
&= \int\int \psi_{\bar{P},\lambda}\big(yva_Q(u)m_Q(u)k_Q(u)\big)e^{-\zeta(H_Q(u))}dudv \\
&= \int\int \psi_{\bar{P},\lambda}(yvu)e^{-\zeta(H_Q(u))}dvdu \\
&= \int_{N_P(\mathbb{R})} \psi_{\bar{P},\lambda}(y\nu)e^{-\zeta(H_Q(\nu))}d\nu,
\end{aligned}$$

since $H_Q(u) = H_Q(vu)$. The last expression equals (6.3), in the special case that $M_R = M$. Since the original expression is just (6.4) in this case, we have justified the claim when $M_R = M$.

We thus have a general identity of (6.3) with (6.4). We can therefore calculate the integral (6.2) by substituting $y = m^{-1}\gamma m$ and $\psi = \mu_R(\Lambda)F_R(\Lambda)$ into (6.4), and then taking the projection onto V^K. The point y actually lies in $M(\mathbb{R})$ rather than $M(\mathbb{R})^1$, but this minor modification of the identity entails simply multiplying (6.4) by the quantity

$$e^{-\rho_P(H_M(m^{-1}\gamma m))} = e^{-\rho_P(H_M(\gamma))}.$$

The vector ψ can be written as

$$\mu_R(\Lambda)F_R(\Lambda) = c_R(1,\Lambda)^{-1}c_{P(R)|P(R)}(1,\Lambda)\mu_{P(R)}(\Lambda)F_1^u(\Lambda),$$

where

$$F_1^u(\Lambda) = {}^0c_{P_1^u|P_1}(w, w^{-1}\Lambda)F_1(w^{-1}\Lambda), \qquad u = w_M^{-1}. \tag{6.5}$$

This follows from the definition (2.13), the formulas

$${}^0c_{P_1^u|P_1}(w, w^{-1}\Lambda) = c_{P_1^u|P_1^u}(1,\Lambda)^{-1}c_{P_1^u|P_1}(w, w^{-1}\Lambda)$$

and

$${}^0c_{P_1^u|P_1}(w, w^{-1}\Lambda)\mu_{P_1}(w^{-1}\Lambda) = \mu_{P_1^u}(\Lambda)\,{}^0c_{P_1^u|P_1}(w, w^{-1}\Lambda),$$

and the fact that $P_1^u = P(R)$. The coefficient of $F_1(w^{-1}\Lambda)$ on the right hand side of (6.5), as an operator valued function of Λ, is known to be a rational function, none of whose singularities intersect $i\mathfrak{a}_R^*$. It follows from our condition on F_1 that $F_1^u(\Lambda)$ is a rapidly decreasing analytic function on a cylindrical neighbourhood of $i\mathfrak{a}_R^*$ in $\mathfrak{a}_{R,\mathbb{C}}^*$ with values in $\mathcal{A}_{\mathrm{cusp}}(M_R, \tau_R)$.

Before we make the substitution into (6.4), let us rewrite the c and μ functions in the formula

$$F_R(\Lambda) = \big(c_R(1,\Lambda)\mu_R(\Lambda)\big)^{-1}\big(c_{P(R)}(1,\Lambda)\mu_{P(R)}(\Lambda)\big)F_1^u(\Lambda)$$

in terms of intertwining operators. Following Harish-Chandra, we can write

$$c_R(1,\Lambda)\mu_R(\Lambda) = J^r_{\bar{R}|R}(\Lambda)\cdot\big(J^r_{R|\bar{R}}(\Lambda)J^r_{\bar{R}|R}(\Lambda)\big)^{-1} = J^r_{R|\bar{R}}(\Lambda)^{-1}.$$

(See [A5, §2].) Similarly, we have

$$c_{P(R)|P(R)}(1,\Lambda)\mu_{P(R)}(\Lambda) = J^r_{P(R)|\bar{P}(\bar{R})}(\Lambda)^{-1}.$$

We then deduce that

$$\begin{aligned}
&J^r_{R|\bar{R}}(\Lambda)J^r_{P(R)|\bar{P}(\bar{R})}(\Lambda)^{-1}\\
&= J^r_{\bar{P}(R)|\bar{P}(\bar{R})}(\Lambda)\big(J^r_{P(R)|\bar{P}(R)}(\Lambda)J^r_{\bar{P}(R)|\bar{P}(\bar{R})}(\Lambda)\big)^{-1}\\
&= J^r_{P(R)|\bar{P}(R)}(\Lambda)^{-1}\\
&= J^r_{Q(R)|\bar{P}(R)}(\Lambda)^{-1}J^r_{P(R)|Q(R)}(\Lambda)^{-1}, \qquad Q\in\mathcal{P}(M),
\end{aligned}$$

from the standard multiplicative and descent properties of unnormalized intertwining operators. The μ-function $\mu_R(\Lambda)$ for R commutes with the operators $J^r_{P(R)|Q(R)}(-\zeta+\Lambda)$ and $J^r_{Q(R)|\bar{P}(R)}(\Lambda)$ in (6.4). We conclude that the projection onto V^K of the value of (6.4) at $y=m^{-1}\gamma m$ and $\psi=\mu_R(\Lambda)F_R(\Lambda)$ is

$$E^K_R\big(m^{-1}\gamma m, \mu_R(\Lambda)J^r_{P(R)|Q(R)}(-\zeta+\Lambda)J^r_{P(R)|Q(R)}(\Lambda)^{-1}F^u_1(\Lambda),\Lambda\big). \tag{6.6}$$

It is the product of this expression with $e^{-\rho_P(H_M(\gamma))}$ that is equal to (6.2). We have thus obtained an expression for (6.2) in terms of intertwining operators. Observe that the factor $e^{-\rho_P(H_M(\gamma))}$ by which (6.6) must be multiplied cancels the second factor in (4.8). This leaves only the first factor $|D^M(\gamma)|^{\frac{1}{2}}$ from (4.8).

We have shown that $\Phi^u_P(\gamma)$ equals the integral over m and Λ, the limit in ζ, and the sum over Q of the product of $|D^M(\gamma)|^{\frac{1}{2}}$, $|W(M_R)|^{-1}$, $\theta_Q(\zeta)^{-1}$ and (6.6). This becomes the product of $|D^M(\gamma)|^{\frac{1}{2}}$ with the integral over $m\in A_M(\mathbb{R})\backslash M(\mathbb{R})$ of

$$|W(M_R)|^{-1}\int_{-\varepsilon_P+i\mathfrak{a}^*_R} E^K_R\big(m^{-1}\gamma m,\mu_R(\Lambda)\mathcal{J}^{r,P}_M(\Lambda)F^u_1(\Lambda),\Lambda\big)d\Lambda,$$

with $\mathcal{J}_M^{r,P}(\Lambda)$ being the operator valued function defined in §5. Let us write

$$(\mathcal{J}_M^{r,P}F_1^u)^\vee(y) = |W^M(M_R)|^{-1}\int_{-\varepsilon_P+i\mathfrak{a}_R^*} E_R\big(y, \mu_R(\Lambda)\mathcal{J}_M^{r,P}(\Lambda)F_1^u(\Lambda), \Lambda\big)d\Lambda,$$

for any point $y \in M(\mathbb{R})$. The function $(\mathcal{J}_M^{r,P}F_1^u)^\vee$ need not be rapidly decreasing on $M(\mathbb{R})$, since the function $\mathcal{J}_M^{r,P}(\Lambda)$ of Λ in the integrand could have poles that meet $i\mathfrak{a}_R^*$. However, its failure to be so is mild. To see this, we write the integral over $(-\varepsilon_P + i\mathfrak{a}_R^*)$ as a double integral

$$(6.7)\qquad \int_{i\mathfrak{a}_R^*/i\mathfrak{a}_M^*}\Big(\int_{-\varepsilon_P+i\mathfrak{a}_M^*} E_R\big(y, \mu_R(\Lambda)\mathcal{J}_M^{r,P}(\Lambda+\lambda)F_1^u(\Lambda+\lambda), \Lambda\big)e^{\lambda(H_M(y))}d\lambda\Big)d\Lambda.$$

We then note that the integral over $(i\mathfrak{a}_R^*/i\mathfrak{a}_M^*)$ can be identified with an obvious variant of the integral in (2.3) (with (G, P_1, x) replaced by (M, R, y)). It follows that for any function $a \in C_c^\infty(\mathfrak{a}_M)$, the product

$$\alpha\big(H_M(y)\big)(\mathcal{J}_M^{r,P}F_1^u)^\vee(y), \qquad\qquad y \in M(\mathbb{R}),$$

belongs to $\mathcal{C}(M, \tau_M)$. Since any conjugacy class in $M(\mathbb{R})$ projects to a point in $\mathfrak{a}_M$, we can form the invariant orbital integral

$$(6.8)\qquad (\mathcal{J}_M^{r,P}F_1^u)_M^{\vee,K}(\gamma) = |D^M(\gamma)|^{\frac{1}{2}}\int_{A_M(\mathbb{R})\backslash M(\mathbb{R})}(\mathcal{J}_M^{r,P}F_1^u)^{\vee,K}(m^{-1}\gamma m)dm$$

of the function

$$(\mathcal{J}_M^{r,P}F_1^u)^{\vee,K} = \pi^K\big((\mathcal{J}_M^{r,P}F_1^u)^\vee\big).$$

We conclude that

$$\Phi_P^u(\gamma) = |W^M(M_R)||W(M_R)|^{-1}(\mathcal{J}_M^{r,P}F_1^u)_M^{\vee,K}(\gamma).$$

Finally, we apply the expansion (5.3) to the V-valued orbital integral (6.8). More precisely, we apply the relevant variant of (5.3), in which G is replaced by M and $T_{\mathrm{temp}}(G)$ is replaced by the set $T_{\mathrm{temp}}(M)/i\mathfrak{a}_M^*$ of $i\mathfrak{a}_M^*$-orbits in $T_{\mathrm{temp}}(M)$, to the orbital integral of the function defined by the outer integral in (6.7). This gives rise to a double integral of a function on the set

$$\{\tau_\lambda :\ \tau \in T_{\mathrm{temp}}(M)/i\mathfrak{a}_M^*,\ \lambda \in (-\varepsilon_P + i\mathfrak{a}_M^*)\}.$$

Notice that the symbol $(\mathcal{J}_M^{r,P} F_1^u)^{\vee,K}$ at this point represents two different objects, the function on $M_R(\mathbb{R})$ above and the function on the right hand side of (5.10). However, an appeal to Lemma 5.1 tells us that the two objects are in fact compatible. Let us write $T_\varepsilon^u(M)$ for the set of elements $\tau \in T_\varepsilon(M)$ that can be represented by a triplet $(M_\tau, \rho_\tau, r_\tau)$ with $M_\tau = M_1^u$. We set $\varepsilon = \varepsilon_P$, as in the statement of the theorem. The orbital integral (6.8) then has an expansion

$$\int_{T_{-\varepsilon}^u(M)} \theta_M(\gamma, \tau^\vee) d_\rho^{-1} e^K \big(\widetilde{R}^r(r_\tau, \rho_\tau) \mathcal{J}_M^{r,P}(\Lambda_\tau) F_1^u(\Lambda_\tau)\big) d\tau, \tag{6.9}$$

in the notation of (5.10). It follows that $\Phi_P^u(\gamma)$ equals the product of

$$|W^M(M_R)||W(M_R)|^{-1}$$

with (6.9). This completes the first stage of the proof.

The second stage of the proof applies to the given function $f \in \mathcal{H}(G)$. We have an expansion

$$f = \sum_{\sigma=(\sigma_1,\sigma_2)} f_\sigma, \qquad \sigma_i \in \Pi(K),$$

as in the proof of Corollary 5.2, which is finite in this case since f itself is K-finite. We need only establish Theorem 6.1 for each of the components $f_\sigma \in \mathcal{H}(G)$. We can therefore assume that $f = f_\sigma$, for a fixed pair $\sigma = (\sigma_1, \sigma_2)$ of irreducible representations of K.

As a Schwartz function, f satisfies the Fourier inversion formula (1.11). It is thus a finite sum of functions $(\widehat{f}_{P_1})^\vee$, where

$$\widehat{f}_{P_1}(\pi_{1,\lambda_1}) = \mathcal{I}_{P_1}(\pi_{1,\lambda_1}, f^\vee), \qquad \pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1,\ \lambda_1 \in i\mathfrak{a}_1^*,$$

for a Levi subgroup M_1 and a parabolic subgroup $P_1 \in \mathcal{P}(M_1)$. The fact that f lies in $\mathcal{H}(G)$ implies that each Schwartz function

$$\lambda_1 \longrightarrow \widehat{f}_{P_1}(\pi_{1,\lambda_1})$$

extends to an entire function on $\mathfrak{a}^*_{1,\mathbb{C}}$ of Paley-Wiener type. This is actually more than we require. We need only assume that $f \in \mathcal{C}(G)$ is such that for each M_1 and π_1, $\widehat{f}_{P_1}(\pi_{1,\lambda_1})$ extends to an analytic function of rapid decrease on a cylindrical neighbourhood of $i\mathfrak{a}^*_1$. In particular, the right hand side of the putative limit (6.1) remains well defined under this condition. The weaker condition on f remains in force if f is replaced by any of the components $(\widehat{f}_{P_1})^\vee$. It would therefore be enough to prove the theorem in the special case that f has a single component $(\widehat{f}_{P_1})^\vee$.

We therefore assume that the function $f = f_\sigma$ equals $(\widehat{f}_{P_1})^\vee$, for a fixed Levi subgroup M_1. We set $\tau = \tau_\sigma$ and

$$F_1(\lambda_1) = \bigoplus_{\pi_1 \in \Pi_{\mathrm{cusp}}(M_1)^1} d_{\pi_1} \psi\big(\widehat{f}_{P_1}(\pi_{1,\lambda_1})\big), \tag{6.10}$$

as in the proof of Corollary 5.2. Then F_1 is an analytic function of rapid decrease on a cylindrical neighbourhood of $i\mathfrak{a}^*_1$ in $\mathfrak{a}^*_{1,\mathbb{C}}$, with values in the finite dimensional vector space $\mathcal{A}_{\mathrm{cusp}}(M_1, \tau_1)$. Following the proof of Corollary 5.2, we use (1.12) and (5.12) to deduce that

$$f^{uT}(x) = \mathrm{tr}\big((F_1^{uT})^\vee(x)\big), \qquad x \in G(\mathbb{R}),$$

for any element $u \in U(M, M_1)$. Since the noninvariant Fourier transform of f is supported on the conjugacy class of M_1, we conclude that

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \big(J_M(\gamma_T, f_T)\big) = \sum_{u \in U(M,M_1)} \Big(\lim_{\substack{T \longrightarrow \infty \\ P,r}} \mathrm{tr}\big(J_M\big(\gamma_T, (F_1^{uT})^\vee\big)\big)\Big). \tag{6.11}$$

It remains only to apply the limit formula we obtained in the first stage of the proof to the summands on the right.

We fix an element $u \in U(M, M_1)$, and adopt the notation of the earlier part of the proof. The function

$$F_{P(R)}(\Lambda) = F_1^u(\Lambda), \qquad \Lambda \in \mathfrak{a}_{R,\mathbb{C}}^*,$$

is defined in terms of F_1 by the relation (6.5). The Fourier transform $\widehat{f}$ of f automatically satisfies a parallel symmetry condition, which can be related to that of (6.5) by (5.13) and [A5, (2.15)]. It follows from (6.10) that

$$F_{P(R)}(\Lambda) = \bigoplus_{\rho \in \Pi_{\mathrm{cusp}}(M_R)^1} d_\rho \psi\big(\widehat{f}_{P(R)}(\rho_\Lambda)\big).$$

Suppose that ρ lies in $\Pi_{\mathrm{cusp}}(M_R)^1$. We shall write

$$S: \ \mathcal{A}_\rho(M_R, \tau_R) \xrightarrow{\ \sim\ } \mathrm{End}_\sigma\big(\mathcal{H}_{P(R)}(\rho)\big)$$

for the inverse of the mapping ψ in (5.11). If ψ_ρ is any vector in $\mathcal{A}_\rho(M_R, \tau_R)$ and $\widetilde{\psi}_\rho = d_\rho \psi_\rho$, we have

$$\mathrm{tr}\big(d_\rho^{-1} e^K(\widetilde{\psi}_\rho)\big) = \mathrm{tr}\big(e^K(\psi_\rho)\big) = \mathrm{tr}\big(E_{P(R)}(1, \psi_\rho, 1)\big) = \mathrm{tr}\big(S(\psi_\rho)\big),$$

by (5.12). We will substitute this formula, with

$$\widetilde{\psi}_\rho = \widetilde{R}^r(r_\tau, \rho_\tau) \mathcal{J}_M^{r,P}(\Lambda_\tau) F_{P(R)}(\Lambda_\tau), \qquad \tau \in T_{-\varepsilon}^u(M),$$

into the expansion (6.9).

As usual, the given element $\tau \in T_{-\varepsilon}^u(M)$ is represented by the triplet $(M_\tau, \rho_\tau, r_\tau)$, while ρ and Λ_τ are as in (5.9). The corresponding function $\widetilde{\psi}_\rho$ depends only on the projection of $F_{P(R)}(\Lambda_\tau)$ onto $\mathcal{A}_\rho(M_R, \tau_R)$. Since this projection equals the product of d_ρ with the function

$$\psi\big(\widehat{f}_{P(R)}(\rho_\tau)\big) = \psi\big(\mathcal{I}_{P(R)}(\rho_\tau, f^\vee)\big),$$

we see that

$$\psi_\rho = \widetilde{R}^r(r_\tau, \rho_\tau) \mathcal{J}_M^{r,P}(\Lambda_\tau) \psi\big(\mathcal{I}_{P(R)}(\rho_\tau, f^\vee)\big).$$

In order to describe $S(\psi_\rho)$, we write π for the induced representation $\mathcal{I}_{R_\tau}(\rho_\tau) = \mathcal{I}_{R_\tau}(\rho_{\Lambda_\tau})$, as in (5.8). It then follows from (5.13), [A10, (R.5)], and the various definitions that

$$\begin{aligned}
&\mathcal{J}_M^{r,P}(\Lambda_\tau)\psi\big(\mathcal{I}_{P(R)}(\rho_\tau, f^\vee)\big)\\
&= \lim_{\zeta\to 0} \sum_{Q\in\mathcal{P}(M)} \big(J^r_{P(R)|Q(R)}(-\zeta+\Lambda_\tau)J^r_{P(R)|Q(R)}(\Lambda_\tau)^{-1}\theta_Q(\zeta)^{-1}\big)\psi\big(\mathcal{I}_P(\pi, f^\vee)\big)\\
&= \lim_{\zeta} \sum_Q \psi\big(J_{P|Q}(\pi_{-\zeta})J_{P|Q}(\pi)^{-1}\mathcal{I}_P(\pi, f^\vee)\big)\theta_Q(\zeta)^{-1}\\
&= \lim_{\zeta} \sum_Q \psi\big(J_{Q|P}(\pi^\vee_\zeta)^\vee\big(J_{Q|P}(\pi^\vee)^{-1}\big)^\vee \mathcal{I}_P(\pi^\vee, f)^\vee\big)\theta_Q(\zeta)^{-1}\\
&= \lim_{\zeta} \sum_Q \psi\big(\big(\mathcal{I}_P(\pi^\vee, f)J_{Q|P}(\pi^\vee)^{-1}J_{Q|P}(\pi^\vee_\zeta)\big)^\vee\big)\theta_Q(\zeta)^{-1}\\
&= \psi\big(\big(\mathcal{I}_P(\pi^\vee, f)\mathcal{J}_M(\pi^\vee, P)\big)^\vee\big).
\end{aligned}$$

We have identified the representation $\mathcal{I}_P(\pi^\vee)$ here with the contragredient of $\mathcal{I}_P(\pi)$, so that $\mathcal{I}_P(\pi, f^\vee)$ equals the transpose $\mathcal{I}_P(\pi^\vee, f)^\vee$ of the operator $\mathcal{I}_P(\pi^\vee, f)$. It follows from (5.14) that

$$\begin{aligned}
\psi_\rho &= \widetilde{R}^r(r_\tau, \rho_\tau)\psi\big(\big(\mathcal{I}_P(\pi^\vee, f)\mathcal{J}_M(\pi^\vee, P)\big)^\vee\big)\\
&= \psi\big(\widetilde{R}(r_\tau, \rho_\tau)\big(\mathcal{I}_P(\pi^\vee, f)\mathcal{J}_M(\pi^\vee, P)\big)^\vee\big)\\
&= \psi\big(\big(\mathcal{I}_P(\pi^\vee, f)\mathcal{J}_M(\pi^\vee, P)\widetilde{R}(r_\tau, \rho_\tau^\vee)\big)^\vee\big),
\end{aligned}$$

since $\widetilde{R}(r_\tau, \rho_\tau)$ is the transpose of the operator $\widetilde{R}(r_\tau, \rho_\tau^\vee)$. We conclude that

$$S(\psi_\rho) = \big(\mathcal{I}_P(\pi^\vee, f)\mathcal{J}_M(\pi^\vee, P)\widetilde{R}(r_\tau, \rho^\vee_\tau)\big)^\vee.$$

We have now established that

$$\begin{aligned}
\mathrm{tr}\big(d_\rho^{-1}e^K(\widetilde{\psi}_\rho)\big) &= \mathrm{tr}\big(S(\psi_\rho)\big)\\
&= \mathrm{tr}\big(\mathcal{I}_P(\pi^\vee, f)\mathcal{J}_M(\pi^\vee, P)\widetilde{R}(r_\tau, \rho^\vee_\tau)\big).
\end{aligned}$$

The last expression in turn can be written as

$$\begin{aligned}
&\mathrm{tr}\big(\widetilde{R}(r_\tau, \rho^\vee_\tau)\mathcal{I}_P(\pi^\vee, f)\mathcal{J}_M(\pi^\vee, P)\big)\\
=&\mathrm{tr}\big(\mathcal{I}_P(\pi^\vee, f)\widetilde{R}(r_\tau, \rho^\vee_\tau)\mathcal{J}_M(\pi^\vee, P)\big)\\
=&\mathrm{tr}\big(\widetilde{R}(r_\tau, \rho^\vee_\tau)\mathcal{J}_M(\pi^\vee, P)\mathcal{I}_P(\pi^\vee, f)\big)\\
&= J_M^P(\tau^\vee, f),
\end{aligned}$$

by definition (5.8). This gives us a formula for the trace of the integrand in (6.9). Making the substitution into (6.9), we see that the trace of (6.9) itself equals

$$\int_{T^u_{-\varepsilon}(M)} \theta(\gamma, \tau^\vee) J^P_M(\tau^\vee, f) d\tau.$$

But $\tau \to \tau^\vee$ is a measure preserving bijection from $T^u_{-\varepsilon}(M)$ to $T^u_\varepsilon(M)$. We conclude from the first part of the proof that

$$\operatorname{tr}\bigl(\Phi^u_P(\gamma)\bigr) = |W^M(M_R)||W(M_R)|^{-1} \int_{T^u_\varepsilon(M)} \theta_M(\gamma, \tau) J^P_M(\tau, f) d\tau.$$

We remind ourselves again that the vectors $\Phi^u_P(\gamma)$ and $e^K(\cdot)$ in (6.9) are now endomorphisms, by virtue of our choice $\tau = \tau_\sigma$, and therefore do have traces.

Our aim in calculating the trace of $\Phi^u_P(\gamma)$ has of course been to be able to apply Lemma 4.3. Combining Lemma 4.3 with (6.11) and our formula for the trace of $\Phi^u_P(\gamma)$, we find that for our given function f, the limit

$$\lim_{\substack{T \longrightarrow \infty \\ P, r}} J_M(\gamma_T, f_T)$$

equals

$$\sum_{u \in U(M, M_1)} |W^M(M_R)||W(M_R)|^{-1} \int_{T^u_\varepsilon(M)} \theta_M(\gamma, \tau) J^P_M(\tau, f) d\tau,$$

uniformly for $\gamma \in \Gamma$. The set $T^u_\varepsilon(M)$ depends only on the Levi subgroup $M^u_1 = M_R$ of M. We can therefore apply the fibration $u \to M^u_1$ of Lemma 1.2 to the sum over u. The limit can thus be written as

$$\sum_{\{M^u_1\}} \int_{T^u_\varepsilon(M)} \theta_M(\gamma, \tau) J^P_M(\tau, f) d\tau,$$

where $\{M^u_1\}$ varies over W^M_0-orbits of Levi subgroups of M. Finally, our choice of f is such that the function

$$J^P_M(\tau, f), \qquad \tau \in T_\varepsilon(M),$$

vanishes unless τ belongs to one of the subsets $T^u_\varepsilon(M)$ of $T_\varepsilon(M)$. It follows that the limit equals the right hand side of (6.1). The limit formula (6.1) thus holds, and is uniform for γ in the relatively compact set Γ. $\square$

We have established our asymptotic formula for weighted orbital integrals. Our ultimate aim, to be addressed in the paper [A13], is to compare distributions on different groups. For this, we require a parallel asymptotic formula for the associated invariant distributions.

In §5, we introduced a (G, M)-family

$$m_Q(\zeta, \pi, P) = \mu_{Q|P}(\pi)\mu_{Q|P}(\pi_{\frac{1}{2}\zeta})^{-1}, \qquad Q \in \mathcal{P}(M),\ \zeta \in i\mathfrak{a}_M^*,$$

of meromorphic functions of $\pi \in \Pi(M)$. The functions in this family are analytic at any representation in the set

$$\Pi_\varepsilon(M) = \big\{\pi_\varepsilon :\ \pi \in \Pi_{\mathrm{temp}}(M),\ \varepsilon = \varepsilon_P\big\}.$$

Suppose that $\tau \in T_\varepsilon(M)$ is represented by a triplet $(M_\tau, \rho_\tau, r_\tau)$. The induced representation $\pi = \mathcal{I}_{R_\tau}(\rho_\tau)$ of $M(\mathbb{R})$ introduced in §5 is a finite direct sum of irreducible representations $\pi_\alpha \in \Pi_\varepsilon(M)$. The functions

$$m_Q(\zeta, \tau, P) = m_Q(\zeta, \pi, P) = m_Q(\zeta, \pi_\alpha, P)$$

then depend only on π, and hence τ, rather than the constituent π_α of π. They give rise to a slowly increasing function

$$m_M(\tau, P) = \lim_{\zeta \to 0} \sum_{Q \in \mathcal{P}(M)} m_Q(\zeta, \tau, P)\theta_Q(\zeta)^{-1}$$

of τ. Let us write

$$I_M^P(\tau, f) = \mathrm{tr}\big(m_M(\tau, P)\mathcal{I}_P(\tau, f)\big) = m_M(\tau, P) f_M(\tau), \tag{6.12}$$

for any function $f \in \mathcal{H}(G)$.

Corollary 6.2. *Suppose that $f \in \mathcal{H}(G)$ and $\varepsilon = \varepsilon_P$ are as in the statement of the theorem. Then*

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} I_M(\gamma_T, f_T) = \int_{T_\varepsilon(M)} \theta_M(\gamma, \tau) I_M^P(\tau, f)\, d\tau, \tag{6.13}$$

uniformly for γ in the relatively compact subset $\Gamma = T_{M,G\text{-reg}}(\mathbb{R})^C$ of $T_M(\mathbb{R})$.

Proof. We shall apply the definition (1.4) to the limit formula (6.1) we have just established. The left hand side of (6.1) equals the sum of the limit on the left hand side of (6.13) with the sum over $L \in \mathcal{L}(M)$, $L \neq G$, of the limits

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \widehat{I}^L_M\big(\gamma_T, \phi_L(f_T)\big). \tag{6.14}$$

We shall apply the formula (6.13) inductively to (6.14).

We first write

$$\widehat{I}^L_M\big(\gamma_T, \phi_L(f_T)\big) = \sum_{\{M_1\}} \sum_{u \in U(M,M_1)} \widehat{I}^L_M\big(\gamma_T, \phi_L(f^{uT})\big),$$

where $\{M_1\} = \{M_1\}_G$ as usual denotes a set of representatives of $G(\mathbb{R})$-conjugacy classes of Levi subgroups M_1 of G. The group $W(M_1)$ acts by left translation on the set $U(M, M_1) = U^G(M, M_1)$. It follows from the identity (5.17) obtained in the proof of Corollary 5.2 that

$$\sum_{\{M_1\}} \sum_u \phi_L(f^{uT}) = \sum_{\{M_1\}} \sum_u \sum_{i=1}^k \phi_L(f)^{w_i u T},$$

for any $L \in \mathcal{L}(M)$. We recall that i indexes a set of representatives $M_{1,i} = \widetilde{w}_i M_1 \widetilde{w}_i^{-1}$ of the L-conjugacy classes of Levi subgroups of L that lie in the G-conjugacy class of M_1. For any i, the mapping $u \to w_i u$ is a bijection from $U^G(M, M_1)$ to $U^G(M, M_{1,i})$. Changing notation, we can therefore write

$$\widehat{I}^L_M\big(\gamma_T, \phi_L(f_T)\big) = \sum_{\{M_1\}_L} \sum_{u \in U^G(M,M_1)} \widehat{I}^L_M\big(\gamma_T, \phi_L(f)^{uT}\big).$$

Consider a summand in which u lies in the complement of $U^L(M, M_1)$ in $U^G(M, M_1)$. We shall estimate it by the inequality (5.16) of Corollary 5.2, with L in place of G, and the chamber $\mathfrak{a}_0^+ \subset \mathfrak{a}^+_{L \cap P_0}$ in place of the cone $c_0 \subset \mathfrak{a}_0^+$. The point T is certainly $(\mathfrak{a}_0^+, uT)$-dominant (relative to L), while the corresponding distance function satisfies

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} d^L_{\mathfrak{a}_0^+}(T, uT) = 0.$$

It follows from (5.16) that

$$\lim_{\substack{T \longrightarrow \infty \\ P,r}} \widehat{I}_M^L\big(\gamma_T, \phi_L(f)^{uT}\big) = 0,$$

for any u in the complement of $U^L(M, M_1)$ of $U^G(M, M_1)$. We see therefore that

$$\begin{aligned} &\lim_{\substack{T \longrightarrow \infty \\ P,r}} \widehat{I}_M^L\big(\gamma_T, \phi_L(f_T)\big) \\ =&\lim_{\substack{T \longrightarrow \infty \\ P,r}} \sum_{\{M_1\}_L} \sum_{u \in U^L(M,M_1)} \widehat{I}_M^L\big(\gamma_T, \phi_L(f)^{uT}\big) \\ =&\lim_{\substack{T \longrightarrow \infty \\ P,r}} \widehat{I}_M^L\big(\gamma_T, \phi_L(f)_T\big). \end{aligned}$$

A small technical complication arises when we apply (6.13) inductively to the last formula for (6.14). The point is that the mapping ϕ_L does not take the space $\mathcal{H}(G)$ into $I\mathcal{H}(L)$. However, ϕ_L does not map the slightly larger space $\mathcal{H}_{\mathrm{ac}}(G)$, introduced in [A6], to its invariant analogue $I\mathcal{H}_{\mathrm{ac}}(L)$ for L. An element in $I\mathcal{H}_{\mathrm{ac}}(L)$ can be regarded as a smooth function ϕ_L on $T_{\mathrm{temp}}(L) \times \mathfrak{a}_L$ that satisfies

$$\phi_L(\tau_\lambda, X) = e^{\lambda(X)} \phi_L(\tau, X), \qquad \lambda \in i\mathfrak{a}_L^*, \ X \in \mathfrak{a}_L,$$

and which as a function τ, belongs to the $i\mathfrak{a}_L^*$-invariant Paley-Wiener space on $T_{\mathrm{temp}}(L)$. The distributions $J_M(\gamma_T, f_T)$ and $I_M(\gamma_T, f_T)$ each depend only on the restriction of f to the closed subset

$$G(\mathbb{R})^{H_G(\gamma)} = \big\{x \in G(\mathbb{R}): \ H_G(x) = H_G(\gamma)\big\}$$

of $G(\mathbb{R})$. For example, the right hand side of the putative formula (6.13) can be written

$$\int_{T_\varepsilon(M)/i\mathfrak{a}_G^*} \theta_M(\gamma, \tau) m_M(\tau, P) f_G\big(\tau, H_G(\gamma)\big) d\tau,$$

where $T_\varepsilon(M)/i\mathfrak{a}_G^*$ is the space of $i\mathfrak{a}_G^*$-orbits in $T_\varepsilon(M)$, and $f_G\big(\tau, H_G(\gamma)\big)$ is the integral over $G(\mathbb{R})^{H_G(\gamma)}$ of the product of f with the virtual character induced from τ. This expression makes sense if f is replaced by a function in $\mathcal{H}_{\mathrm{ac}}(G)$. The resulting modification of (6.13) can be applied inductively to the function $\phi_L(f)$ in $I\mathcal{H}_{\mathrm{ac}}(L)$, and holds uniformly for γ in Γ. This leads to an expression

$$\int_{T_\varepsilon(M)/i\mathfrak{a}_L^*} \theta_M(\gamma,\tau)m_M(\tau, P\cap L)\phi_L\big(f,\tau^L,H_L(\gamma)\big)d\tau$$

for the limit (6.14). But $\phi_L\big(f,\tau^L,H_L(\gamma)\big)$ is equal to the integral

$$J_L\big(\tau^L,H_L(\gamma),f\big) = \int_{i\mathfrak{a}_L^*} J_L(\tau_\lambda^L,f)e^{-\lambda(H_L(\gamma))}d\lambda.$$

It follows from Fourier inversion on the group $\mathfrak{a}_L$ that (6.14) equals

$$\int_{T_\varepsilon(M)} \theta_M^L(\gamma,\tau)m_M(\tau,P\cap L)J_L(\tau^L,f)d\tau.$$

Let us now add the right hand side of (6.13) to the sum over $L\neq G$ of (6.14). This yields an expression

$$\sum_{L\in\mathcal{L}(M)}\int_{T_\varepsilon(M)} \theta_M(\gamma,\tau)m_M(\tau,P\cap L)J_L(\tau^L,f)d\tau. \tag{6.15}$$

Applying the usual formula [A3, Lemma 6.3] to the product (5.4) of (G,M)-families, we see from the definitions (5.7) and (5.8) that

$$\sum_{L\in\mathcal{L}(M)} m_M(\tau,P\cap L)J_L(\tau^L,f) = J_M^P(\tau,f).$$

It follows that (6.15) equals the right hand side of (6.1). The point of Theorem 6.1 was of course to establish the equality of the left and right hand sides of (6.1). Given what we have just proved, this yields the equality of the left and right hand sides of (6.13). The limit formula (6.13) thus holds, and is uniform for γ in the relatively compact set Γ. $\square$

We have completed the proof of our two asymptotic formulas. The invariant formula (6.13) of Corollary 6.2 is the result we shall use in the next paper [A13]. We have established it in the form most suitable for application, rather than aim for optimal generality. There are three ways in which it could be extended. We shall discuss these in turn, limiting ourselves in each case to a few sketchy remarks.

We have taken the torus T_M in M to be elliptic. This is a superficial constraint, imposed only to insure that the set $\Gamma = T_{M,G\text{-reg}}(\mathbb{R})^C$ be relatively compact in $T_M(\mathbb{R})$. Suppose that T_1 is an arbitrary maximal torus in M over $\mathbb{R}$, and that Γ_1 is any subset of $T_{1,G\text{-reg}}(\mathbb{R})$ that is relatively compact in $T_1(\mathbb{R})$. The proof of (6.13) we have established for elliptic T_M over the course of the paper works also for T_1 and Γ_1, with only occasional changes in notation. Alternatively, one can apply formulas of descent ([A7, Proposition 7.1], [A7, Theorem 8.1]) to reduce the general form of (6.13) to the elliptic case.

We have taken f to be a function in the Hecke algebra. This constraint is more substantial. There are two steps to be taken in order to obtain a formula that applies to f in the Schwartz space. The first would be to show that the limit on the left hand side of (6.13) exists uniformly for $f \in \mathcal{C}(G)$. The second would be to transform the right hand side of (6.13) to a tempered linear form in f. Together, they would yield a version of (6.13) that applies to any $f \in \mathcal{C}(G)$.

For the first step, it would be necessary to strengthen Lemma 4.3. One would need a uniform estimate for the limit (4.15) of this lemma that is similar to the estimate (4.17) of Lemma 4.4. More precisely, one would want to prove the existence of a continuous seminorm $\|\cdot\|_1$ on $\mathcal{C}(G)$ and a positive function $c(T)$ that approaches 0 as T approaches infinity in $\mathfrak{a}_P^r$ such that

$$\|J_M(\gamma_T, f_1^{uT}) - \Phi_P^u(\gamma)\| \leq c(T)\|f_1\|, \qquad f_1 = F^\vee,\ u \in U(M, M_1),$$

for any (τ, V), $\gamma \in \Gamma$, $T \in \mathfrak{a}_P^r$, and any $F \in \mathcal{C}\big(i\mathfrak{a}_1, \mathcal{A}_{\text{cusp}}(M_1, \tau_1)\big)$ that satisfies the symmetry condition (2.2). (The seminorms $\|\cdot\|$ and $\|\cdot\|_1$ on each side of this inequality are meant to follow the conventions of their counterparts in (4.17).) An estimate of this nature ought to be accessible with a more detailed analysis of the arguments of §2–§4. It would serve as a companion to the uniform estimate of Lemma 4.4. Together with the consequence Corollary 5.2 of Lemma 4.4, it would likely yield a proof that the limit on the left hand side of (6.13) exists uniformly for $f \in \mathcal{C}(G)$.

For the second step, we would need to transform the contour of integration on the right hand side of (6.13). The function $\tau \to m_M(\tau, P)$ in (6.13) has poles of order 1 at certain singular hyperplanes in $T_{\mathrm{temp}}(M)$. A deformation of the contour of integration from $T_\varepsilon(M)$ to $T_{\mathrm{temp}}(M)$ would consequently lead to multidimensional residues, thereby contributing a sum of integrals over subsets of $T_{\mathrm{temp}}(M)$ of lesser dimension. I have not analyzed the combinatorics of the process. They are probably simpler than those of the derivation of the spectral side of the local trace formula, but they also seem to be slightly different.

The local trace formula incidentally is relevant to the topics of this paper. I was not able to use it to simplify any of the arguments here. However, a tempered formulation of (6.13) would have features in common with [A10, (4.1)], a formula obtained directly from the local trace formula. We recall that [A10, (4.1)] gives qualitative description for the Fourier transform of $I_M(\gamma, f)$. If for no other reason, it would be worthwhile to carry out the change of contour in (6.13) in order to be able to compare the structure of the resulting formula with that of [A10, (4.1)]. Such a comparison might give new interpretations for the coefficients $i(\tau)$ [A10, p. 182] that occur in both the local and global trace formulas.

Finally, we have taken the underlying field to be the real numbers. It would have been quite feasible to work with an arbitrary local field F of characteristic 0 instead of $\mathbb{R}$. Our reason for not doing so is twofold. First of all, the application in [A13] uses differential equations, and therefore works only for real groups. Secondly, the p-adic case seems to merit closer inspection. Recall that Harish-Chandra's asymptotic estimates for Eisenstein integrals are simpler for p-adic groups. For example, a supercuspidal Eisenstein integral is actually equal to its constant term in some asymptotic region. Do such properties extend to the asymptotic formulas of this paper? An affirmative answer could conceivably have implications for the local trace formula. Suppose that one chooses the two test functions in the local trace formula to be unramified. The spectral side takes its usual simple form, while the geometric side becomes a sum over Levi subgroups of inner products of unramified weighted orbital integrals. Suitable asymptotic formulas might allow one to compute some of the terms in these inner products.

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