HARMONIC ANALYSIS OF INVARIANT DISTRIBUTIONS

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Suppose that G is a reductive Lie group, with Lie algebra $\mathfrak{g}$. To be safe, let us assume that G is the set of real points of a reductive algebraic group defined over $\mathbb{R}$. Harish-Chandra has defined the Schwartz space, $\mathcal{C}(G)$, on G. It is a certain space of functions on G such that

$$C_c^{\infty}(G) \subset \mathcal{C}(G) \subset L^2(G).$$

In each case the inclusion is a continuous map from one space onto a dense subspace of the second. A distribution, F, on G is called tempered if it extends to a continuous linear functional on $\mathcal{C}(G)$. F is said to be invariant if for fixed $f \in C_c^{\infty}(G)$, $\langle F, f^y \rangle$ is independent of $y \in G$. Here

$$f^y(x) = f(y^{-1}xy).$$

EXAMPLE: Suppose that T is a Cartan subgroup of G with Lie algebra $\mathfrak{t}$. If $\gamma \in T_{reg}$, the set of points in T whose centralizer is T, define

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{t}}.$$

Harish-Chandra has shown that the map which sends $f \in \mathcal{C}(G)$ to

$$F(\gamma, f) = F^{G,T}(\gamma, f) = |D(\gamma)|^{1/2} \int_{T/G} f(x^{-1}\gamma x)\,dx$$

is well defined, and is a tempered invariant distribution on G. The distributions so obtained, known as the orbital integrals of f, play a central role in the harmonic analysis

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on G. They also arise naturally in the study of $L^2(\Gamma\backslash G)$, where Γ is a discrete co-compact subgroup of G.

PROBLEM: Define the notion of Fourier transform for an invariant tempered distribution.

In this lecture we shall describe a solution to this problem. We shall then use our solution to define a wide class of invariant tempered distributions, of which the orbital integrals are special cases.

Let K be a fixed maximal compact subgroup of G. It the set of fixed points of a Cartan involution θ of G. Suppose that P is a parabolic subgroup of G. Then $P = NM$, and $M = M^1 \cdot A$, where N is the unipotent radical of P, and M^1 and A are uniquely determined θ-stable reductive groups such that M^1 has compact center and A is a vector group. Any $m \in M$ can be written $m^1 \exp H$, where $m^1 \in M^1$ and H belongs to $\mathfrak{a}$, the Lie algebra of A. We denote H by $H_M(m)$. Since $G = PK$, any x in G can be written

$$x = nmk, \qquad n \in N, \quad m \in M, \quad k \in K.$$

It is customary to denote the vector $H_M(m)$ by $H_P(x)$. Suppose that σ is an irreducible representation of M. We can pull σ back from $M \cong N\backslash P$ to P and then induce to G. The result is a representation $\tilde{I}_P(\sigma)$ of G on a Hilbert space $\tilde{\mathcal{H}}_P(\sigma)$ of functions on G, which is unitary if σ is unitary. Any function in $\tilde{\mathcal{H}}_P(\sigma)$ is uniquely determined by its restriction to K. Let $\mathcal{H}_P(\sigma)$ be the space of functions on K obtained in this way, and let $I_P(\sigma)$ be the corresponding representation of G on $\mathcal{H}_P(\sigma)$. The reason for introducing this latter space is that it does not change if σ is twisted by a character of A. In other words, if

$$\sigma_\Lambda(m) = \sigma(m)\, e^{\Lambda(H_M(m))}, \qquad m \in M,$$

for any quasi-character Λ of $\mathfrak{a}$, the representations

$$\{I_P(\sigma_\Lambda) : \Lambda \in \operatorname{Hom}(\mathfrak{a}, \mathbb{C})\}$$

all act on the same space.

A representation π of G is said to be <u>tempered</u> if its character is a tempered distribution. This implies that for $f \in \mathcal{C}(G)$,

$$\operatorname{ch}_\pi(f) = \operatorname{tr}\left(\int_G f(x)\pi(x)dx\right)$$

is well defined. We obtain a map from $\mathcal{C}(G)$ to the space of complex valued functions on $\mathcal{E}_T(G)$, the set of classes of irreducible tempered representations of G. The problem is to characterize the image of this map.

It is known that π is tempered if and only if it is a subrepresentation of some $I_P(\sigma)$, where σ is an irreducible unitary representation of M which is square integrable modulo A. Let Δ_M be any second order left and right invariant differential operator on M such that for all such σ the operator $\sigma(\Delta_M)$, a priori a scalar, is actually positive. Then if π is any subrepresentation of $I_P(\sigma)$, define

$$\|\pi\| = \sigma(\Delta_M) .$$

It is easy to show that this number depends only on the tempered representation π and not on P and σ. For any P, and any $\sigma \in \mathcal{E}_T(M)$, we can define $\|\sigma\|$ the same way. Suppose that s is a complex valued function on $\mathcal{E}_T(G)$. If P is a parabolic subgroup, and $\sigma \in \mathcal{E}_T(M)$, then $I_P(\sigma)$ is a finite sum of tempered representations of G. We define $s(I_P(\sigma))$ additively. If $D = D_\Lambda$ is a differential operator of constant coefficients on the real vector space $\mathrm{Hom}(\mathfrak{a}, i\mathbb{R})$, and n is a positive integer, define

$$\|s\|_{D,n} = \sup_{\sigma \in \mathcal{E}_T(M)} (\|\sigma\|^n |D_\Lambda\, s(I_P(\sigma_\Lambda))_{\Lambda=0}|)$$

if all the derivatives in this formula exist. Otherwise, set $\|s\|_{D,n} = \infty$. Let $\mathcal{I}(\hat{G})$ be the set of all functions

$$s: \mathcal{E}_T(G) \to \mathbb{C}$$

such that

$$\|s\|_{D,n} < \infty$$

for all P, D and n. With the topology defined by the semi-norms $\|\ \|_{D,n}$, $\mathcal{I}(\hat{G})$ becomes a Frechet space.

<u>THEOREM 1</u>: The map

$$ch: (f,\pi) \to ch_\pi(f), \qquad f \in \mathcal{C}(G), \quad \pi \in \mathcal{E}_T(G),$$

is a continuous surjection from $\mathcal{C}(G)$ onto $\mathcal{I}(\hat{G})$.

<u>COROLLARY</u>: The transpose

$$ch': \mathcal{I}'(\hat{G}) \to \mathcal{C}'(G)$$

is a continuous injection.

THEOREM 2: The image of ch' is exactly the space of invariant tempered distributions on G.

COROLLARY: For $f \in \mathcal{C}(G)$ suppose that either

(a) $ch_\pi(f) = 0$ for every $\pi \in \mathcal{E}_T(G)$, or

(b) $F^{G,T}(\gamma, f) = 0$ for every Cartan subgroup T and all $\gamma \in T_{reg}$

Then $I(f) = 0$ for every invariant tempered distribution I.

The corollary is easy to prove. Recall that if $\pi \in \mathcal{E}_T(G)$,

$$ch_\pi(f) = \sum_{\{T\}} \int_{T_{reg}} \Phi_\pi^T(\gamma) F^{G,T}(\gamma, f) d\gamma,$$

where $\{T\}$ is a set of representatives for conjugacy classes of Cartan subgroups of G, and $\Phi_\pi^T(\gamma)$ is a bounded function on T. Therefore, (b) implies (a). But

$$I(f) = (ch'(S))(f) = S(ch\ f),$$

for some $S \in \mathcal{I}'(G)$. Therefore (a) implies that $I(f) = 0$.

The proofs of Theorems 1 and 2 are more difficult and will appear elsewhere. However, they give a solution to our problem. If I is a tempered invariant distribution on G, we define its Fourier transform to be $(ch')^{-1}(I)$. It is an element in the dual space of $\mathcal{I}(\hat{G})$.

Suppose that A is a vector subgroup of G such that $\theta A = A$. Let M be the centralizer of A in G and let $\mathfrak{a}$ be the Lie algebra of A. A is called a <u>special subgroup</u> of G if it is the split component of a parabolic subgroup P. Suppose that this is the case and let $\mathcal{P}(A)$ be the set of all such P. If $P \varepsilon \mathcal{P}(A)$, let Φ_P be the set of simple roots of (P,A). Then

$$\mathfrak{a}_P^+ = \{H \varepsilon \mathfrak{a}: \alpha(H) > 0, \quad \alpha \varepsilon \Phi_P\}$$

is called the <u>positive chamber</u> of P. As P varies throughout $\mathcal{P}(A)$, the associated positive chambers are disjoint and dense in $\mathfrak{a}$. Groups P and P' in $\mathcal{P}(A)$ are said to be <u>adjacent</u> if their chambers share a common wall.

Let us identify $\mathfrak{a}$ with its dual space by means of a suitable positive definite bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{a}$. Suppose that

$$\{c_P(\Lambda): P \varepsilon \mathcal{P}(A), \quad \Lambda \varepsilon \mathfrak{a}_{\mathbb{C}}\}$$

is a family of analytic functions. We shall call it a <u>compatible family</u> if whenever P and P' are adjacent and Λ is in their common wall,

$$c_P(\Lambda) = c_{P'}(\Lambda).$$

<u>LEMMA 1</u>: If the family is compatible,

$$\sum_{P\varepsilon \mathcal{P}(A)} c_P(\Lambda) \ (\Pi_{\alpha\varepsilon\Phi_P} \langle\alpha, \Lambda\rangle)^{-1}$$

is an analytic function of Λ.

PROOF: The only possible singularities of the function are $\langle\beta,\Lambda\rangle = 0$ for roots β of (G,A). Given β the only summands for which $\langle\beta,\Lambda\rangle = 0$ is a singularity occur in adjacent pairs, P and P'. Moreover, we can assume $\Phi_P \bigcup (-\Phi_{P'}) = \{\beta\}$. The contribution in the above sum from (P,P') is

$$\langle\beta,\Lambda\rangle^{-1}\{c_P(\Lambda)(\prod_{\alpha\in\Phi_P\setminus\{\beta\}}\langle\alpha,\Lambda\rangle)^{-1} - c_{P'}(\Lambda)(\prod_{\alpha\in\Phi_{P'}\setminus\{-\beta\}}\langle\alpha,\Lambda\rangle)^{-1}\}.$$

If Λ is in the common wall of P and P', the expression in the curly brackets is 0. It follows that $\langle\beta,\Lambda\rangle = 0$ is not a singularity. []

<u>EXAMPLE</u>: Fix $x \in G$. It is easy to check that

$$\{c_P(\Lambda) = e^{\langle \Lambda, H_P(x)\rangle} : P \in P(A)\}$$

is a compatible family. Therefore

$$v(x) = \lim_{\Lambda\to 0} \sum_{P\in P(A)} e^{\langle\Lambda,H_P(x)\rangle} (\prod_{\alpha\in\Phi_P}\langle\beta,\Lambda\rangle)^{-1}$$

is well defined. If $m \in M$,

$$H_P(mx) = H_P(x) + H_M(m),$$

so that

$$v(mx) = e^{\langle 0,H_M(m)\rangle} v(x) = v(x).$$

Suppose that T is a Cartan subgroup in M, and $\gamma \in T_{reg}$. It can be shown that

$$|D(\gamma)|^{1/2}\int_{T\backslash G} f(x^{-1}\gamma x)v(x)dx, \qquad f \in \mathcal{C}(G),$$

is a tempered distribution. However, it is not invariant. We would like to modify it to obtain an invariant distribution.

LEMMA 2: Suppose that σ is an irreducible unitary representation of M. Then there exist unitary operators

$$R_{P'|P}(\sigma): \mathcal{H}_P(\sigma) \to \mathcal{H}_{P'}(\sigma), \qquad P,P' \in \mathcal{P}(A),$$

such that

(i) $\quad R_{P'|P}(\sigma)I_P(\sigma,x) = I_{P'}(\sigma,x)R_{P'|P}(\sigma), \qquad x \in G,$

(ii) $\quad R_{P''|P}(\sigma) = R_{P''|P'}(\sigma)R_{P'|P}(\sigma), \qquad P,P',P'' \in \mathcal{P}(A),$

(iii) If P and P' are adjacent and Λ lies in their common wall, $R_{P'|P}(\sigma_\Lambda) = R_{P'|P}(\sigma)$.

Suppose that $P_0 \in \mathcal{P}(A)$, $f \in \mathcal{C}(G)$, and $\sigma \in \mathcal{E}_T(M)$. Then

$$c_P(\Lambda) = \mathrm{tr}(R_{P|P_0}(\sigma)^{-1}R_{P|P_0}(\sigma_\Lambda)I_{P_0}(\sigma,f))$$

is a compatible family. Therefore

$$\phi_A(F,\sigma) = \lim_{\Lambda\to 0} \sum_P c_P(\Lambda)(\prod_{\alpha\in\Phi_P} \langle\alpha,\Lambda\rangle)^{-1}$$

is well defined. It is independent of P_0.

LEMMA 3: The map

$$\sigma \to \phi_A(f,\sigma), \qquad \sigma \in \mathcal{E}_T(M),$$

is in $\mathcal{I}(M)$.

It follows from Theorem 1 that there is a function $\phi_A(f)$ in $\mathcal{C}(M)$ such that

$$\mathrm{ch}_{\sigma}(\phi_A(f)) = \phi_A(f,\sigma)$$

for every $\sigma \in \mathcal{E}_T(M)$. $\phi_A(f)$ is not uniquely defined, but it follows from Theorem 2 that if I is any invariant tempered distribution on M, $I(\phi_A(f))$ is uniquely defined. Suppose that $\gamma \in T_{reg}$. We define a number $F^{G,T,A}(\gamma,f)$ inductively by

$$\sum_{\{A^*:A^*\subset A\}} F^{M^*,T,A}(\gamma,\phi_{A^*}(f)) = |D(\gamma)|^{1/2} \int_{T\backslash G} f(x^{-1}\gamma x)v(x)dx.$$

The sum on the left is over the finite set of special subgroups which are contained in A. We are assuming inductively that we have defined all the summands except the one for which A^* is the split component of the center of G. In this latter case, of course,

$$F^{M^*,T,A}(\gamma,\phi_{A^*}(f)) = F^{G,T,A}(\gamma,f).$$

Notice that if A is the split component of the center of G there is only one term in the sum. In this case $F^{G,T,A}(\gamma,f)$ is just the orbital integral $F^{G,T}(\gamma,f)$ defined earlier.

<u>THEOREM 3</u>: The map

$$F^{G,T,A}(\gamma): f \to F^{G,T,A}(\gamma,f)$$

is an invariant tempered distribution on G.

Notice that this theorem is necessary for our inductive definition to work. The theorem, along with Lemmas 2 and 3, is proved in [1]. The distributions $F^{G,T',A}(\gamma)$ seem to be the natural replacements of the orbital integrals when one tries to study $L^2(\Gamma\backslash G)$ and $\Gamma\backslash G$ is assumed only to have finite volume.

Reference

[1] J. Arthur, On the invariant distribution associated to weighted orbital integrals, preprint.

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