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THE FOURIER TRANSFORM OF WEIGHTED ORBITAL INTEGRALS ON SL(2, $\mathbb{R}$)

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1. INTRODUCTION. A study of the adelic version of the Selberg trace formula leads naturally to the analysis of certain tempered distributions on reductive groups over local fields [1a], [4], [8]. The invariant distributions which arise in this context appear mainly as ordinary orbital integrals and their limits. The Fourier analysis of ordinary orbital integrals has been studied extensively over the past decade, and, in the case of real reductive groups, this analysis may now be regarded as essentially complete [2], [7b], [10].

The situation for p-adic groups is much less satisfactory, and considerable work remains to be done in that case (see [9] for more details).

Along with the ordinary orbital integrals, certain non-invariant distributions called weighted orbital integrals arise as additional terms in the trace formula. The Fourier analysis of weighted orbital integrals is more complicated than that of the ordinary orbital integrals. However, it is likely that, to fully apply the general trace formula, one will have to understand these distributions and their Fourier transforms.

Let G be an acceptable, real semisimple Lie group. Let A be the split component of a parabolic subgroup of G, and suppose that $T = T_I T_{\mathbb{R}}$ is a Cartan subgroup of G such that $A \subset T_{\mathbb{R}}$. If $v(x)$, $x \in G$, is the weight factor associated to the class $\mathcal{P}(A)$ of parabolic subgroups of G [1c], then the weighted orbital integral associated to A is defined as

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$$T_f(h) = \epsilon_{\mathbb{R}}(h)\Delta(h)\int_{G/T_{\mathbb{R}}} f(xhx^{-1})\,v(x)\,d\dot{x}\,, \quad h \in T'_{\mathbb{R}}\,. \tag{1.1}$$

Here, $f \in \mathcal{C}(G)$, the Schwartz space of G, $\Delta(h)$ is the usual discriminant, $\epsilon_{\mathbb{R}}(h) = \pm 1$, and $d\dot{x}$ is an invariant measure on $G/T_{\mathbb{R}}$.

The properties of the distributions $f \mapsto T_f(h)$ have been studied extensively by Arthur [1b, c]. In particular, Arthur computes the Fourier transform of T_f when $f \in \mathcal{C}_o(G)$, the space of cusp forms on G. In another direction, under the assumption that $A \neq T_{\mathbb{R}}$, Herb [7a] has determined the Fourier transform of T_f on a subspace of $\mathcal{C}(G)$ which is strictly larger than $\mathcal{C}_o(G)$. In the cases considered by Arthur and Herb, the distribution T_f is actually an invariant distribution when restricted to the appropriate subspace of $\mathcal{C}(G)$. Thus, the Fourier transform may be written as a "linear combination" of tempered characters of G.

The distribution T_f is decidedly non-invariant on $\mathcal{C}(G)$. There is, at present, no general theory concerning the Fourier transforms of tempered, non-invariant distributions. We approach the problem as follows.

Let $(\pi, \mathcal{H}_\pi) \in \hat{G}_{temp}$, the space of (equivalence classes of) irreducible, tempered representations of G. We set

$$\pi(f) = \int_G f(x)\,\pi(x)\,dx, \quad f \in \mathcal{C}(G)\,. \tag{1.2}$$

Then $\pi(f)$ is of trace class.

Now, if Λ is a non-invariant, tempered distribution on G, we seek a "distribution" $\mathfrak{F}\Lambda$ on $\hat{G}_{temp}$ such that

$$(\mathfrak{F}\Lambda)(\mathfrak{F}f) = \Lambda(f), \quad f \in \mathcal{C}(G)\,, \tag{1.3}$$

where $\mathfrak{F}f(\pi) = \pi(f)$ for $\pi \in \hat{G}_{temp}$. More specifically, for each $\pi \in \hat{G}_{temp}$, we expect that there is an operator $A_\Lambda(\pi)$ on $\mathcal{H}_\pi$ and a measure $d_\Lambda(\pi)$ such that

$$\Lambda(f) = \int_{\hat{G}_{temp}} \operatorname{tr}\{A_\Lambda(\pi)\,\pi(f)\}\,d_\Lambda(\pi), \quad f \in \mathcal{C}(G)\,. \tag{1.4}$$

We note that the operator $A_\Lambda(\pi)$ and the measure $d_\Lambda(\pi)$ are not well-defined independently of each other.

The goal of this paper is to derive the full Fourier transform of the distribution T_f for the group $G = SL_2(\mathbb{R})$. As noted above, Arthur [1b, c] has computed the Fourier transform of this distribution when $f \in \mathcal{C}_o(G)$. We therefore restrict our attention to functions in the orthogonal complement of the space of cusp forms. In particular, we compute the Fourier transform of T_f when f is a wave packet corresponding to a principal series representation of G.

The representation theory of G is well-known. In order to state our results, we first define the following subgroups of G. Let

$$
\begin{aligned}
K &= SO(2,\mathbb{R}) = \left\{ t_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \le \theta \le 2\pi \right\}; \\
N &= \left\{ n_y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}; \\
\overline{N} &= \left\{ \bar{n}_y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} : y \in \mathbb{R} \right\}; \\
A &= \left\{ h_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}; \\
A_I &= \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}; \\
P &= A_I AN .
\end{aligned}
\tag{1.5}
$$

If $x \in G$ is decomposed as $x = k\bar{n}a$, $k \in K$, $\bar{n} \in \overline{N}$, $a \in A$, then we define

$$
v(x) = v(\bar{n}) = \frac{1}{2} \log(1+y^2), \quad \bar{n} = \bar{n}_y . \tag{1.6}
$$

The function $v(x)$ is the weight factor used to define the weighted orbital integral associated to the Cartan subgroup $H = A_I A$ of G (see [1b]).

For $f \in \mathcal{C}(G)$, the weighted orbital integral of f is given by

$$
T_f(\gamma h_t) = |e^t - e^{-t}| \int_{G/A} f(x\gamma h_t x^{-1})\, v(x)\, d\dot{x}, \quad \gamma \in A_I,\ t \neq 0 , \tag{1.7}
$$

where $d\dot{x}$ is a suitably normalized, G-invariant measure on G/A. This integral converges, and the map $f \mapsto T_f(\gamma h_t)$ defines a tempered distribution on G (see [1b]).

The Schwartz space $\mathcal{C}(G)$ decomposes naturally into a direct sum of subspaces determined by the various series of irreducible tempered representations of G. These series are:

(1) $\hat{G}_d$, the discrete series of G, parameterized by the non-trivial characters of K, which, in this case, is also a compact Cartan subgroup of G;

(2) $\hat{G}_c$, the principal series of G, parameterized by a character $\chi \in \hat{A}_I$ and a real number λ which determines a character of A by the formula $h_t \mapsto e^{i\lambda t}$. We denote the corresponding representation of G by $\pi^{\chi,\lambda}$. For fixed χ, the representations $\pi^{\chi,\lambda}$ and $\pi^{\chi,-\lambda}$ are equivalent, and, if χ is non-trivial, the representation $\pi^{\chi,0}$ splits into two irreducible components. If χ is the trivial character on A_I, we write $\pi^{\chi,0} = \pi^{+,0}$.

The representation $\pi^{\chi,\lambda}$ may be induced from P, $\pi^{\chi,\lambda} = \mathrm{Ind}_P^G(\chi \otimes e^{i\lambda} \otimes 1)$, and $\pi^{\chi,\lambda}$ is equivalent to $\overline{\pi}^{\chi,\lambda}$ where $\overline{\pi}^{\chi,\lambda} = \mathrm{Ind}_{\overline{P}}^G(\chi \otimes e^{i\lambda} \otimes 1)$, $\overline{P} = A_I A\overline{N}$. We denote by $M_\chi(\lambda)$ a suitably normalized intertwining operator for $\pi^{\chi,\lambda}$ and $\overline{\pi}^{\chi,\lambda}$, and by $M'_\chi(\lambda)$ the derivative of $M_\chi(\lambda)$ with respect to λ.

We now state the main theorem of this paper.

<u>THEOREM 1.8</u>. Let f be a K-finite function in $\mathcal{C}(G)$, $\gamma \in A_I$, $t \neq 0$. Then

$$T_f(\gamma h_t) = -|e^t - e^{-t}| \sum_{\pi \in \hat{G}_d} \Theta_\pi(\gamma h_t)\,\mathrm{tr}\,\pi(f)$$

$$+ \sum_{\chi \in \hat{A}_I} \chi(\gamma) \int_{-\infty}^{\infty} \varphi_\lambda(t)\,\mathrm{tr}\,\pi^{\chi,\lambda}(f)\,d\lambda$$

$$+ \sqrt{-1} \sum_{\chi \in \hat{A}_I} \chi(\gamma)\,\mathrm{P.V.} \int_{-\infty}^{\infty} e^{-i\lambda|t|}\,\mathrm{tr}\,\{M'_\chi(\lambda)M_\chi(\lambda)^{-1}\pi^{\chi,\lambda}(f)\}\,d\lambda$$

$$+ \pi\,\mathrm{tr}\,\pi^{+,0}(f) \ .$$

Here Θ_π is the character of the discrete series representation π, the (scalar-valued) function φ_λ is the solution of an inhomogeneous second order differential equation (see (2.19)), and the "π" preceding the trace in the last term is the real number $\pi \sim 22/7$.

It is clear from the above formula for T_f that the non-invariance of the distribution is embodied in the third term. (The principal value integral in this term is made precise in (2.20).) The expression in the first term represents the Fourier transform of T_f for $f \in \mathcal{C}_o(G)$. We shall derive the last three terms by working with wave packets f determined by a fixed K-type. In this case, the resulting operators $\pi^{\chi,\lambda}(f)$ are diagonal with respect to an appropriate basis and have exactly one non-zero entry. This simplifies the computations considerably.

An announcement of this theorem and a more detailed expository account of the Fourier transform and weighted orbital integrals appears in [7c].

In section 2 notation and background information are given, and Theorem 1.8 is restated for wave packets determined by a fixed K-type as Theorem 2.18. This theorem is proved in section 3. Finally, in section 4 we compute the Fourier transform of a singular distribution associated to the weighted orbital integral.

Our approach to the determination of the Fourier transform of weighted orbital integrals was outlined by the first author a number of years ago. More recently, Warner [11] has extended the results of Theorm 1.8 to semisimple Lie groups of real rank one. The singular distributions derived in section 4, which also play a role in the trace formula, seem more difficult to derive in the general (rank one) case. At this writing, these singular distributions have been determined only for SL_2.

2. <u>PRELIMINARIES</u>. Let $G = SL(2,\mathbb{R})$, the group of two-by-two matrices with real entries and determinant one. We will use the notation of (1.5) for subgroups and elements of G. Each $x \in G$ can be decomposed uniquely according to the Iwasawa decompositions $G = KNA$ or $G = K\overline{N}A$ as

$$x = \kappa(x)\, n(x)\, h(x) = \overline{\kappa}(x)\, \overline{n}(x)\, \overline{h}(x) \tag{2.1}$$

where $\kappa(x), \overline{\kappa}(x) \in K$, $n(x) \in N$, $\overline{n}(x) \in \overline{N}$, and $h(x), \overline{h}(x) \in A$. If $h(x) = h_t$ as in (1.5), write $H(x) = t$. Similarly, if $\overline{h}(x) = h_t$, write $\overline{H}(x) = t$.

Haar measures dk and $d\overline{n}$ on K and $\overline{N}$ respectively are normalized so that K has total volume one and so that $\int_{\overline{N}} e^{-2H(\overline{n})}\, d\overline{n} = 1$. Let $d\dot{x}$ denote the G-invariant measure on the quotient G/A which corresponds to the product measure $dk\,d\overline{n}$ on $K\overline{N}$. This makes precise the formula for $T_f(\gamma h_t)$ given in (1.7).

Using the left-invariance under K of the weighting function v of (1.6), we can also write

$$T_f(\gamma h_t) = |e^t - e^{-t}| \int_{\overline{N}} \bar{f}(\bar{n}\gamma h_t \bar{n}^{-1})\, v(\bar{n})\, d\bar{n} \tag{2.2}$$

where

$$\bar{f}(x) = \int_K f(kxk^{-1})\, dk \tag{2.3}$$

is the average of f over K. Thus, we see that it is sufficient to evaluate T_f for functions invariant under conjugation by K.

We will also use the familiar (unweighted) orbital integral defined for $f \in \mathcal{C}(G)$ by

$$F_f(\gamma h_t) = |e^t - e^{-t}| \int_{G/A} f(x\gamma h_t x^{-1})\, d\dot{x}\ , \quad \gamma \in A_I,\ t \neq 0\ . \tag{2.4}$$

The following properties of the weighted orbital integral are among those proved in [1b]. For fixed $f \in \mathcal{C}(G)$, $\gamma \in A_I$, $t \mapsto T_f(\gamma h_t)$ is an even function which is smooth for $t \neq 0$. (The behavior of $T_f(\gamma h_t)$ at $t = 0$ will be discussed in section 4.) Further, let z denote the left and right invariant differential operator on G given by $z = XY + YX + H^2 + 1$ where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then for all $f \in \mathcal{C}(G)$, $\gamma \in A_I$, $t \neq 0$,

$$\frac{d^2}{dt^2} T_f(\gamma h_t) = T_{zf}(\gamma h_t) + (\sinh t)^{-2} F_f(\gamma h_t)\ . \tag{2.5}$$

We now review some definitions and formulas of Harish-Chandra from [6a, c] regarding induced representations and wave packets. Let $\overline{P} = A_I A \overline{N}$, $P = A_I AN$. For $\chi \in \hat{A}_I$, let $\mathcal{H}^\chi = \{g \in L^2(K) : g(k\gamma) = \chi(\gamma)\, g(k)$ for all $\gamma \in A_I$, $k \in K\}$. Then for $\lambda \in \mathbb{R}$, the unitary principal series representation $\pi^{\chi,\lambda} = \operatorname{Ind}_P^G(\chi \otimes e^{i\lambda} \otimes 1)$ can be realized on $\mathcal{H}^\chi$ by

$$[\pi^{\chi,\lambda}(x)\, g](k) = e^{-(i\lambda+1)H(x^{-1}k)}\, g(\kappa(x^{-1}k)),\ x \in G,\ k \in K\ . \tag{2.6}$$

For $m \in Z$, let ω_m be the character of K given by $\omega_m(t_\theta) = e^{im\theta}$, $0 \le \theta \le 2\pi$.

Let $Z_\chi = \{m \in Z: \omega_m(\gamma) = \chi(\gamma)$ for all $\gamma \in A_I\}$. Then $\{\omega_m : m \in Z_\chi\}$ is a basis for $\mathcal{H}^\chi$ and for $m \in Z_\chi$ we define the Eisenstein integral

$$(2.7) \qquad E(m:\lambda:x) = \langle \pi^{\chi,\lambda}(x)\omega_m, \omega_m\rangle \ .$$

The matrix coefficient $E(m:\lambda)$ is not Schwartz class. In order to get an element of $\mathcal{C}(G)$ we take $\alpha \in \mathcal{C}(\mathbb{R})$, the Schwartz space on $\mathbb{R}$, and form the wave packet

$$(2.8) \qquad f(x) = f(\alpha : m : x) = \int_{-\infty}^{\infty} \alpha(\lambda) E(m:\lambda:x)\mu_\chi(\lambda)\, d\lambda$$

where $\mu_\chi(\lambda)$ is the Plancherel measure corresponding to the representation $\pi^{\chi,\lambda}$.

Since $E(m:\lambda)$ is an eigenfunction of the differential operator z of (2.5) with eigenvalue $p(\lambda) = -\lambda^2$, f satisfies the equation

$$(2.9) \qquad zf(\alpha:m) = f(p\alpha:m) \ .$$

Further, for $k_1, k_2 \in K$, $x \in G$,

$$(2.10) \qquad f(k_1xk_2) = \omega_m(k_1k_2)\, f(x) \ .$$

In particular, f is invariant under conjugation by K, which would not have been the case if we had started with an off-diagonal matrix coefficient. These wave packets span the space of K-central functions orthogonal to the space of cusp forms.

A further consequence of the transformation property of f under K together with the fact that A_I is central in G is that for f as in (2.8),

$$(2.11) \qquad T_f(\gamma h_t) = \chi(\gamma)\, T_f(h_t)\ , \quad \gamma \in A_I\ , \ t \neq 0 \ .$$

Thus it is enough to evaluate $T_f(h_t)$, $t \neq 0$.

For $f = f(\alpha:m)$ as in (2.8), the operators $\pi(f)$ defined in (1.2) are particularly simple. First, for $\pi \in \hat{G}_d$, $\pi(f) \equiv 0$. Also, for $\chi' \in \hat{A}_I$, $\chi' \neq \chi$, we have $\pi^{\chi',\lambda}(f) \equiv 0$ for all $\lambda \in \mathbb{R}$. Finally, for $m' \in Z_\chi$,

$$(2.12) \qquad \pi^{\chi,\lambda}(f)\omega_{m'} = \begin{cases} 0 & \text{if } m' \neq m \\ (\alpha(\lambda) + \alpha(-\lambda))\omega_m & \text{if } m' = m \end{cases}$$

so that $\pi^{\chi,\lambda}(f)$ has only one non-zero entry and it is on the diagonal.

Recall that for all $\chi \in \hat{A}_I$, $\lambda \in \mathbb{R}$, $\pi^{\chi,\lambda}$ is unitarily equivalent to the representation $\bar{\pi}^{\chi,\lambda} = \operatorname{Ind}_{\bar{P}}^{G}(\chi \otimes e^{i\lambda} \otimes 1)$ defined on $\mathcal{H}^\chi$ by

$$(2.13) \qquad [\bar{\pi}^{\chi,\lambda}(x)g](k) = e^{-(i\lambda-1)\bar{H}(x^{-1}k)} g(\bar{\kappa}(x^{-1}k)), \quad x \in G, \ k \in K .$$

Let $M^\chi(\lambda)$ denote the intertwining operator on $\mathcal{H}^\chi$ which satisfies $M^\chi(\lambda)\pi^{\chi,\lambda}(x) = \bar{\pi}^{\chi,\lambda}(x)M^\chi(\lambda)$ for all $x \in G$. With respect to the basis $\{\omega_m : m \in Z_\chi\}$, the intertwining operators are also diagonal matrices with

$$(2.14) \qquad M^\chi(\lambda)\,\omega_m = c_m(\lambda)\,\omega_m , \quad m \in Z_\chi .$$

Here $c_m(\lambda)$ is the c-function which is the meromorphic continuation to the real axis of the analytic function defined for $\mu \in \mathbb{C}$ with $\operatorname{Im} \mu < 0$ by

$$(2.15) \qquad c_m(\mu) = \int_{\bar{N}} \omega_m(\kappa(\bar{n}))\, e^{(-i\mu-1)H(\bar{n})}\, d\bar{n} .$$

We will need to know the poles and residues of $d_m(\lambda) = c_m'(\lambda)\, c_m(\lambda)^{-1}$ along the real axis. Using formulas of Cohn [3], we get

$$(2.16) \qquad c_m(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{i\lambda+1}{2}\right)}{\Gamma\left(\frac{i\lambda+1+m}{2}\right)\Gamma\left(\frac{i\lambda+1-m}{2}\right)}$$

Taking the logarithmic derivative, we find that

$$(2.17) \qquad d_m(\lambda) = \frac{i}{2}\left[\psi\left(\frac{i\lambda}{2}\right) + \psi\left(\frac{i\lambda+1}{2}\right) - \psi\left(\frac{i\lambda+1+m}{2}\right) - \psi\left(\frac{i\lambda+1-m}{2}\right)\right]$$

where $\psi(\mu) = \Gamma'(\mu)\Gamma(\mu)^{-1}$, $\mu \in \mathbb{C}$. The properties of ψ can be found in [5]. It has simple poles only at $\mu = 0, -1, -2, \ldots$ so that $d_m(\lambda)$ is analytic for $\lambda \neq 0$. If m is even, the term $(i/2)\psi[i\lambda/2]$ contributes a simple pole at $\lambda = 0$ with residue -1 and all other terms are analytic at $\lambda = 0$. If m is odd, we find using the identity $\psi(\mu+1) = \psi(\mu) + 1/\mu$ that all poles at $\lambda = 0$ cancel.

Combining the above formulas for the operators $\pi(f)$ and $M^\chi(\lambda)$, it is easy to see that Theorem 1.8 can be restated as follows for wave packets.

<u>THEOREM 2.18</u>. Let $f = f(\alpha : m)$ where $\alpha \in \mathcal{C}(\mathbb{R})$ and $m \in Z$. Let $t \neq 0$. Then

$$T_f(h_t) = \int_{-\infty}^{\infty} [\alpha(\lambda) + \alpha(-\lambda)]\, \varphi_\lambda(t)\, d\lambda$$

$$+ i\ \text{P.V.} \int_{-\infty}^{\infty} [\alpha(\lambda) + \alpha(-\lambda)]\, e^{-i\lambda|t|}\, c_m'(\lambda)\, c_m(\lambda)^{-1}\, d\lambda$$

$$+ \begin{cases} 2\pi\alpha(0) & m \text{ even} \\ 0 & m \text{ odd} \end{cases} .$$

The function $\varphi_\lambda(t)$ in the first term is defined for $t \neq 0$ by

$$\varphi_\lambda(t) = \frac{1}{\lambda} \int_{|t|}^{\infty} \sin \lambda(u - |t|) \cos \lambda u\, (\sinh u)^{-2} du\,, \quad \lambda \neq 0\,; \tag{2.19a}$$

$$\varphi_0(t) = \lim_{\lambda \to 0} \varphi_\lambda(t) = \int_{|t|}^{\infty} (u - |t|)(\sinh u)^{-2} du\ . \tag{2.19b}$$

Note that both integrals converge absolutely for $t \neq 0$, and that there is a constant C so that $|\varphi_\lambda(t)| \le C/|\sinh t|$ for all $\lambda \in \mathbb{R}$, $t \neq 0$.

For m odd, the principal value is not required in the second term since $c_m'(\lambda)\, c_m(\lambda)^{-1}$ is analytic at $\lambda = 0$. For m even, we define

$$\text{P.V.} \int_{-\infty}^{\infty} [\alpha(\lambda) + \alpha(-\lambda)]\, e^{-i\lambda|t|}\, c_m'(\lambda)\, c_m(\lambda)^{-1}\, d\lambda$$

$$= \lim_{\epsilon \downarrow 0} \int_{|\lambda|>\epsilon} [\alpha(\lambda) + \alpha(-\lambda)]\, e^{-i\lambda|t|}\, c_m'(\lambda)\, c_m(\lambda)^{-1}\, d\lambda \tag{2.20}$$

$$= \int_{-\infty}^{\infty} \alpha(\lambda) [e^{-i\lambda|t|}\, c_m'(\lambda)\, c_m(\lambda)^{-1} + e^{i\lambda|t|}\, c_m'(-\lambda)\, c_m(-\lambda)^{-1}]\, d\lambda\,.$$

Note that the integrand in the last expression extends to an analytic function at $\lambda = 0$ since the simple poles cancel.

3. PROOF OF THE MAIN THEOREM. Fix $\alpha \in \mathcal{C}(\mathbb{R})$ and $m \in \mathbb{Z}$. Let $f = f(\alpha : m)$ be defined as in (2.8). For $t \neq 0$ write

$$T(t : \alpha : m) = T_f(h_t)\ ; \tag{3.1a}$$

$$F(t : \alpha : m) = F_f(h_t)\ ; \tag{3.1b}$$

(3.1c) $$R(t:\alpha:m) = i\,\mathrm{P.V.}\int_{-\infty}^{\infty}[\alpha(\lambda)+\alpha(-\lambda)]e^{-i\lambda|t|}\,c_m'(\lambda)\,c_m(\lambda)^{-1}\,d\lambda + \begin{cases} 2\pi\alpha(0) & m \text{ even} \\ 0 & m \text{ odd}\end{cases}.$$

The main step in the proof of Theorem 2.18 is to show that $R(t:\alpha:m)$ gives the asymptotic behavior of $T(t:\alpha:m)$ as $|t|\to\infty$ and that the distribution on $\mathbb{R}$ given by $\alpha\to T(t:\alpha:m) - R(t:\alpha:m)$ is given by integration against a continuous function of λ. These results are contained in the following lemma which will be proved at the end of this section as Lemmas 3.10 and 3.11.

LEMMA 3.2. For all $t\neq 0$,

$$T(t:m:\alpha) - R(t:m:\alpha) = 2\int_{-\infty}^{\infty}\alpha(\lambda)\,\varphi(t:\lambda:m)\,d\lambda$$

where $\varphi(t:\lambda:m)$ is a continuous function of at most polynomial growth in the λ variable satisfying $\lim_{|t|\to\infty}\varphi(t:\lambda:m) = 0$, uniformly on compacta of λ.

Assuming the lemma, it only remains to show that $\varphi(t:\lambda:m)$ is the function $\varphi_\lambda(t)$ defined in (2.19). To show this we use the differential equation (2.5) satisfied by $T(t:\alpha:m)$ which can be written using (2.9) as

(3.3) $$\frac{d^2}{dt^2}\,T(t:\alpha:m) = T(t:p\alpha:m) + (\sinh t)^{-2}F(t:\alpha:m).$$

Clearly $R(t:\alpha:m)$ satisfies the corresponding homogeneous differential equation

(3.4) $$\frac{d^2}{dt^2}\,R(t:\alpha:m) = R(t:p\alpha:m).$$

The Fourier inversion formula for the orbital integral F_f (see [10]) can be used to write

(3.5) $$F(t:\alpha:m) = \int_{-\infty}^{\infty} 2\alpha(\lambda)\cos\lambda t\,d\lambda.$$

Thus the function φ given by Lemma 3.2 must be a solution of the equation

$$\frac{d^2}{dt^2}\varphi(t:\lambda:m) = -\lambda^2\varphi(t:\lambda:m) + \cos\lambda t\,(\sinh t)^{-2} \quad . \tag{3.6}$$

But $\varphi_\lambda(t)$ is the unique solution of this equation satisfying $\lim_{|t|\to\infty} \varphi_\lambda(t) = 0$.

In order to prove Lemma 3.2 we must derive new formulas for both $T(t:\alpha:m)$ and $R(t:\alpha:m)$ and then use the asymptotics of the Eisenstein integral. Since both T and R are even, we may as well assume that $t > 0$.

Since $f = f(\alpha:m)$ is K-central we can write, using (1.6), (2.2), and a standard change of variables on $\overline{N}$,

$$T(t:\alpha:m) = |e^t - e^{-t}|\int_{\overline{N}} f(\bar{n}h_t\bar{n}^{-1})\,v(\bar{n})\,d\bar{n} = \frac{|e^t - e^{-t}|}{\pi}\int_{-\infty}^{\infty} f(\bar{n}_y h_t \bar{n}_y^{-1})\frac{1}{2}\log(1+y^2)dy$$

$$= \frac{e^t}{\pi}\int_{-\infty}^{\infty} f(\bar{n}_y h_t)\,\frac{1}{2}\log(1 + (1 - e^{-2t})^{-2}y^2)\,dy \quad .$$

Now using the definition of f in (2.8), we obtain

$$T(t:\alpha:m) = \frac{e^t}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(\lambda)\mu_\chi(\lambda)\,E(m:\lambda:\bar{n}_y h_t)\frac{1}{2}\log(1 + (1 - e^{-2t})^{-2}y^2)\,d\lambda\,dy \ . \tag{3.7a}$$

Using the same argument, but leaving out the weight factor $v(\bar{n})$, we obtain

$$F(t:\alpha:m) = \frac{e^t}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(\lambda)\mu_\chi(\lambda)\,E(m:\lambda:\bar{n}_y h_t)\,d\lambda\,dy \ . \tag{3.7b}$$

Now as $t \to +\infty$, $\frac{1}{2}\log(1 + (1 - e^{-2t})^{-2}y^2) \to \frac{1}{2}\log(1 + y^2) = H(\bar{n}_y)$. Also, $\bar{n}h_t = \kappa(\bar{n})h(\bar{n})h_t^{-1}n(\bar{n})h_t \to \kappa(\bar{n})h(\bar{n})h_t$. Finally, $e^tE(m:\lambda:\kappa(\bar{n})h(\bar{n})h_t)$ is asymptotic as $t \to \infty$ to $\omega_m(\kappa(\bar{n}))e^{-H(\bar{n})}E_P(m:\lambda:H(\bar{n}) + t)$ where for any $t > 0$ the constant term $E_P(m:\lambda:t)$ is defined by

$$E_P(m:\lambda:t) = c_m(\lambda)e^{i\lambda t} + c_m(-\lambda)e^{-i\lambda t} \quad . \tag{3.8}$$

(The c-functions $c_m(\pm\lambda)$ are given by (2.16).) Thus we would expect that as $t \to +\infty$, $T(t:\alpha:m) \to \tilde{R}(t:\alpha:m)$ where

$$(3.9)\qquad \tilde{R}(t:\alpha:m) = \int_{\bar{N}}\int_{-\infty}^{\infty}\alpha(\lambda)\mu_\chi(\lambda)\,\omega_m(\kappa(\bar{n})e^{-H(\bar{n})}E_P(m:\lambda:H(\bar{n})+t)\,H(\bar{n})\,d\lambda\,d\bar{n}\,.$$

LEMMA 3.10. The integral defining $\tilde{R}$ in (3.9) converges as an iterated integral, and for $t \neq 0$,

$$T(t:m:\alpha) - \tilde{R}(t:m:\alpha) = 2\int_{-\infty}^{\infty}\alpha(\lambda)\,\tilde{\varphi}(t:\lambda:m)\,d\lambda$$

where $\tilde{\varphi}$ is a continuous function of polynomial growth in λ, satisfying $\lim_{|t|\to\infty}\tilde{\varphi}(t:\lambda:m) = 0$, uniformly on compacta of λ.

Proof. Using (3.7) we can write

$$T(t:\alpha:m) - R(t:\alpha:m) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(\lambda)\,\mu_\chi(\lambda)\,B(t:m:\lambda:y)\,d\lambda\,dy$$

where

$$B(t:m:\lambda:y) = e^t E(m:\lambda:\bar{n}_y h_t)\,\frac{1}{2}\log\left(1+(1-e^{-2t})^{-2}y^2\right)$$

$$- e^{-H(\bar{n}_y)}\,\omega_m(\kappa(\bar{n}_y))\,E_P(m:\lambda:H(\bar{n}_y)+t)\,\frac{1}{2}\log(1+y^2)\ .$$

Write $a_t = (1-e^{-2t})^{-2}$ and write $\bar{n}_y h_t = k_1 h_v k_2$ according to the Cartan decomposition, where $k_1, k_2 \in K$, $v \geq 0$. Then

$$v = v(y,t) = \log\{(\cosh^2 t + e^{2t}y^2/4)^{1/2} + (\sinh^2 t + e^{2t}y^2/4)^{1/2}\}$$

and $k_1 k_2 = t_\theta$ where $\theta = \theta(y,t)$ satisfies

$$e^{i\theta} = (1+e^{-2t} - iy)/[(1+e^{-2t})^2 + y^2]^{1/2}\ .$$

$B = B(t:m:\lambda:y)$ can be written as a sum of the following five terms:

$$B_1 = e^t E(m:\lambda:\bar{n}_y h_t)\left[\frac{1}{2}\log(1+a_t y^2) - \frac{1}{2}\log(1+y^2) - \frac{1}{2}\log a_t\right];$$

$$B_2 = \frac{1}{2}\log a_t\; e^t E(m:\lambda:\bar{n}_y h_t)\,;$$

$$B_3 = \frac{1}{2}\log(1+y^2)[e^t E(m:\lambda:\bar{n}_y h_t) - e^{(t-v)}\,\omega_m(t_\theta)\,E_P(m:\lambda:v)];$$

$$B_4 = \frac{1}{2}\log(1+y^2)\,\omega_m(t_\theta)[e^{(t-v)}E_P(m:\lambda:v) - e^{-H(\bar{n}_y)}E_P(m:\lambda:H(\bar{n}_y)+t)];$$

$$B_5 = \frac{1}{2}\log(1+y^2)\,e^{-H(\bar{n}_y)}E_P(m:\lambda:H(\bar{n}_y)+t)[\omega_m(t_\theta) - \omega_m(\kappa(\bar{n}_y))]\,.$$

We will show that there are functions $\varphi_i(t:\lambda:m)$, $1 \le i \le 5$, corresponding to these five terms, the sum of which yields the function $\tilde{\varphi}(t:\lambda:m)$ required in the lemma. For $t > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{2}\log\left[\frac{1+a_t y^2}{a_t + a_t y^2}\right] dy = -\pi e^{-2t}$$

converges absolutely. Further, $e^t E(m:\lambda:\bar{n}_y h_t)$ is bounded for all $t > 0$, $y \in \mathbb{R}$. Thus we can exchange limits of integration to write

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\lambda)\mu_\chi(\lambda)\,B_1(t:m:\lambda:y)\,d\lambda\,dy = 2\int_{-\infty}^{\infty} \alpha(\lambda)\,\varphi_1(t:\lambda:m)\,d\lambda$$

where

$$\varphi_1(t:\lambda:m) = \frac{1}{2\pi}\mu_\chi(\lambda)\int_{-\infty}^{\infty} e^t E(m:\lambda:\bar{n}_y h_t)\,\frac{1}{2}\log\left[\frac{1+a_t y^2}{a_t + a_t y^2}\right] dy$$

is smooth and of polynomial growth as a function of λ. Further, for λ in any compact set, there is a constant C so that $|\varphi_1(t:\lambda:m)| \le C e^{-2t}$ for all $t > 0$.

Using (3.5) and (3.7b), we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\lambda)\,\mu_\chi(\lambda)\,B_2(t:m:\lambda:y)\,d\lambda\,dy = \frac{1}{2\pi}\log a_t\,F(t:\alpha:m)$$

$$= \log a_t \int_{-\infty}^{\infty} \alpha(\lambda)\cos\lambda t\,d\lambda\,.$$

Thus we can take $\varphi_2(t:\lambda:m) = \frac{1}{2}\log a_t \cos\lambda t$.

It folllows from the estimate of Harish-Chandra for the constant term [6a] that there are a continuous seminorm ν an $C^\infty(G)$ and an $\epsilon > 0$ so that

$|e^t E(m:\lambda:\bar{n}_y h_t) - E_P(m:\lambda:t)| \leq \nu(E(m:\lambda))e^{-\epsilon t}$ for all $t > 0$. Writing $\bar{n}_y h_t = k_1 h_v k_2$, $t_\theta = k_1 k_2$, as above, $E(m:\lambda:\bar{n}_y h_t) = \omega_m(t_\theta) E(m:\lambda:h_v)$. Thus $|e^t E(m:\lambda:\bar{n}_y h_t) - e^{(t-v)} \omega_m(t_\theta) E_P(m:\lambda:h_v)| \leq \nu(E(m:\lambda)) e^{(t-v)} e^{-\epsilon v}$ for all $t > 0$, $y \in \mathbb{R}$. But

$$e^{(t-v)} e^{-\epsilon v} = e^{-\epsilon t} g(y,t)^{-1-\epsilon}$$

where

$$g(y,t) = e^{v-t} = \frac{1}{2}\left\{[(1+e^{-2t})^2 + y^2]^{1/2} + [(1-e^{-2t})^2 + y^2]^{1/2}\right\} .$$

Note that for all $t > 0$, $y \in \mathbb{R}$, $g(y,t) \geq \frac{1}{2}(1+y^2)^{1/2}$. Thus

$$\int_{-\infty}^{\infty} \frac{1}{2}\log(1+y^2)\, g(y,t)^{-1-\epsilon}\, dy \leq \int_{-\infty}^{\infty} \frac{1}{2}\log(1+y^2)\left(\frac{2}{(1+y^2)^{1/2}}\right)^{1+\epsilon} dy$$

which converges absolutely for any $\epsilon > 0$ and is independent of t. Thus we can exchange the order of integration to obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\lambda)\,\mu_\chi(\lambda)\, B_3(t:m:\lambda)\, d\lambda\, dy = 2\int_{-\infty}^{\infty} \alpha(\lambda)\varphi_3(t:\lambda:m)\, d\lambda$$

where $|\varphi_3(t:\lambda:m)| \leq C e^{-\epsilon t} \mu_\chi(\lambda)\nu(E(m:\lambda))$ for some constant C independent of $\lambda \in \mathbb{R}$ and $t > 0$.

In the fourth term, we can write

$$|e^{t-v} E_P(m:\lambda:v) - e^{-H(\bar{n}_y)} E_P(m:\lambda:H(\bar{n}_y)+t)|$$

$$\leq \left| c_m(\lambda) e^{i\lambda t}\left(g(y,t)^{-1+i\lambda} - (1+y^2)^{\frac{-1+i\lambda}{2}}\right)\right|$$

$$+ \left| c_m(-\lambda) e^{-i\lambda t}\left(g(y,t)^{-1-i\lambda} - (1+y^2)^{\frac{-1-i\lambda}{2}}\right)\right|$$

$$\leq \left| c_m(\lambda)\left(g(y,t)^{-1+i\lambda} - (1+y^2)^{\frac{-1+i\lambda}{2}}\right)\right| .$$

Using the formula for $g(y,t)$, one checks that there is a constant C so that for all $t > 0$, $y, \lambda \in \mathbb{R}$,

$$\left| g(y,t)^{-1+i\lambda} - (1+y^2)^{\frac{-1+i\lambda}{2}} \right| \le \frac{C|1+i\lambda|\, e^{-2t}}{1+y^2} .$$

Thus

$$\int_{-\infty}^{\infty} \frac{1}{2}\log(1+y^2) \left| g(y,t)^{-1+i\lambda} - (1+y^2)^{\frac{-1+i\lambda}{2}} \right| dy$$

$$\le C\,|1+i\lambda|\; e^{-2t} \int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(1+y^2)}{1+y^2}\, dy$$

so that the integral converges absolutely. Thus again we can exchange order of integration to obtain

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(\lambda)\mu_\chi(\lambda)\, B_4(t:m:\lambda:y)\, d\lambda\, dy = 2 \int_{-\infty}^{\infty} \alpha(\lambda)\, \varphi_4(t:\lambda:m)\, d\lambda$$

where $|\varphi_4(t:\lambda:m)| \le C|c_m(\lambda)|\,\mu_\chi(\lambda)|\,1+i\lambda|\, e^{-2t}$ for some constant C.

Finally, we write $\kappa(\bar{n}_y) = t_\varphi$ where

$$e^{i\varphi} = \frac{1-iy}{(1+y^2)^{1/2}}$$

and recall that

$$e^{i\theta} = \frac{1+e^{-2t} - iy}{[(1+e^{-2t})^2 + y^2]^{1/2}} .$$

Then for all $t > 0$, $y \in R$,

$$|\omega_m(t_\theta) - \omega_m(t_\varphi)| \le \frac{C\,|m|\, e^{-2t}}{(1+y^2)^{1/2}} .$$

Thus

$$|B_5(t:m:\lambda:y)| \le C\,|m|\; |c_m(\lambda)|\, e^{-2t}\, \frac{\log(1+y^2)}{1+y^2} ,$$

and hence is absolutely integrable with respect to y, so that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(\lambda)\mu_\chi(\lambda)\, B_5(t:m:\lambda:y)\, d\lambda\, dy = 2 \int_{-\infty}^{\infty} \alpha(\lambda)\, \varphi_5(t:\lambda:m)\, d\lambda$$

with $|\varphi_5(t:\lambda:m)| \le C|m|\,|c_m(\lambda)|\,\mu_\chi(\lambda)e^{-2t}$ for some constant C.

Q.E.D.

In order to complete the proof of Lemma 3.2, and hence of Theorem 2.18, it now suffices to prove the following lemma.

LEMMA 3.11. For all $t > 0$, $\tilde{R}(t:\alpha:m) = R(t:\alpha:m)$.

Proof. Using the definition (3.8) of E_P we can write

$$\tilde{R}(t:\alpha:m) = \int_{\bar{N}}\int_{-\infty}^{\infty} \alpha(\lambda)\,\mu_\chi(\lambda)\,c_m(\lambda)\,e^{i\lambda t}\,\omega_m(\kappa(\bar{n}))\,H(\bar{n})\,e^{(i\lambda-1)H(\bar{n})}d\lambda\,d\bar{n}$$

$$+\int_{\bar{N}}\int_{-\infty}^{\infty} \alpha(\lambda)\,\mu_\chi(\lambda)\,c_m(-\lambda)\,e^{-i\lambda t}\,\omega_m(\kappa(\bar{n}))\,H(\bar{n})e^{(-i\lambda-1)H(\bar{n})}d\lambda\,d\bar{n}\ .$$

The inside integrals

$$\int_{-\infty}^{\infty} \alpha(\lambda)\,\mu_\chi(\lambda)\,c_m(\pm\lambda)\,e^{\pm i\lambda(t+H(\bar{n}))}d\lambda$$

are absolutely convergent since $\mu_\chi(\lambda)c_m(\pm\lambda)$ is analytic for all $\lambda \in \mathbb{R}$ and of polynomial growth at infinity [6c]. However, for fixed λ,

$$\int_{\bar{N}} \omega_m(\kappa(\bar{n}))\,H(\bar{n})\,e^{(\pm i\lambda-1)H(\bar{n})}d\bar{n}$$

does not converge. In order to change the order of integration we must think of λ as $\operatorname{Re}\mu$, $\mu \in \mathbb{C}$. Then $\mu_\chi(\lambda)c_m(\lambda) = c_m(-\lambda)^{-1}$ and $\mu_\chi(\lambda)c_m(-\lambda) = c_m(\lambda)^{-1}$ extend to holomorphic functions on a strip about $\operatorname{Im}\mu = 0$, say on $|\operatorname{Im}\mu| < 2\epsilon$ for some $\epsilon > 0$, [6c]. Assume that α is the restriction to the real axis of a holomorphic function which decays rapidly as $|\operatorname{Re}\mu| \to \infty$ uniformly in $|\operatorname{Im}\mu| \le 2\epsilon$. Then we can shift contours of integration to write

$$\tilde{R}(t:\alpha:m) = \int_{\bar{N}}\int_{-\infty}^{\infty} \alpha(\lambda+i\epsilon)c_m(-\lambda-i\epsilon)^{-1}\,e^{(i\lambda-\epsilon)t}\,\omega_m(\kappa(\bar{n}))H(\bar{n})e^{(i\lambda-\epsilon-1)H(\bar{n})}d\lambda\,d\bar{n}$$

$$+\int_{\bar{N}}\int_{-\infty}^{\infty}\alpha(\lambda-i\epsilon)c_m(\lambda-i\epsilon)^{-1}e^{(-i\lambda-\epsilon)t}\omega_m(\kappa(\bar{n}))H(\bar{n})e^{(-i\lambda-\epsilon-1)H(\bar{n})}d\lambda\,d\bar{n}\,.$$

But, from (2.15), for $\epsilon > 0$,

$$\int_{\bar{N}}\omega_m(\kappa(\bar{n}))\,H(\bar{n})\,e^{(\pm i\lambda-\epsilon-1)H(\bar{n})}\,d\bar{n}$$

converges absolutely to $ic_m'(\mp\lambda-i\epsilon)$ where the prime denotes the derivative with respect to λ. Thus the order of integration can be reversed to write

$$\tilde{R}(t:\lambda:m) = i\int_{-\infty}^{\infty}\alpha(\lambda+i\epsilon)\,c_m(-\lambda-i\epsilon)^{-1}\,c_m'(-\lambda-i\epsilon)\,e^{(i\lambda-\epsilon)t}\,d\lambda$$

$$+\,i\int_{-\infty}^{\infty}\alpha(\lambda-i\epsilon)\,c_m(\lambda-i\epsilon)^{-1}\,c_m'(\lambda-i\epsilon)\,e^{(-i\lambda-\epsilon)t}\,d\lambda\,.$$

For $\mu\in\mathbb{C}$ write $h_1(\mu) = i\,\alpha(\mu)\,c_m(-\mu)^{-1}\,c_m(-\mu)\,e^{i\mu t}$ and $h_2(\mu) = i\,\alpha(\mu)\,c_m(\mu)^{-1}\,c_m'(\mu)\,e^{-i\mu t}$. Then both h_1 and h_2 have at worst simple poles at $\mu=0$, but are otherwise holomorphic on the strip $|\operatorname{Im}\mu| < 2\epsilon$. Further, h_1+h_2 is holomorphic throughout this region. Thus

$$\int_{-\infty}^{\infty}h_1(\lambda+i\epsilon)\,d\lambda + \int_{-\infty}^{\infty}h_2(\lambda-i\epsilon)\,d\lambda = \int_{-\infty}^{\infty}(h_1+h_2)(\lambda+i\epsilon)\,d\lambda$$

$$+\,2\pi i\,\operatorname{Res}_{\mu=0}h_2(\mu) = \int_{-\infty}^{\infty}(h_1+h_2)(\lambda)\,d\lambda$$

$$+\begin{cases}2\pi\alpha(0) & m \text{ even}\\ 0 & m \text{ odd}\end{cases}.$$

We now have

$$\tilde{R}(t:\alpha:m) = i\int_{-\infty}^{\infty}\alpha(\lambda)\left[c_m(-\lambda)^{-1}c_m'(-\lambda)e^{i\lambda t}+c_m(\lambda)^{-1}c_m'(\lambda)e^{-i\lambda t}\right]d\lambda$$

$$+\begin{cases}2\pi\alpha(0) & m \text{ even}\\ 0 & m \text{ odd}\end{cases}$$

which, using (3.1c) and (2.20), is equal to $R(t:\alpha:m)$. This proves the lemma when α is the restriction of a holomorphic function as above. But such α form a dense subset of $\mathcal{C}(\mathbb{R})$ and for each $t \neq 0$, $\alpha \mapsto R(t:\alpha:m)$ is a tempered distribution. Using (3.10), we obtain

$$\tilde{R}(t:\alpha:m) = T(t:\alpha:m) + 2 \int_{-\infty}^{\infty} \alpha(\lambda)\,\tilde{\varphi}(t:\lambda:m)\,d\lambda$$

also gives a tempered distribution since $\alpha \mapsto T(t:\alpha:m)$ is tempered. Thus $R(t:\alpha:m) = \tilde{R}(t:\alpha:m)$ for all $\alpha \in \mathcal{C}(\mathbb{R})$.

Q.E.D.

4. THE SINGULAR WEIGHTED ORBITAL INTEGRAL. We now look at the behavior of $T_f(h_t)$ as $t \to 0$. Recall from (3.7) that we can write

$$(4.1a) \qquad T_f(h_t) = \frac{e^t}{\pi} \int_{-\infty}^{\infty} f(\bar{n}_y h_t)\, \frac{1}{2} \log(1 + (1 - e^{-2t})^{-2} y^2)\, dy\,, \quad t \neq 0$$

$$(4.1b) \qquad F_f(h_t) = \frac{e^t}{\pi} \int_{-\infty}^{\infty} f(\bar{n}_y h_t)\, dy\,.$$

Define $S_f(h_t) = T_f(h_t) + \frac{1}{2}\log(1 - e^{-2t})^2 F_f(h_t)$, $t \neq 0$, as in [1b]. Using (4.1), we can write

$$(4.2) \qquad S_f(h_t) = \frac{e^t}{\pi} \int_{-\infty}^{\infty} f(\bar{n}_y h_t)\, \frac{1}{2} \log((1 - e^{-2t})^2 + y^2)\, dy\,.$$

This shows that $S_f(h_t)$ has a well-behaved limit as $t \to 0$ given by

$$(4.3) \qquad \lim_{t \to 0} S_f(h_t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\bar{n}_y) \log|y|\, dy\,.$$

Arthur defines the "singular" weighted orbital integral

$$(4.4) \qquad T_f(1) = \lim_{t \to 0} S_f(h_t)\,.$$

This term also appears in the Selberg trace formula. (See (9.2) on p. 384 of [1a].)

The Fourier transform of $T_f(1)$ can be easily computed from that of $T_f(h_t)$, $t \neq 0$. As before, suppose $f = f(\alpha : m)$, $\alpha \in C(\mathbb{R})$, $m \in Z$, is a wave packet. Write $S(t:\alpha:m) = S_f(h_t)$. Recall that Theorem 2.18 can be written as

$$T(t:\alpha:m) = R(t:\alpha:m) + 2\int_{-\infty}^{\infty} \alpha(\lambda)\, \varphi_\lambda(t)\, d\lambda \quad .$$

Now

$$\lim_{t\to 0} R(t:\alpha:m) = i\ \text{P.V.} \int_{-\infty}^{\infty} [\alpha(\lambda)+\alpha(-\lambda)]\, c_m(\lambda)^{-1} d\lambda + \begin{cases} 2\pi\alpha(0) & m \text{ even} \\ 0 & m \text{ odd} \end{cases} .$$

Thus to find $\lim_{t\to 0} S(t:\alpha:m)$ it suffices to evaluate $\lim_{t\to 0} [S(t:\alpha:m) - R(t:\alpha:m)]$

$$= \lim_{t\to 0} \left[2\int_{-\infty}^{\infty} \alpha(\lambda)\, \varphi_\lambda(t)\, d\lambda + \frac{1}{2}\log(1-e^{-2t})^2\, F(t:\alpha:m) \right]$$

$$= \lim_{t\to 0}\ 2\int_{-\infty}^{\infty} \alpha(\lambda)\, h_\lambda(t)\, d\lambda$$

where, using (3.5), we have

$$h_\lambda(t) = \varphi_\lambda(t) + \frac{1}{2}\log(1-e^{-2t})^2 \cos\lambda t \quad .$$

Using the differential equation (3.6) and behavior at infinity characterizing φ_λ, we see that for $t > 0$, $h_\lambda(t)$ is the unique solution of the equation

$$\text{(4.5)} \qquad \frac{d^2}{dt^2} h_\lambda(t) = -\lambda^2 h_\lambda(t) - \frac{4\lambda \sin\lambda t}{e^{2t}-1}$$

satisfying $\lim_{t\to+\infty} h_\lambda(t) = 0$. Thus for $t > 0$ we can write

$$\text{(4.6)} \qquad h_\lambda(t) = -4\int_t^{\infty} \frac{\sin(\lambda(u-t))\, \sin(\lambda u)}{e^{2u}-1}\, du \quad .$$

Thus

$$\lim_{t \downarrow 0} h_\lambda(t) = h_\lambda(0) = -4 \int_0^\infty \frac{\sin^2 \lambda u}{e^{2u} - 1}$$

is well-behaved. We have proved the following proposition.

PROPOSITION 4.7. Let $f = f(\alpha : m)$ where $\alpha \in \mathcal{C}(\mathbb{R})$ and $m \in Z$. Then

$$T_f(1) = \int_{-\infty}^{\infty} [\alpha(\lambda) + \alpha(-\lambda)] h_\lambda(0)\, d\lambda$$

$$+ i\, \text{P.V.} \int_{-\infty}^{\infty} [\alpha(\lambda) + \alpha(-\lambda)] c_m'(\lambda) c_m(\lambda)^{-1}\, d\lambda$$

$$+ \begin{cases} 2\pi\alpha(0) & m \text{ even} \\ 0 & m \text{ odd} \end{cases} .$$

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