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The Work of Ngô Bao Châu

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Abstract

Ngô Bao Châu has been awarded a Fields Medal for his proof of the fundamental lemma. I shall try to describe the role of the fundamental lemma in the theory of automorphic forms. I hope that this will make it clear why the result will be a cornerstone of the subject. I will also try to give some sense of Ngô's proof. It is a profound and beautiful argument, built on insights mathematicians have contributed for over thirty years.

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The Formal Statement

Here is the statement of Ngô's primary theorem. It is taken from the beginning of the introduction of his paper [N2].

Théorème 1. *Soient k un corps fini à q éléments, $\mathcal{O}$ un anneau de valuation discrète complet de corps résiduel k et F son corps des fractions. Soit G un schéma en groupes réductifs au-dessus de $\mathcal{O}$ dont le nombre de Coxeter multiplié par deux est plus petit que la caractéristique de k. Soient (κ, ρ_κ) une donnée endoscopique de G au-dessus de $\mathcal{O}$ et H le schéma en groupes endoscopiques associé.*

On a l'égalité entre la κ-intégrale orbitale et l'intégrale orbitale stable

$$\Delta_G(a)\mathbf{O}_a^\kappa(1_{\mathfrak{g}}, \mathrm{d}t) = \Delta_H(a_H)\mathbf{SO}_{a_H}(1_{\mathfrak{h}}, \mathrm{d}t) \tag{1}$$

associées aux classes de conjugaison stable semi-simples régulières a et a_H de

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$\mathfrak{g}(F)$ et $\mathfrak{h}(F)$ qui se correspondent, aux fonctions caractéristiques $1_{\mathfrak{g}}$ et $1_{\mathfrak{h}}$ des compacts $\mathfrak{g}(\mathcal{O})$ et $\mathfrak{h}(\mathcal{O})$ dans $\mathfrak{g}(F)$ et $\mathfrak{h}(F)$ et où on a noté

$$\Delta_G(a) = q^{-\mathrm{val}(\mathfrak{D}_G(a))/2} \text{ et } \Delta_H(a_H) = q^{-\mathrm{val}(\mathfrak{D}_H(a_H))/2}$$

$\mathfrak{D}_G$ and $\mathfrak{D}_H$ étant les fonctions discriminant de G et de H.

In §1.11 of his paper, Ngô describes the various objects of his assertion in precise terms. At this point we simply note that the "orbital integrals" he refers to are integrals of locally constant functions of compact support. The assertion is therefore an identity of sums taken over two finite sets. Observe however that there is one such identity for every pair (a, a_H) of "regular orbits". As a approaches a singular point, the size of the two finite sets increases without bound, and so therefore does the complexity of the identity. Langlands called it the *fundamental lemma* when he first encountered the problem in the 1970's. It was clearly fundamental, since he saw that it would be an inescapable precondition for any of the serious applications of the trace formula he had in mind. He called it a lemma because it seemed to be simply a family of combinatorial identities, which would soon be proved. Subsequent developments, which culminated in Ngô's proof, have revealed it to be much more. The solution draws on some of the deepest ideas in modern algebraic geometry.

Ngô's theorem is an infinitesimal form of the fundamental lemma, since it applies to the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of the groups G and H. However, Waldspurger had previously used methods of descent to reduce the fundamental lemma for groups to its Lie algebra variant [W3]. Ngô's geometric methods actually apply only to fields of positive characteristic, but again Waldspurger had earlier shown that it suffices to treat this case [W1].[1] Therefore Ngô's theorem does imply the fundamental lemma that has preoccupied mathematicians in automorphic forms since it was first conjectured by Langlands in the 1970's.

I would like to thank Steve Kudla for some helpful suggestions.

Automorphic Forms and the Langlands Programme

To see the importance of the fundamental lemma, we need to recall its place in the theory of automorphic forms. *Automorphic forms* are eigenforms of a commuting family of natural operators attached to reductive algebraic groups. The corresponding eigenvalues are of great arithmetic significance. In fact, the

[1] Another proof of this reduction was subsequently established by Cluckers, Hales and Loeser, by completely different methods of motivic integration.

information they contain is believed to represent a unifying force for large parts of number theory and arithmetic geometry. The *Langlands programme* summarizes much of this, in a collection of interlocking conjectures and theorems that govern automorphic forms and their associated eigenvalues. It explains precisely how a theory with roots in harmonic analysis on algebraic groups can characterize some of the deepest objects of arithmetic. There has been substantial progress in the Langlands programme since its origins in a letter from Langlands to Weil in 1967. However, its deepest parts remain elusive.

The operators that act on automorphic forms are differential operators (Laplace-Beltrami operators) and their combinatorial p-adic analogues (Hecke operators). They are best studied implicitly in terms of group representations. One takes G to be a connected reductive algebraic group over a number field F, and R to be the representation of $G(\mathbb{A})$ by right translation on the Hilbert space $L^2\big(G(F)\backslash G(\mathbb{A})\big)$. We recall that $G(\mathbb{A})$ is the group of points in G with values in the ring $\mathbb{A} = \mathbb{A}_F$ of adèles of F, a locally compact group in which the diagonal image of $G(F)$ is discrete. Automorphic forms, roughly speaking, are functions on $G(F)\backslash G(\mathbb{A})$ that generate irreducible subrepresentations of R, which are in turn known as *automorphic representations*. Their role is similar to that of the much more elementary functions

$$e^{inx}, \qquad n \in \mathbb{Z},\ x \in \mathbb{Z}\backslash\mathbb{R},$$

in the theory of Fourier series. We can think of x as a *geometric* variable, which ranges over the underlying domain, and n as a *spectral* variable, whose automorphic analogue contains hidden arithmetic information.

The centre of the Langlands programme is the principle of functoriality. It postulates a reciprocity law for the spectral data in automorphic representations of different groups G and H, for any L-homomorphism $\rho :\ {}^LH \to {}^LG$ between their L-groups. We recall that ${}^LG$ is a complex, nonconnected group, whose identity component $\widehat{G}$ can be regarded as a complex dual group of G. There is a special case of this that is of independent interest. It occurs when H is an *endoscopic group* for G, which roughly speaking, means that ρ maps $\widehat{H}$ injectively onto the connected centralizer of a semisimple element of $\widehat{G}$. The theory of endoscopy, due also to Langlands, is a separate series of conjectures that includes more than just the special case of functoriality. Its primary role is to describe the internal structure of automorphic representations of G in terms of automorphic representations of its smaller endoscopic groups H. The fundamental lemma arises when one tries to use the trace formula to relate the automorphic representations of G with those of its endoscopic groups.[2]

[2]Endoscopic groups should actually be replaced by *endoscopic data*, objects with slightly more structure, but I will ignore this point.

The Trace Formula and Transfer

The trace formula for G is an identity that relates spectral data with geometric data. The idea, due to Selberg, is to analyze the operator

$$R(f) = \int_{G(\mathbb{A})} f(y)\, R(y)\, \mathrm{d}y$$

on $L^2\big(G(F)\backslash G(\mathbb{A})\big)$ attached to a variable test function f on $G(\mathbb{A})$. One observes that $R(f)$ is an integral operator, with kernel

$$K(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\,\gamma y), \qquad x, y \in G(\mathbb{A}).$$

One then tries to obtain an explicit formula by expressing the trace of $R(f)$ as the integral

$$\int_{G(F)\backslash G(\mathbb{A})} \sum_{\gamma \in G(F)} f(x^{-1}\,\gamma x)\, \mathrm{d}x$$

of the kernel over the diagonal. The formal outcome is an identity

$$\sum_{\{\gamma\}} \int_{G_\gamma(F)\backslash G(\mathbb{A})} f(x^{-1}\,\gamma x)\, \mathrm{d}x = \sum_{\pi} \mathrm{tr}\big(\pi(f)\big), \tag{2}$$

where $\{\gamma\}$ ranges over the conjugacy classes in $G(F)$, $G_\gamma(F)$ is the centralizer of γ in $G(F)$, and π ranges over automorphic representations.

The situation is actually more complicated. Unless $G(F)\backslash G(\mathbb{A})$ is compact, a condition that fails in the most critical cases, $R(f)$ is not of trace class, and neither side converges. One is forced first to truncate the two sides in a consistent way, and then to evaluate the resulting integrals explicitly. It becomes an elaborate process, but one that eventually leads to a rigorous formula with many new terms on each side [A1]. However, the original terms in (2) remain the same in case π occurs in the *discrete* part of the spectral decomposition of R, and γ is *anisotropic* in the strong sense that G_γ is a maximal torus in G with $G_\gamma(F)\backslash G_\gamma(\mathbb{A})$ compact. If γ is anisotropic, and f is a product of functions f_v on the completions $G(F_v)$ of $G(F)$ at valuations v on F, the corresponding integral in (2) can be written

$$\begin{aligned}
&\int_{G_\gamma(F)\backslash G(\mathbb{A})} f(x^{-1}\,\gamma x)\, \mathrm{d}x \\
&\quad = \mathrm{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\big) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\,\gamma x)\, \mathrm{d}x \\
&\quad = \mathrm{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\big) \prod_v \int_{G_\gamma(F_v)\backslash G(F_v)} f_v(x_v^{-1}\,\gamma x_v)\, \mathrm{d}x_v.
\end{aligned}$$

The factor

$$\mathbf{O}_\gamma(f_v) = \mathbf{O}_\gamma(f_v, \mathrm{d}t_v) = \int_{G_\gamma(F_v)\backslash G(F_v)} f_v(x_v^{-1}\,\gamma\,x_v)\,\mathrm{d}x_v$$

is the "orbital integral" of f_v over the conjugacy class of γ in $G(F_v)$. It depends on a choice of Haar measure $\mathrm{d}t_v$ on $T(F_v) = G_\gamma(F_v)$, as well as the underlying Haar measure $\mathrm{d}x_v$ on $G(F_v)$, and makes sense if γ is replaced by any element $\gamma_v \in G(F_v)$ that is *strongly regular*, in the sense that G_{γ_v} is any maximal torus.

The goal is to compare automorphic spectral data on different groups G and H by establishing relations among the geometric terms on the left hand sides of their associated trace formulas. This presupposes the existence of a suitable transfer correspondence $f \to f^H$ of test functions from $G(\mathbb{A})$ to $H(\mathbb{A})$. The idea here is to define the transfer locally at each completion v by asking that the orbital integrals of f_v^H match those of f_v. Test functions are of course smooth functions of compact support, a condition that for the totally disconnected group $G(F_v)$ at a p-adic place v becomes the requirement that f_v be locally constant and compactly supported. The problem is to show for both real and p-adic places v that f_v^H, defined only in terms of conjugacy classes in $H(F_v)$, really is the family of orbital integrals of a smooth function of compact support on $H(F_v)$.

The transfer of functions is a complex matter, which I have had to oversimplify. It is founded on a corresponding transfer mapping $\gamma_{H,v} \to \gamma_v$ of strongly regular conjugacy classes over v from any local endoscopic group H for G to G itself. But this only makes sense for *stable* (strongly regular) conjugacy classes, which in the case of G are defined as the intersections of $G(F_v)$ with conjugacy classes in the group $G(\overline{F}_v)$ over an algebraic closure $\overline{F}_v$. A *stable* orbital integral of f_v is the sum of ordinary orbital integrals over the finite set of conjugacy classes in a stable conjugacy class. Given f_v, H and $\gamma_{H,v}$, Langlands and Shelstad set $\mathbf{SO}_{\gamma_{H,v}}(f_v^H)$ equal to a certain linear combination of orbital integrals of f_v over the finite set of conjugacy classes in the stable image γ_v of $\gamma_{H,v}$. The coefficients are subtle but explicit functions, which they introduce and call transfer factors [LS]. They then conjecture that as the notation suggests, $\{\mathbf{SO}_{\gamma_{H,v}}(f_v^H)\}$ is the set of stable orbital integrals of a smooth, compactly supported function f_v^H on $H(F_v)$.

We can at last say what the fundamental lemma is. For a test function $f = \prod\limits_v f_v$ on $G(\mathbb{A})$ to be *globally* smooth and compactly supported, it must satisfy one further condition. For almost all p-adic places v, f_v must equal the characteristic function 1_{G_v} of an (open) hyperspecial maximal compact subgroup K_v of $G(F_v)$. The fundamental lemma is the natural variant at these places of the Langlands-Shelstad transfer conjecture. It asserts that if f_v equals 1_{G_v}, we can actually take f_v^H to be an associated characteristic function 1_{H_v} on $H(F_v)$. It is in these terms that we understand the identity (1) in Ngô's theorem. We of course have to replace 1_{G_v} and 1_{H_v} by their analogues $1_{\mathfrak{g}_v}$ and $1_{\mathfrak{h}_v}$ on the Lie algebras $\mathfrak{g}(F_v)$ and $\mathfrak{h}(F_v)$ of $G(F_v)$ and $H(F_v)$, and the mapping

$\gamma_{H,v} \to \gamma_v$ by a corresponding transfer mapping $a_{H,v} \to a_v$ of stable adjoint orbits. The superscript κ on the left hand side of (1) is an index that determines an endoscopic group $H = H^\kappa$ for G over F_v by a well defined procedure. It also determines a corresponding linear combination of orbital integrals (called a κ-orbital integral) on $\mathfrak{g}(F_v)$, indexed by the $G(F_v)$-orbits in the stable orbit a_v. The coefficients depend in a very simple way on κ, and when normalized by the quotient $\Delta_G(\,\cdot\,)\,\Delta_H(\,\cdot\,)^{-1}$ of discriminant functions, represent the specialization of the general Langlands-Shelstad transfer factors to the Lie algebra $\mathfrak{g}(F_v)$. The term on the left hand side of (1) is a κ-orbital integral of $1_{\mathfrak{g}_v}$, and the term on the right hand side is a corresponding stable orbital integral of $1_{\mathfrak{h}_v}$.

The Hitchin Fibration

We have observed that local information, in the form of the Langlands-Shelstad transfer conjecture and the fundamental lemma, is a requirement for the comparison of global trace formulas. However, it is sometimes also possible to go in the opposite direction, and to deduce local information from global trace formulas. The most important such result is due to Waldspurger. In 1995, he used a special case of the trace formula to prove that the fundamental lemma implies the Langlands-Shelstad transfer conjecture for p-adic places v [W1]. (The archimedean places v had been treated by local means earlier by Shelstad. See [S].) The fundamental lemma would thus yield the full global transfer mapping $f \to f^H$. It is indeed fundamental!

Ngô had a wonderful idea for applying global methods to the fundamental lemma itself. He observed that the Hitchin fibration [H], which Hitchin had introduced for the study of the moduli space of vector bundles on a Riemann surface, was related to the geometric side of the trace formula. His idea applies to the field $F = k(X)$ of rational functions on a (smooth, projective) curve X over a finite field of large characteristic. This is a global field, which combines the arithmetic properties of a number field with the geometric properties of the field of meromorphic functions on a Riemann surface, and for which both the trace formula and the Hitchin fibration have meaning. Ngô takes G to be a quasisplit group scheme over X. His version of the Hitchin fibration also depends on a suitable divisor D of large degree on X.

The total space of the Hitchin fibration $\mathcal{M} \to \mathcal{A}$ is an algebraic (Artin) stack[3] $\mathcal{M}$ over k. To any scheme S over k, it attaches the groupoid $\mathcal{M}(S)$ of Higgs pairs (E, ϕ), where E is a G-torsor over $X \times S$, and $\phi \in H^0\big(X \times S,\, \mathrm{Ad}(E) \otimes \mathcal{O}_X(D)\big)$ is a section of the vector bundle $\mathrm{Ad}(E)$ obtained from the adjoint representation of G on its Lie algebra $\mathfrak{g}$, twisted by the line bundle $\mathcal{O}_X(D)$. Ngô observed that in the case $S = \mathrm{Spec}(k)$, the definitions

[3] I am little uncomfortable discussing objects in which I do not have much experience. I apologize in advance for any inaccuracies.

lead to a formal identity

$$\sum_{\xi}\left(\sum_{\{a\}}\int_{G_a^{\xi}(F)\backslash G^{\xi}(\mathbb{A})} f_D\big(\mathrm{Ad}(x)^{-1}\,a\big)\,\mathrm{d}x\right) = |\{\mathcal{M}(k)\}|, \tag{3}$$

whose right hand side equals the number of isomorphism classes in the groupoid $\mathcal{M}(k)$ [N1, §1]. On the left hand side, ξ ranges over the set $\ker^1(F, G)$ of locally trivial elements in $H^1(F, G)$, a set that frequently equals $\{1\}$, and G^{ξ} is an inner twist of G by ξ, equipped with a trivialization over each local field F_v, with Lie algebra $\mathfrak{g}^{\xi}$. Also, $\{a\}$ ranges over the $G^{\xi}(F)$ orbits in $\mathfrak{g}^{\xi}(F)$, and $G_a^{\xi}(F)$ is the stabilizer of a in $G^{\xi}(F)$, while

$$f_D = \bigotimes_v f_{D,v},$$

where v ranges over the valuations of F (which is to say the closed points of X) and $f_{D,v}$ is the characteristic function in $\mathfrak{g}^{\xi}(F_v)$ of the open compact subgroup $\varpi_v^{-d_v(D)}\,\mathfrak{g}^{\xi}(\mathcal{O}_v)$.

The expression in the brackets in (3) is the analogue for the Lie algebra $\mathfrak{g}^{\xi}$ of the left hand geometric side of (2). It is to be regarded in the same way as (2), as part of a formal identity between two sums that both diverge. On the other hand, as in (2), the sum over the subset of orbits $\{a\}$ that are anisotropic actually does converge.

The base $\mathcal{A}$ of the Hitchin fibration is an affine space over k. As a functor, it assigns to any S the set

$$\mathcal{A}(S) = \bigoplus_{i=1}^{r} H^0\big(X \times S,\, \mathcal{O}_X(e_i D)\big),$$

where $e_1, \ldots, e_r$ are the degrees of the generators of the polynomial algebra of G-invariant polynomials on $\mathfrak{g}$. Roughly speaking, the set $\mathcal{A}(k)$ attached to $S = \mathrm{Spec}(k)$ parametrizes the stable $G(\mathbb{A})$-orbits in $\mathfrak{g}(\mathbb{A})$ that have representatives in $\mathfrak{g}(F)$, and intersect the support of the function f_D. The Chevalley mapping from $\mathfrak{g}$ to its affine quotient $\mathfrak{g}/G$ determines a morphism h from $\mathcal{M}$ to $\mathcal{A}$ over k. This is the Hitchin fibration. Ngô uses it to isolate the orbital integrals that occur on the left hand side of (3). In particular, he works with the open subscheme $\mathcal{A}^{\mathrm{ani}}$ of $\mathcal{A}$ that represents orbits that are anisotropic over $\overline{k}$. The restriction

$$h^{\mathrm{ani}} :\ \mathcal{M}^{\mathrm{ani}} \longrightarrow \mathcal{A}^{\mathrm{ani}}, \qquad \mathcal{M}^{\mathrm{ani}} = h^{-1}(\mathcal{A}^{\mathrm{ani}}) = \mathcal{M} \times_{\mathcal{A}} \mathcal{A}^{\mathrm{ani}}, \tag{4}$$

of the morphism h to $\mathcal{A}^{\mathrm{ani}}$ is then proper and smooth, a reflection of the fact that the stabilizer in G of any anisotropic point $a \in \mathfrak{g}(F)$ is an anisotropic torus over the maximal unramified extension of F. (See [N2, §4].)

Affine Springer Fibres

The Hitchin fibration can be regarded as a "geometrization" of a part of the global trace formula. It opens the door to some of the most powerful techniques of algebraic geometry. Ngô uses it in conjunction with another geometrization, which had been introduced earlier, and applies to the fibres $\mathcal{M}_a$ of the Hitchin fibration. This is the interpretation of the local orbital integral

$$\mathbf{O}_{\gamma_v}(1_{\mathfrak{g}_v}) = \int_{G_{a_v}(F_v)\backslash G(F_v)} 1_{\mathfrak{g}_v}\big(\mathrm{Ad}(x_v)^{-1}\, a_v\big)\, \mathrm{d}x_v$$

in terms of affine Springer fibres.

The original Springer fibre of a nilpotent element N in a complex semisimple Lie algebra is the variety of Borel subalgebras (or more generally, of parabolic subalgebras in a given adjoint orbit under the associated group) that contain N. It was used by Springer to classify irreducible representations of Weyl groups. The affine Springer fibre of a topologically unipotent (regular, semisimple) element $a_v \in \mathfrak{g}(F_v)$, relative to the adjoint orbit of the lattice $\mathfrak{g}(\mathcal{O}_v)$, is the set

$$\mathcal{M}_v(a,k) = \big\{x_v \in G(F_v)/G(\mathcal{O}_v) :\ \mathrm{Ad}(x_v)^{-1}\, a_v \in \mathfrak{g}(\mathcal{O}_v)\big\}$$

of lattices in the orbit that contain a_v. Suppose for example that a_v is anisotropic over F_v, in the strong sense that the centralizer $G_{a_v}(F_v)$ is compact. If one takes the compact (abelian) groups $G_{a_v}(F_v)$ and $\mathfrak{g}(\mathcal{O}_v)$ to have Haar measure 1, one sees immediately that $\mathbf{O}_{a_v}(1_{\mathfrak{g}_v})$ equals the order $|\mathcal{M}_v(a,k)|$ of $\mathcal{M}_v(a,k)$. (Topologically unipotent means that the linear operator $\mathrm{ad}(a_v)^n$ on $\mathfrak{g}(F_v)$ approaches 0 as n approaches infinity. In general, the closer a_v is to 0, the larger is the set $\mathcal{M}_v(a,k)$, and the more complex the orbital integral $\mathbf{O}_{a_v}(1_{\mathfrak{g}_v})$.)

Kazhdan and Lusztig introduced affine Springer fibres in 1988, and established some of their geometric properties [KL]. In particular, they proved that $\mathcal{M}_v(a,k)$ is the set of k-points of an inductive limit $\mathcal{M}_v(a)$ of schemes over k. (It is this ind-scheme that is really called the affine Springer fibre.) Their results also imply that if a_v is anisotropic over the maximal unramified extension of F_v, $\mathcal{M}_v(a)$ is in fact a scheme.

The study of these objects was then taken up by Goresky, Kottwitz and MacPherson. Their strategy was to obtain information about the orbital integral $|\mathcal{M}_v(a,k)|$ from some version of the Lefschetz fixed point formula. They realized that relations among orbital integrals could sometimes be extracted from cohomology groups of affine Springer fibres $\mathcal{M}_v(a)$ and $\mathcal{M}_v(a_H)$, for the two different groups G and H. Following this strategy, they were able to establish the identity (1) for certain pairs $(a_v, a_{H,v})$ attached to unramified maximal tori [GKM]. Goresky, Kottwitz and MacPherson actually worked with certain equivariant cohomology groups. Laumon and Ngô later added a deformation argument, which allowed them to prove the fundamental lemma for unitary groups [LN]. However, the equivariant cohomology groups that led to these results are not available in general.

It was Ngô's introduction of the global Hitchin fibration that broke the impasse. He formulated the affine Springer fibre $\mathcal{M}_v(a)$ as a functor of schemes S over k, in order that it be compatible with the relevant Hitchin fibre $\mathcal{M}_a$ [N2, §3.2]. He also introduced a third object to mediate between the two kinds of fibre. It is a Picard stack $\mathcal{P} \to \mathcal{A}$, which acts on $\mathcal{M}$, and represents the natural symmetries of the Hitchin fibration. Ngô attached this object to the group scheme J over $\mathcal{A}$ obtained from the G-centralizers of regular elements in $\mathfrak{g}$, and the Kostant section from semisimple conjugacy classes to regular elements.

The stack $\mathcal{P}$ plays a critical role. Ngô used it to formulate the precise relation between the Hitchin fibre $\mathcal{M}_a$ at any $a \in \mathcal{A}^{\mathrm{ani}}(\overline{k})$ with the relevant affine Springer fibres $\mathcal{M}_v(a)$ [N2, Proposition 4.15.1]. Perhaps more surprising is the fact that as a group object in the category of stacks, $\mathcal{P}$ governs the stabilization of anisotropic Hitchin fibres $\mathcal{M}_a$. Ngô analyses the characters $\{\kappa\}$ on the abelian groups of connected components $\pi_0(\mathcal{P}_a)$. He shows that they are essentially the geometric analogues of objects that were used to stabilize the anisotropic part of the trace formula.

Stabilization

Could one possibly establish the fundamental lemma from the trace formula? Any such attempts have always foundered on the lack of a transfer of unit functions 1_{G_v} to 1_{H_v} by orbital integrals. In some sense, however, this is exactly what Ngô does. It is not the trace formula for automorphic forms that he uses, but the Grothendieck-Lefschetz trace formula of algebraic geometry. Moreover, it is the "spectral" side of this trace formula that he transfers from $\mathfrak{g}$ to $\mathfrak{h}$ (the Lie algebras of G and H), in the form of data from cohomology, rather than its "geometric" side, in the form of data given by fixed points of Frobenius endomorphisms. This is in keeping with the general strategy of Goresky, Kottwitz and MacPherson. The difference here is that Ngô begins with perverse cohomology attached to the global Hitchin fibration, rather than the ordinary equivariant cohomology of a local affine Springer fibre.

Stabilization refers to the operation of writing the trace formula for G, or rather each of its terms $I(f)$, as a linear combination

$$I(f) = \sum_H \iota(G, H)\, S^H(f^H) \tag{5}$$

of stable distributions on the endoscopic groups H of G over F. (A stable distribution is a linear form whose values depend only on the stable orbital integrals of the given test function. The resulting identity of stable distributions for any given H, obtained by induction on $\dim(H)$ from (5) and the trace formula for G, is known as the stable trace formula.) The process is most transparent for the anisotropic terms[4]

$$I^{\text{ani}}(f) = \sum_{\{\gamma\}} \text{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\big) \cdot \prod_v \big(\mathbf{O}_\gamma(f_v)\big), \tag{6}$$

in which $\{\gamma\}$ ranges over the set of anisotropic conjugacy classes in $G(F)$. It was carried out in this case by Langlands [L] and Kottwitz [K2], assuming the existence of the global transfer mapping $f \to f^H$ (which Waldspurger later reduced to the fundamental lemma). This is reviewed by Ngô in the first chapter (§1.13) of his paper [N2].

The idea for the stabilization of (6) can be described very roughly as follows. One first groups the conjugacy classes $\{\gamma\}$ into stable conjugacy classes $\{\gamma\}_{\text{st}}$ in $G(F)$, for representatives γ attached to anisotropic tori $T = G_\gamma$. The problem is to quantify the obstruction for the contribution of $\{\gamma\}_{\text{st}}$ to be a stable distribution on $G(\mathbb{A})$. For any v, the set of $G(F_v)$-conjugacy classes in the stable conjugacy class of γ in $G(F_v)$ is bijective with the set

$$\ker\big(H^1(F_v, G) \;\longrightarrow\; H^1(F_v, T)\big)$$

of elements in the finite abelian group $H^1(F_v, T)$ whose image in the Galois cohomology set $H^1(F_v, G)$ is trivial. Let me assume for simplicity in this description that G is simply connected. The set $H^1(F_v, G)$ is then trivial for any p-adic place v, and becomes a concern only when v is archimedean. The obstruction for $\{\gamma\}_{\text{st}}$ is thus closely related to the abelian group

$$\text{coker}\left(H^1(F, T) \;\longrightarrow\; \bigoplus_v H^1(F_v, T)\right).$$

The next step is to apply Fourier inversion to this last group. According to Tate-Nakayama duality theory, its dual group of characters κ is isomorphic to $\widehat{T}^\Gamma$, the group of elements in the complex dual torus $\widehat{T}$ that are invariant under the natural action of the global Galois group $\Gamma = \text{Gal}(\overline{F}\backslash F)$. On the other hand, each $\kappa \in \widehat{T}^\Gamma$ maps to a semisimple element in the complex dual group $\widehat{G}$, which can be used to define an endoscopic group $H = H^\kappa$ for G. One accounts for the local archimedean sets $H^1(F_v, G)$ simply by defining the local contribution of a complementary element in $H^1(F_v, T)$ to be 0. In this way, one obtains a global contribution to (6) for any κ. It is a global κ-orbital integral, whose local factor at almost any v appears on the left hand side of the identity in the fundamental lemma.

One completes the stabilization of (6) by grouping the indices (T, κ) into equivalence classes that map to a given H. The corresponding contributions to

[4] This expression only makes sense if the split component A_G of G is trivial. In general, one must include A_G in the volume factors.

the right hand side of (6) become the summands of H in (5). Notice that the summands with $\kappa = 1$ correspond to the endoscopic group H with $\widehat{H} = \widehat{G}$ (a quasisplit inner form G^* of G). Like all of the other summands, they are defined directly. This is in contrast to the more exotic parts $I(f)$ of the trace formula [A1, §29], where the contribution of $H = G^*$ (known as the stable part $I_{\mathrm{st}}(f)$ of $I(f)$ in case $G = G^*$ is already quasisplit) can only be constructed from (5) indirectly by induction on $\dim(H)$.

The heart of Ngô's proof is an analogue of the stabilization of (6) for the geometrically anisotropic part (4) of the Hitchin fibration.[5] This does not depend on the transfer of functions, and is therefore unconditional. Ngô formulates it as an identity of the $\{\kappa\}$-component $(\,\cdot\,)_\kappa$ of an object attached to G with the stable component $(\,\cdot\,)_{\mathrm{st}}$ of a similar object for the corresponding endoscopic group. I will only be able to describe his steps in the most general of terms.

Since $\mathcal{M}^{\mathrm{ani}}$ is a smooth Deligne-Mumford stack, the purity theorems of [D] and [BBD] can be applied to the proper morphism h^{ani} in (4). They yield an isomorphism

$$h_*^{\mathrm{ani}}\,\overline{\mathbb{Q}}_\ell \cong \bigoplus_n {}^pH^n\,(h_*^{\mathrm{ani}}\,\overline{\mathbb{Q}}_\ell)\,[-n], \tag{7}$$

whose left hand side is a priori only an object in the derived category $D_c^b(\mathcal{A})$ of the bounded complexes of sheaves on $\mathcal{A}$ with constructible cohomology, but whose right hand summands are pure objects in the more manageable abelian subcategory of perverse sheaves on $\mathcal{A}$. Ngô then considers the action of the stack $\mathcal{P}^{\mathrm{ani}}$ over $\mathcal{A}^{\mathrm{ani}}$ on either side. Appealing to a homotopy argument, he observes that this action factors through the quotient $\pi_0(\mathcal{P}^{\mathrm{ani}})$ of connected components, a sheaf of finite abelian groups on $\mathcal{A}^{\mathrm{ani}}$. As we noted earlier, an analysis of this sheaf then leads him to the dual characters $\{\kappa\}$ that were part of the stabilization of (6), and relative to which one can take equivariant components ${}^pH^n(f_*^{\mathrm{ani}}\,\overline{\mathbb{Q}}_\ell)_\kappa$ of the summands in (7). On the other hand, if H corresponds to κ, we have the morphism ν from $\mathcal{A}_H$ to $\mathcal{A}$ that comes from the embedding $\widehat{H} \subset \widehat{G}$ of two dual groups of equal rank. It provides a pullback mapping of sheaves from $\mathcal{A}$ to $\mathcal{A}_H$. Ngô's stabilization of (4) then takes the form of an isomorphism

$$\nu^*\left(\bigoplus_n {}^pH^n(h_*^{\mathrm{ani}}\,\overline{\mathbb{Q}})_\kappa\,[2r]\,(r)\right) \cong \bigoplus_n {}^pH^n\,(h_{H,*}^{\mathrm{ani}}\,\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}, \tag{8}$$

for a degree shift $[2r]$ and Tate twist (r) attached to a certain positive integer $r = r_H^G(D)$. (See [N2, Theorem 6.4.2].)

[5] Recall that the left hand side of (3) differs from that of (2) in having a supplementary sum over $\xi \in \ker^1(F, G)$. This is part of the structure of the Hitchin fibration. But it also actually leads to a slight simplification of the stabilization of (6) by Langlands and Kottwitz. (See [N2, §1.13].)

Ngô's "geometric stabilization" identity (8) , whose statement I have oversimplified slightly,[6] is a key theorem. In particular, it leads directly to the fundamental lemma. For it implies a similar identity for the stalks of the sheaves at a point $a_H \in \mathcal{A}_H$ (with image $a \in \mathcal{A}$ under ν). After some further analysis, the application of a theorem of proper base change reduces what is left to an endoscopic identity for the cohomology of affine Springer fibres. This is exactly what Goresky, Kottwitz and MacPherson had been working towards. Once it is available, an application of the Grothendieck-Lefschetz trace formula gives a relation among points on affine Springer fibres, which leads to the fundamental lemma. (See [LN, §3.10] for example.)

However, it is more accurate to say that the (global) stabilization identity (8) is *parallel* to the (local) fundamental lemma. Ngô actually had to prove the two theorems together. In a series of steps, which alternate between local and global arguments, and go back and forth between the two theorems, he treats special cases that become increasingly more general, until the proof of both theorems is at last complete. Everything of course depends on the original divisor D on X, which in Ngô's argument is allowed to vary in such a way that its degree approaches infinity. The main technical result that goes into the proof of (8) is a theorem on the support of the sheaves on the left hand side. As I understand it, this is highly dependent on the fact that these objects are actually perverse sheaves.

Further Remarks

I should also mention two important generalizations of the fundamental lemma. One is the "twisted fundamental lemma" conjectured by Kottwitz and Shelstad, which will be needed for any endoscopic comparison that includes the twisted trace formula. Waldspurger [W3] had reduced this conjecture to the primary theorem of Ngô, together with a variant [N2, Théorème 2] of (1) that Ngô proves by the same methods. Another is the "weighted fundamental lemma", which applies to the more general geometric terms in the trace formula that are obtained by truncation. It is needed for any endoscopic comparisons that do not impose unsatisfactory local constraints on the automorphic representations. Once again, Waldspurger had reduced the conjectural identity to its analogue for a Lie algebra over a local field of positive characteristic. Chaudouard and Laumon have recently proved the weighted fundamental lemma for Lie algebras by extending the methods of Ngô to other terms in the trace formula [CL]. This has been a serious enterprise, which requires a geometrization of analytic truncation methods in order to deal with the failure of the full Hitchin fibration $\mathcal{M} \to \mathcal{A}$ to be proper. In any case, all forms of the fundamental lemma have

[6] The isomorphism is between the semisimplifications of the graded perverse sheaves. Moreover, ν, h^{ani} and h_H^{ani} should be replaced by their preimages $\widetilde{\nu}$, $\widetilde{h}^{\mathrm{ani}}$ and $\widetilde{h}_H^{\mathrm{ani}}$ relative to certain finite morphisms.

now been proved, including the most general "twisted, weighted fundamental lemma".

I have emphasized the role of transfer in the comparison of trace formulas. This is likely to lead to a classification of automorphic representations for many groups G, beginning with orthogonal and symplectic groups [A2], according to Langlands' conjectural theory of endoscopy. The fundamental lemma also has other important applications. For example, its proof fills a longstanding gap in the theory of Shimura varieties. Kottwitz observed some years ago that the key geometric terms in the Grothendieck-Lefschetz formula for a Shimura variety are actually twisted orbital integrals [K1]. The twisted fundamental lemma now allows a comparison of these terms with corresponding terms in the stable trace formula. (See [K3].) This in turn leads to reciprocity laws between the arithmetic data in the cohomology of many such varieties with the spectral data in automorphic forms.

This completes my report. It will be clear that Ngô's proof is deep and difficult. What may be less clear is the enormous scope of his methods. The many diverse geometric objects he introduces are all completely natural. That they so closely reflect objects from the trace formula and local harmonic analysis, and fit together so beautifully in Ngô's proof, is truly remarkable.

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