SOME PROBLEMS IN LOCAL HARMONIC ANALYSIS

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INTRODUCTION

The purpose of this article is to discuss some questions in the harmonic analysis of real and p-adic groups. We shall be particularly concerned with the properties of a certain family of invariant distributions. These distributions arose naturally in a global context, as the terms on the geometric side of the trace formula. However, they are purely local objects, which include the ordinary invariant orbital integrals. One of our aims is to describe how the distributions also arise in a local context. They appear as the terms on the geometric side of a new trace formula, which is simpler than the original one, and is the solution of a natural question in local harmonic analysis. The local trace formula seems to be a promising tool. It might have implications for the difficult local problems which are holding up progress in automorphic forms.

We have organized the paper loosely around three general problems. We shall describe the problems and the distributions together in §1. This section is entirely expository. In §2, which is also largely expository, we shall discuss the role of the distributions in the local trace formula. Finally, in §3, we shall see how the local trace formula can be applied to some questions in local harmonic analysis. We shall sketch an application to each of the three general problems of §1. These results are all more or less immediate consequences of the same kind of approximation argument. It remains to be seen whether a deeper study of the local trace formula will lead to further applications.

1. WEIGHTED ORBITAL INTEGRALS AND WEIGHTED CHARACTERS

Let G be a connected reductive algebraic group over a local field F. We assume that F is of characteristic 0. Then F equals the real field $\mathbb{R}$, or a p-adic field $\mathbb{Q}_p$, or a finite extension of one of these. In particular, our discussion of the group $G(F)$ of rational points applies to both real and p-adic groups.

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We fix a suitable maximal compact subgroup K of $G(F)$. We can then form the Hecke algebra

$$\mathcal{H}(G) = \mathcal{H}(G(F), K)$$

of smooth, compactly supported, K-finite functions on $G(F)$. The Hecke algebra is contained in the space $C_c^\infty(G(F))$ of smooth functions of compact support (the two spaces are in fact equal in the p-adic case), and $C_c^\infty(G(F))$ is contained in Harish-Chandra's Schwartz space $\mathcal{C}(G(F))$. There are natural topologies on the three spaces for which the embeddings

$$\mathcal{H}(G) \subset C_c^\infty(G(F)) \subset \mathcal{C}(G(F))$$

are continuous, and have dense image.

By a *distribution* on $\mathcal{H}(G)$, we shall mean a continuous linear functional I on $\mathcal{H}(G)$. This is a slight abuse of terminology, for it is only those I which extend to continuous linear functionals on $C_c^\infty(G(F))$ which are distributions on $G(F)$. The functionals which in addition extend continuously to $\mathcal{C}(G(F))$ are of course the *tempered* distributions on $G(F)$. The functional I is said to be *invariant* if

$$I(f * g) = I(g * f), \qquad f, g \in \mathcal{H}(G).$$

In case I extends to $C_c^\infty(G(F))$, this condition is easily seen to be equivalent to the more familiar property

$$I(f^y) = I(f), \qquad f \in C_c^\infty(G(F)),\ y \in G(F),$$

where $f^y(x) = f(yxy^{-1})$. The most fundamental invariant distributions are the two families of orbital integrals and tempered characters.

Recall that orbital integrals are parametrized by points γ in $G_{\mathrm{reg}}(F)$, the set of regular semisimple elements in $G(F)$. They are defined by integrals

$$I_G(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx, \qquad f \in \mathcal{H}(G),$$

where G_γ is the centralizer of γ in G, and

$$D(\gamma) = \det\big(1 - Ad(\gamma)_{\mathfrak{g}/\mathfrak{g}_\gamma}\big)$$

is the Weyl discriminant. Orbital integrals are invariant distributions, which have been shown by Harish-Chandra to be tempered. They are basic objects in local harmonic analysis. Orbital integrals are also very important for the theory of automorphic forms, for they are the terms on the geometric side of the Selberg trace formula for compact quotient.

The tempered characters

$$I_G(\pi, f) = \operatorname{tr}(\pi(f)), \qquad f \in \mathcal{H}(G),$$

can be regarded as dual analogues of orbital integrals. They too are invariant tempered distributions, which are parametrized by the set $\Pi_{\text{temp}}(G(F))$ of (equivalence classes of) irreducible tempered representations of $G(F)$. It is convenient to think of $I_G(\pi, f)$ as a transform as well as a distribution. We therefore define a map

$$f \longrightarrow f_G,$$

from $\mathcal{H}(G)$ to the space of complex valued functions on $\Pi_{\text{temp}}(G(F))$, by setting

$$f_G(\pi) = I_G(\pi, f) = \operatorname{tr}(\pi(f)), \qquad \pi \in \Pi_{\text{temp}}(G(F)).$$

An invariant distribution I on $\mathcal{H}(G)$ is said to be *supported on characters* if $I(f) = 0$ for every function $f \in \mathcal{H}(G)$ such that $f_G = 0$. Our first problem is a classification question, which we mention for the sake of general orientation.

Problem A. *Show that any invariant distribution I on $\mathcal{H}(G)$ is supported on characters.*

Remark 1. In the p-adic case, the problem was solved by Kazhdan [18], who used global methods (specifically, a simple form of the global trace formula) to show that orbital integrals are supported on characters. Harish-Chandra had earlier given an argument based on Shalika germs which reduced the question to the case of orbital integrals. For real groups, the problem has not been solved in its present form.

Remark 2. For tempered distributions on the Schwartz space, the analogous question can be answered for general real groups by using the characterization of $\mathcal{C}(G(F))$ [1] under the full Fourier transform. On the other hand, for p-adic groups this version of the problem has been solved only for $GL(n)$ [22].

One reason for considering the Hecke algebra is that there is a nice characterization [12], [14] of the space

$$\mathcal{I}(G) = \{f_G : f \in \mathcal{H}(G)\}$$

of functions on $\Pi_{\text{temp}}(G(F))$. This leads to a natural topology on $\mathcal{I}(G)$ in terms of the co-ordinates of the domain $\Pi_{\text{temp}}(G(F))$, for which the map $f \to f_G$ is continuous. One checks that if I is supported on characters, there is a unique distribution $\hat{I}$ on $\mathcal{I}(G)$ such that

$$\hat{I}(f_G) = I(f), \qquad f \in \mathcal{H}(G).$$

The next problem we state informally as

Problem B. *Given some natural invariant distribution I which is supported on characters, deduce information about $\hat{I}$.*

Remark 1. One does not generally expect to be able to compute $\hat{I}$ explicitly. Instead, one could try to determine the qualitative properties of $\hat{I}$. For example, if I is tempered, one could ask whether $\hat{I}$ is a function on the space $\Pi_{\text{temp}}(G(F))$. Given Harish-Chandra's Plancherel formula one can construct a variety from $\Pi_{\text{temp}}(G(F))$ which is a disjoint union of Euclidean spaces (F-Archimedean) or compact tori (the p-adic case). For a tempered I, one could try to determine explicitly the singular support of $\hat{I}$.

Remark 2. Suppose that

$$I = I_G(\gamma), \qquad \gamma \in G_{\text{reg}}(F).$$

Using results of Shelstad on L-indistinguishability, R. Herb has computed $\hat{I}$ in the case $F = \mathbb{R}$. For p-adic groups, the problem in this case is very important, but little is known. In particular, even though I is tempered, there is to my knowledge no general result on the singular support of $\hat{I}$.

The orbital integrals $I_G(\gamma)$ are part of a larger family of invariant distributions. Suppose that M is a Levi component of some parabolic subgroup of G defined over F. The set $\mathcal{P}(M)$ of all parabolic subgroups with Levi component M is in bijective correspondence with the chambers in the real vector space

$$\mathfrak{a}_M = Hom(X(M)_F, \mathbb{R}) \xrightarrow{\sim} Hom(X(A_M), \mathbb{R}),$$

where $X(\cdot)$ stands for the module of rational characters, and A_M is the split component of the center of M. For any group $P = MN_P$ in $\mathcal{P}(M)$, there is the usual map

$$H_P : G(F) \longrightarrow \mathfrak{a}_P = \mathfrak{a}_M$$

that comes from the decomposition

$$G(F) = M(F)N_P(F)K.$$

Suppose $x \in G(F)$. Let $\Pi_M(x)$ be the convex hull in $\mathfrak{a}_M$ of the finite set

$$\{H_P(x) : P \in \mathcal{P}(M)\}.$$

We write $v_M(x) = v_M^G(x)$ for the volume in $\mathfrak{a}_M/\mathfrak{a}_G$ of the projection of $\Pi_M(x)$.

The convex polytopes $\Pi_M(x)$ are nice objects whose geometric properties are tied up with the structure of G. For example, there is a bijection

$$Q \in \mathcal{F}(M) \longrightarrow \Pi_M^Q(x)$$

between the finite set $\mathcal{F}(M)$ of parabolic subgroups which contain M, and the *facets* of the polytope $\Pi_M(x)$. The facet $\Pi_M^Q(x)$ is equal to the convex hull of the set

$$\{H_P(x) : P \in \mathcal{P}(M),\ P \subset Q\}.$$

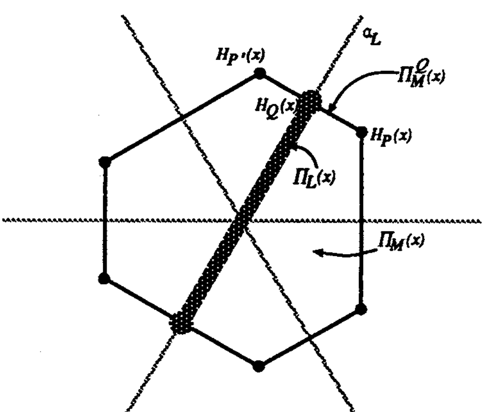
We write $v_M^Q(x)$ for the volume of the projection of $\Pi_M^Q(x)$ onto $\mathfrak{a}_M/\mathfrak{a}_Q$. Another property of $\Pi_M(x)$ concerns the finite set $\mathcal{L}(M)$ of Levi subgroups L of G which contain M. These are in bijective correspondence with the vector subspaces $\mathfrak{a}_L$ of $\mathfrak{a}_M$ which are orthogonal complements of facets, or rather, orthogonal complements of affine spaces generated by facets. There is thus a bijection

$$L \in \mathcal{L}(M) \longrightarrow \Pi_L(x)$$

between the finite set of Levi subgroups which contain M, and convex polytopes obtained by projecting $\Pi_M(x)$ onto orthogonal complements of facets. More precisely, $\Pi_L(x)$ is the projection of $\Pi_M(x)$ onto the orthogonal complement $\mathfrak{a}_L$ of the affine space generated by any of the facets $\{\Pi_M^Q(x) : Q \in \mathcal{P}(L)\}$. The notation makes sense, for $\Pi_L(x)$ is just the convex hull of $\{H_Q(x) : Q \in \mathcal{P}(L)\}$, which is the polytope associated to L in its own right. In particular, $v_L(x)$ is the volume of the projection of $\Pi_M(x)$ onto $\mathfrak{a}_L/\mathfrak{a}_G$.

The usual diagram for $G = SL(3)$ is a useful reminder of these relationships. Taking M to be minimal, we identify $\mathfrak{a}_M$ as a Euclidean space with the plane. There are six chambers and six minimal parabolic subgroups $P \in \mathcal{P}(M)$. There are three subspaces $\mathfrak{a}_L$ of dimension 1, and for each such L there are two maximal parabolic subgroups $Q \in \mathcal{P}(L)$.

Figure 1.



In general, the function $v_M(x)$ can be used to define a noninvariant measure on any G-regular class in $M(F)$. Observe first that if m is any point in $M(F)$, the polytope $\Pi_M(mx)$ is the translate of $\Pi_M(x)$ by a vector $H_M(m)$. Therefore, $v_M(mx)$ equals $v_M(x)$. Now, suppose that γ belongs to

$M(F) \cap G_{\mathrm{reg}}(F)$. Then G_γ is contained in M, so that $v_M(x)$ is left invariant under $G_\gamma(F)$. One can therefore define the weighted orbital integral

$$J_M(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)v_M(x)dx,$$

for any function $f \in \mathcal{H}(G)$.

Although it is a generalization of the ordinary orbital integral, the weighted orbital integral $J_M(\gamma, f)$ is not invariant. There is in fact a rather explicit formula for the lack of invariance.

It can be shown that $J_M(\gamma)$ is a tempered distribution, so to analyze its lack of invariance it suffices to look at $J_M(\gamma, f^y)$ for any element $y \in G(F)$. Changing variables in the integral over x, we first write

$$J_M(\gamma, f^y) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)v_M(xy)dx.$$

If P belongs to $\mathcal{P}(M)$, set

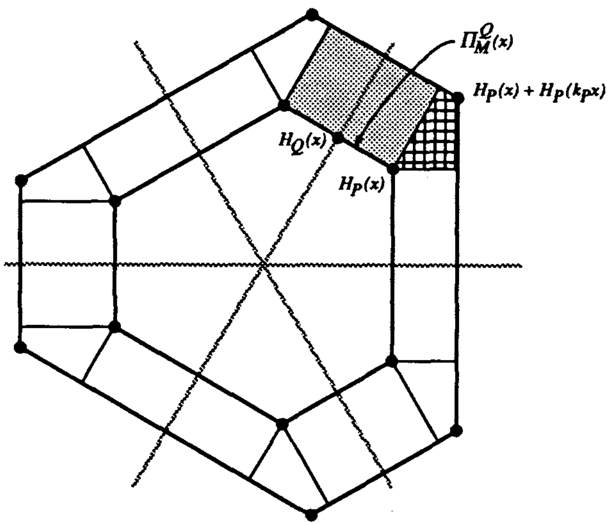
$$x = n_P\, m_P\, k_P, \qquad n_P \in N_P(F),\ m_P \in M(F),\ k_P \in K.$$

Then

$$H_P(xy) = H_P(m_P) + H_P(k_P y) = H_P(x) + H_P(k_P y).$$

To sketch how to evaluate $v_M(xy)$, we argue geometrically from the diagram for $SL(3)$. For simplicity, assume that for each $P \in \mathcal{P}(M)$, the point $H_P(k_P y)$ lies in the chamber of P. Then for $SL(3)$ we have Figure 2:

Figure 2.



The inner hexagon is $\Pi_M(x)$, while the outer hexagon is just $\Pi_M(xy)$. Let $u_P(k_P y)$ denote the area of the hatched quadrilateral. This region is separated from the shaded rectangle by a line segment whose length equals the analogous number $u_Q(k_Q y)$ for the maximal parabolic subgroup Q. The other side of the rectangle is $\Pi_M^Q(x)$, a line segment of length $v_M^Q(x)$, so the area of the shaded rectangle is the product of $u_Q(k_Q y)$ with $v_M^Q(x)$. In this way, we can account for the area of each of the pieces that comprise $\Pi_M(xy)$. We obtain a formula

$$v_M(xy) \;=\; \sum_{Q\in\mathcal{F}(M)} v_M^Q(x)u_Q(k_Q y),$$

which expresses $v_M(xy)$ as a sum of mixed volumes. Substituting this expansion into the integral above, we obtain the formula

$$J_M(\gamma, f^y) \;=\; \sum_{Q\in\mathcal{F}(M)} J_M^{M_Q}(\gamma, f_{Q,y}), \tag{1.1}$$

where

$$f_{Q,y}(m) \;=\; \delta_Q(m)^{\frac{1}{2}}\int_K\int_{N_Q(F)} f(k^{-1}mnk)u_Q(ky)dn\,dk$$

for any $m\in M_Q(F)$ [2, Lemma 8.2]. Here M_Q is the Levi component of Q which contains M, and δ_Q is the modular function of $Q(F)$. Notice that the summand in (1.1) with $M = G$ equals $J_M(\gamma, f)$. We therefore can write (1.1) as

$$J_M(\gamma, f^y) - J_M(\gamma, f) \;=\; \sum_{Q\neq G} J_M^{M_Q}(\gamma, f_{Q,y}),$$

more clearly displaying it as an obstruction to the invariance of $J_M(\gamma)$.

Just as the tempered characters are dual to the invariant orbital integrals, there are dual analogues of weighted orbital integrals, which can be regarded as weighted (tempered) characters. If $\pi\in\Pi_{\mathrm{temp}}(M(F))$ and $\Lambda\in i\mathfrak{a}_M^*$, the representation

$$\pi_\Lambda(m) = e^{\Lambda(H_M(m))}\pi(m), \qquad m\in M(F),$$

also belongs to $\Pi_{\mathrm{temp}}(M(F))$. We can form the (parabolically) induced representations

$$\mathcal{I}_P(\pi_\Lambda), \qquad P\in\mathcal{P}(M),$$

of $G(F)$, and the normalized intertwining operators

$$R_{Q|P}(\pi_\Lambda):\ \mathcal{I}_P(\pi_\Lambda)\longrightarrow\mathcal{I}_Q(\pi_\Lambda), \qquad P,Q\in\mathcal{P}(M),$$

between them [6, §1-2]. Both $\mathcal{I}_P(\pi_\Lambda)$ and $R_{Q|P}(\pi_\Lambda)$ are analytic operator valued functions of $\Lambda \in i\mathfrak{a}_M^*$. For any $P \in \mathcal{P}(M)$, set

$$\mathcal{R}_M(\pi, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} R_{Q|P}(\pi)^{-1} R_{Q|P}(\pi_\Lambda) \theta_Q(\Lambda)^{-1} ,$$

where if $\Delta_Q^\vee$ denotes the set of simple "co-roots" of Q,

$$\theta_Q(\Lambda) = \mathrm{vol}\big(\mathfrak{a}_M/\mathbb{Z}(\Delta_Q^\vee) + \mathfrak{a}_G\big)^{-1} \prod_{\alpha^\vee \in \Delta_Q^\vee} \Lambda(\alpha^\vee) .$$

It is a simple matter to show that the limit exists [2, Lemma 6.3]. Consequently $\mathcal{R}_M(\pi, P)$ is a well defined operator on the underlying space of $\mathcal{I}_P(\pi)$. For example, if P is a maximal parabolic with simple root $\alpha, \mathcal{R}_M(\pi, P)$ is a constant multiple of the logarithmic derivative

$$R_{\overline{P}|P}(\pi)^{-1} \cdot \lim_{z \to 0} \Big(\frac{d}{dz} R_{\overline{P}|P}(\pi_{z\alpha})\Big) .$$

In general, the weighted character of a function $f \in \mathcal{H}(G)$ is defined as the trace

$$J_M(\pi, f) = \mathrm{tr}\big(\mathcal{R}_M(\pi, P) \mathcal{I}_P(\pi, f)\big) .$$

As a spectral analogue of $J_M(\gamma, f)$, the distribution $J_M(\pi, f)$ is not invariant. However, the considerations that lead to the formula (1.1) can be adapted to the study of $J_M(\pi, f^y)$. Observing first that

$$v_M(x) = \lim_{\Lambda \to 0} \Big(\int_{\Pi_M(x)/\mathfrak{a}_G} e^{\Lambda(H)} dH \Big) ,$$

one can then rewrite the right hand limit in a general form that is similar to the expression for $\mathcal{R}_M(\pi, P)$. The mixed volume expansion for $v_M(xy)$ translates into a special case of an expansion that applies to certain functions of Λ, and in particular, to the functions of which both $v_M(x)$ and $\mathcal{R}_M(\pi, P)$ are the limits. This provides an expansion for

$$J_M(\pi, f^y) = \mathrm{tr}\big(\mathcal{I}_P(\pi, y) \mathcal{R}_M(\pi, P) \mathcal{I}_P(\pi, y)^{-1} \mathcal{I}_P(\pi, f)\big)$$

as a sum over groups $Q \in \mathcal{F}(M)$. The result is a formula

$$J_M(\pi, f^y) = \sum_{Q \in \mathcal{F}(M)} J_M^{M_Q}(\pi, f_{Q,y}) \tag{1.2}$$

which is parallel to (1.1) [2, Lemma 8.3].

Because the distributions $\{J_M(\gamma)\}$ and $\{J_M(\pi)\}$ have similar behaviour (1.1) and (1.2) under conjugation, we might suspect that they are related to each other by invariant distributions. This is indeed the case. One must first interpret $J_M(\pi)$ as a transform

$$\phi_M(f) = \phi_M^G(f) : \pi \longrightarrow J_M(\pi, f), \qquad \pi \in \Pi_{\text{temp}}(M(F)),$$

that maps functions on $G(F)$ to functions on $\Pi_{\text{temp}}(M(F))$. There is a technical problem that the function $\phi_M(f)$ does not belong to $\mathcal{I}(M)$. For example, in the case of maximal parabolic P, the logarithmic derivative

$$R_{\overline{P}|P}(\pi_{z\alpha})^{-1} \frac{d}{dz} R_{\overline{P}|P}(\pi_{z\alpha}), \qquad z \in i\mathbb{R},$$

will have poles when the variable z is extended to the whole complex plane. This means that $J_M(\pi, f)$ is not a Paley-Wiener function in the co-ordinates of π. However, the problem is not serious. Let $\mathcal{H}_{ac}(G)$ be the space of smooth, K-finite functions on $G(F)$ whose restrictions to the fibres of the map $H_G : G(F) \to \mathfrak{a}_G$ all have compact support. One can define a version of the map $f \to f_G$ for $f \in \mathcal{H}_{ac}(G)$, and a variant of the main theorem in [12] and [14] provides a characterization of the image

$$\mathcal{I}_{ac}(G) = \{f_G : f \in \mathcal{H}_{ac}(G)\} .$$

(See [5, Appendix].) It can then be shown that ϕ_M maps $\mathcal{H}_{ac}(G)$ continuously to $\mathcal{I}_{ac}(M)$ [6, Theorem 12.1].

Theorem. [4, §2] *There are unique invariant distributions*

$$I_M(\gamma) = I_M^G(\gamma)$$

on $\mathcal{H}_{ac}(G)$ which are supported on characters, and such that

$$J_M(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^L(\gamma, \phi_L(f)) .$$

Proof sketch. We shall sketch the formal part of the proof. We assume inductively that $I_M^L(\gamma)$ has been defined, and has the required properties, for any $L \in \mathcal{L}(M)$ with $L \neq G$. The required distribution can then be defined uniquely by the formula

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{\{L \in \mathcal{L}(M) : L \neq G\}} \hat{I}_M^L(\gamma, \phi_L(f)) .$$

We shall show that $I_M(\gamma)$ is invariant.

The formulas (1.1) and (1.2) cannot actually be applied as they stand, for f^y is not a K-finite function. However, since the formulas have simple variants which pertain to K-finite functions [4, §2], we shall ignore this difficulty. In particular, we shall interpret (1.2) as as formula

$$\phi_M(f^y) = \sum_{Q\in\mathcal{F}(M)} \phi_M^{M_Q}(f_{Q,y})$$

for the map ϕ_M. Together with (1.1) this yields

$$\begin{aligned} I_M(\gamma, f^y) - I_M(\gamma, f) &\\ = \sum_{\{Q\in\mathcal{F}(M):Q\neq G\}} & J_M^{M_Q}(\gamma, f_{Q,y}) \\ & - \sum_{L\in\mathcal{L}(M)} \sum_{\{Q\in\mathcal{F}(L):Q\neq G\}} \hat{I}_M^L(\gamma, \phi_L^{M_Q}(f_{Q,y})) \\ = \sum_{\{Q\in\mathcal{F}(M):Q\neq G\}} & \Big(J_M^{M_Q}(\gamma, f_{Q,y}) \\ & - \sum_{\{L\in\mathcal{L}(M):L\subset M_Q\}} \hat{I}_M^L(\gamma, \phi_L^{M_Q}(f_{Q,y}))\Big) . \end{aligned}$$

Applying the induction hypothesis to M_Q, we see that the last expression vanishes. Therefore $I_M(\gamma)$ is invariant.

Since the distributions satisfy the required formula by definition, it remains only to show that $I_M(\gamma)$ is supported on characters. This was done by global means in [5, Theorem 5.1]. We shall later sketch how it can also be established by purely local means. □

We thus obtain a family $\{I_M(\gamma)\}$ of invariant distributions on $\mathcal{H}(G)$ which are parame-trized by Levi subgroups M and G-regular conjugacy classes γ in $M(F)$. These distributions should be regarded as the true generalizations of the orbital integrals $\{I_G(\gamma)\}$. They are important in the theory of automorphic forms, for they are the terms on the geometric side of the global trace formula when the quotient is assumed only to have finite volume. They are also intimately tied up with local harmonic analysis, as we shall presently see. Thus, the distributions are natural objects for which it is appropriate to ask questions as in Problem B. What is the "discrete part" of $\hat{I}_M(\gamma)$? That is, what are the values taken by $\hat{I}_M(\gamma)$ on the discrete components of $\Pi_{\text{temp}}(G(F))$? (Actually, we mean the discrete components of the variety attached to $\Pi_{\text{temp}}(G(F))$ by taking into account the reducibility of induced representations.) What is the singular support of $\hat{I}_M(\gamma)$ on the continuous components? More generally, can one compute $\hat{I}_M(\gamma)$ explicitly, modulo smooth functions on the continuous components?

We should remark that there are twisted versions of the various objects discussed above. One can account for this generalization by taking G to be a connected component of a nonconnected reductive group. The only ingredient that is lacking is a characterization of the analogous space $\mathcal{I}(G)$ when $F = \mathbb{R}$. (The p-adic twisted trace Paley-Wiener theorem has been established by Rogawski [23].)

The third problem we mention is to relate the distributions to the theory of endoscopy.

Problem C.

(a) *If G is a connected, quasi-split group, construct stable invariant distributions $SI_M^G(\gamma)$ on $\mathcal{H}(G)$ from the distributions $I_M^G(\gamma)$.*

(b) *If G is any connected group, or a component of a nonconnected group, establish identities between the distributions $\{I_M^G(\gamma)\}$ and $\{S\hat{I}_{M_H}^H(\gamma_H, f^H)\}$, where H ranges over endoscopic data for G, and $f \to f^H$ is the conjectured Langlands-Shelstad transfer mapping.*

Remark 1. The problem is very difficult. For example, when F is p-adic, the important special case that $M = G$ includes the "fundamental lemma", which is far from being solved. This problem is certainly the most important of the three for automorphic forms. A solution could be combined with the global trace formula to yield a general theory of endoscopy for automorphic forms, and in particular, many reciprocity laws between automorphic representations on different groups.

Remark 2. The endoscopic side of each identity would be a certain finite linear combination of distributions $\{S\hat{I}_{M_H}^H(\gamma_H, f^H)\}$. It should not be difficult to describe the coefficients of each such linear combination explicitly, but this has not been done. We refer the reader to the original article [21], and perhaps also [8, §3], for a discussion of the undefined terms in the statement of the problem.

Remark 3. For groups of general rank, there are only two cases of the problem that have been solved.

(i) G the multiplicative group of a central simple algebra.

(ii) G a connected component of the semi-direct product

$$\mathrm{Res}_{E/F}(GL(n)) \rtimes \mathrm{Gal}(E/F) ,$$

where E/F is a finite cyclic extension.

The solution in each case is contained Theorem A of [11, §II.5], a result which was proved by global methods.

2. The local trace formula

The connection of the distributions $\{I_M(\gamma)\}$ with harmonic analysis is through a local version of the trace formula. The distributions occur on the geometric side, in much the same way that they occur in the global trace formula. We shall describe the local trace formula in this section.

In the next section we shall sketch how it can be used to give information about each of the three problems. The noninvariant version of the local trace formula is proved in the preprint [10]. The details of the invariant version described here, as well as the applications, will be given elsewhere.

The formula begins with a problem suggested by Kazhdan. Consider the regular representation

$$\big(R(u,y)\phi\big)(x) = \phi(u^{-1}xy), \qquad u,y \in G(F), \quad \phi \in L^2\big(G(F)\big),$$

of $G(F) \times G(F)$ on the Hilbert space $L^2\big(G(F)\big)$. Consider also a function in $\mathcal{H}(G \times G)$ of the form

$$f(y_1,y_2) = f_1(y_1)f_2(y_2), \qquad y_i \in G(F), \quad f_i \in \mathcal{H}(G).$$

(From now on, f will denote a function of $G(F) \times G(F)$, rather than on $G(F)$ as before.) Then $R(f)$ is an operator on $L^2\big(G(F)\big)$, which maps a function ϕ to the function

$$\begin{aligned}\big(R(f)\phi\big)(x) &= \int_{G(F)}\int_{G(F)} f_1(u)f_2(y)\phi(u^{-1}xy)\,du\,dy \\ &= \int_{G(F)}\int_{G(F)} f_1(xu)f_2(uy)\phi(y)\,du\,dy \\ &= \int_{G(F)} K(x,y)\phi(y)\,dy,\end{aligned}$$

where

$$K(x,y) = \int_{G(F)} f_1(xu)f_2(uy)\,du.$$

Thus $R(f)$ is an integral operator with smooth kernel $K(x,y)$. Now by the Plancherel formula we know that

$$R = R_{\text{disc}} \oplus R_{\text{cont}},$$

where R_{disc} is a direct sum of square integrable representations of $G(F) \times G(F)$, and R_{cont} is a subrepresentation of R which decomposes continuously. The problem is to find an explicit formula for the trace of $R_{\text{disc}}(f)$.

Suppose for a moment that G is semisimple. On the one hand,

$$\operatorname{tr}\big(R_{\text{disc}}(f)\big) = \sum_{\sigma} \operatorname{tr}\big(\sigma^\vee(f_1)\big)\operatorname{tr}\big(\sigma(f_2)\big),$$

where σ is summed over the discrete series $G(F)$, and $\sigma^\vee$ denotes the contragredient of σ. Since there are only finitely many discrete series that contain a given K-type, the sum can be taken over a finite set. On the other hand, if $K_{\mathrm{cont}}(x,y)$ is the kernel of $R_{\mathrm{cont}}(f)$, $R_{\mathrm{disc}}(f)$ is an integral operator whose kernel is given by the difference of $K(x,y)$ and $K_{\mathrm{cont}}(x,y)$. In particular,

$$\mathrm{tr}\big(R_{\mathrm{disc}}(f)\big) = \int_{G(F)} \big(K(x,x) - K_{\mathrm{cont}}(x,x)\big)dx .$$

The idea is to use the formula above for $K(x,x)$, and the formula for $K_{\mathrm{cont}}(x,y)$ provided by Harish-Chandra's Plancherel theorem, to get a second expression for $\mathrm{tr}\big(R_{\mathrm{disc}}(f)\big)$.

Returning to the case of reductive G, we fix a suitable minimal Levi subgroup M_0 of G. Consider first the formula for $K(x,x)$, which after a change of variables becomes

$$K(x,x) = \int_{G(F)} f_1(u)f_2(x^{-1}ux)du .$$

The Weyl integration formula gives an expansion of this into integrals over conjugacy classes. Let $\Gamma_{ell}\big(G(F)\big)$ be the set of conjugacy classes $\{\gamma\}$ in $G(F)$ such that the centralizer of γ in $G(F)$ is compact modulo $A_G(F)$. Any G-regular conjugacy class in $G(F)$ is the image of a class $\{\gamma\}$ in $\Gamma_{ell}\big(M(F)\big)$ for some Levi subgroup M which contains M_0. The pair $(M,\{\gamma\})$ is uniquely determined only modulo the action of the Weyl group W_0^G of (G, A_{M_0}), so the number of such pairs equals $|W_0^G|\,|W_0^M|^{-1}$. The Weyl integration formula can therefore be interpreted as an expansion

$$\sum_M |W_0^M|\,|W_0^G|^{-1} \int_{\Gamma_{ell}(M(F))} |D(\gamma)| \Big(\int_{A_M(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1)f_2(x^{-1}x_1^{-1}\gamma x_1 x)dx_1\Big)d\gamma \tag{2.1}$$

for $K(x,x)$. The sum here is over the groups $M \in \mathcal{L}(M_0)$. The measure $d\gamma$ is supported on the G-regular classes in $\Gamma_{ell}\big(M(F)\big)$, and is determined in the usual way by a Haar measure on the torus that centralizes γ.

The contribution from $K_{\mathrm{cont}}(x,x)$ can be regarded as a second expansion for $K(x,x)$ in terms of spectral data. Let $\Pi_2\big(G(F)\big)$ be the set of (equivalence classes of) irreducible unitary representations of $G(F)$ which are square integrable modulo $A_G(F)$. We obtain a measure $d\sigma$ on $\Pi_2\big(G(F)\big)$ by transferring a suitable measure on $i\mathfrak{a}_G^*$ by means of the action $\sigma \to \sigma_\Lambda$. (See [16,§2].) Harish-Chandra's Plancherel theorem [15], [16] is easily seen to yield an expansion

$$\sum_M |W_0^M|\,|W_0^G|^{-1} \int_{\Pi_2(M(F))} m(\sigma) \Big(\sum_S \mathrm{tr}\big(\mathcal{I}_P(\sigma,x)S(f)\big)\overline{\mathrm{tr}\big(\mathcal{I}_P(\sigma,x)S\big)}\Big)d\sigma, \tag{2.2}$$

where $m(\sigma)$ is the Plancherel density and

$$S(f) = \mathcal{I}_P(\sigma, f_2) S \mathcal{I}_P(\sigma, f_1^\vee),$$

the function $f_1^\vee$ being defined by $f_1^\vee(x_1) = f_1(x_1^{-1})$. The outer sum is over $M \in \mathcal{L}(M_0)$ as in (2.1), and for a given M, P stands for any group $P(M)$. The inner sum is over $S \in \mathcal{B}_P(\sigma)$, a fixed K-finite, orthonormal basis of the space of Hilbert-Schmidt operators on the underlying space of $\mathcal{I}_P(\sigma)$.

Notice the formal similarity of the two expansions (2.1) and (2.2). The terms with $M = G$ are simplest in each case, and are easily seen to be integrable functions of x in $A_G(F)\backslash G(F)$. Their integrals are equal to

$$\int_{\Gamma_{ell}(G(F))} |D(\gamma)| \Big(\int_{A_G(F)\backslash G(F)} f_1(x_1^{-1}\gamma x_1) dx_1 \int_{A_G(F)\backslash G(F)} f_2(x_2^{-1}\gamma x_2) dx_2 \Big) d\gamma \qquad (2.3)$$

and

$$\int_{\Pi_2(G(F))} \operatorname{tr}\big(\sigma^\vee(f_1)\big) \operatorname{tr}\big(\sigma(f_2)\big) d\sigma \qquad (2.4)$$

respectively. In particular, if G is semisimple, the trace of $R_{\mathrm{disc}}(f)$ equals (2.4), and can consequently be expressed as the sum of (2.3) with a "parabolic term", consisting of the remaining contributions to (2.1) and (2.2). The parabolic term is of course much harder to compute. It equals the integral over $x \in A_G(F)\backslash G(F)$ of the difference of the expressions obtained from (2.1) and (2.2) by taking the sums only over $M \neq G$. None of the terms in (2.1) and (2.2) with $M \neq G$ is integrable. How then can the difference of the expressions yield a function whose integral is computable? The answer lies in a truncation process, which in the end works out surprisingly well.

Let $P_0 \in \mathcal{P}(M_0)$ be fixed minimal parabolic subgroup. The truncation depends on a point T in the chamber $\mathfrak{a}_{P_0}^+$ which is very regular, in the sense that the number

$$d(T) = \max_{\alpha \in \Delta_{P_0}} \alpha(T)$$

is large. According to the polar decomposition, $G(F)$ equals $KM_0(F)K$. Let $u(x,T)$ be the characteristic function of the set of points

$$k_1 m k_2, \qquad m \in A_G(F)\backslash M_0(F), \quad k_1, k_2 \in K,$$

in $A_G(F)\backslash G(F)$ such that $H_{M_0}(m)$ lies in the convex hull of

$$\{sT : \; s \in W_0^G\},$$

taken modulo $\mathfrak{a}_G$. Then $u(x,T)$ is the characteristic function of a large compact subset of $A_G(F)\backslash G(F)$. In particular, the integral

$$K^T(f) = \int_{A_G(F)\backslash G(F)} K(x,x)\, u(x,T) dx$$

converges. We shall outline the three steps by which one obtains an explicit formula from $K^T(f)$.

The first step is to study the geometric and spectral expansions of $K^T(f)$ as functions of T. These are obtained from (2.1) and (2.2) by multiplying each expression with $u(x,T)$, and then integrating over x in $A_G(F)\backslash G(F)$. If $F = \mathbb{R}$, one shows that $K^T(f)$ is asymptotic to a polynomial $p_0(T,f)$ in T as $d(T)$ approaches infinity. If F is p-adic, we take T to be in the lattice

$$\mathfrak{a}_{M_0,F} = H_{M_0}(M_0(F))$$

in $\mathfrak{a}_{M_0}$. In this case it turns out that $K^T(f)$ is asymptotic to a function

$$\sum_{k=0}^{N} p_k(T,f)e^{\zeta_k(T)},$$

where $\zeta_1 = 0$, $\zeta_1, \ldots, \zeta_N$ are distinct points in the the compact torus

$$i\mathfrak{a}^*_{M_0}/Hom(\mathfrak{a}_{M_0,F}, 2\pi i\mathbb{Z}),$$

and each $p_k(T,f)$ is a polynomial in T. In each case, the "constant term"

$$\tilde{J}(f) = p_0(0,f)$$

of $K^T(f)$ is well defined.

The second step is to calculate $\tilde{J}(f)$ explicitly. More precisely, one must evaluate the terms in the geometric and spectral expansions of $\tilde{J}(f)$. The calculations on the spectral side are the more difficult, and were suggested by work of Waldspurger [24], who carried out the process for p-adic spherical functions on $GL(n)$. The contributions to the final formula of the terms with $M = G$ in (2.1) and (2.2) remain as before, the expressions (2.3) and (2.4). However, the contributions from $M \neq G$ are more elaborate. Their principal ingredients are essentially the weighted orbital integrals and weighted characters discussed above. It is of course the identity of the two expansions of $\tilde{J}(f)$ that yields the noninvariant trace formula. The final result is stated and proved in [10, Theorem 12.1].

The third step is to convert the noninvariant formula into an invariant local trace formula. This is a relatively simple matter, which follows the analogous procedure used in the global trace formula [5, §2]. The final result is an identity between two expansions

$$\sum_M |W_0^M|\,|W_0^G|^{-1}\,(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{ell}(M)} I_M(\gamma,f)d\gamma \tag{2.5}$$

and

$$\sum_M |W_0^M|\,|W_0^G|^{-1}\,(-1)^{\dim(A_M/A_G)} \int_{\Pi_{\mathrm{disc}}(M)} a^M_{\mathrm{disc}}(\pi)\,i_M(\pi,f)d\pi\,, \tag{2.6}$$

whose terms we describe as follows.

In the geometric expansion (2.5), $\Gamma_{ell}(M)$ stands for the set of conjugacy classes in $M(F) \times M(F)$ of the form (γ, γ), where γ is a conjugacy class in $\Gamma_{ell}(M(F))$. The integrand $I_M(\gamma, f)$ is the invariant distribution discussed above, but defined for the function f on $G(F) \times G(F)$, and with the weight function taken relative to the diagonal image of $\mathfrak{a}_M$ in $\mathfrak{a}_M \oplus \mathfrak{a}_M$, rather than the full space $\mathfrak{a}_M \oplus \mathfrak{a}_M$. This distribution can in fact be decomposed in terms of the original distributions on $G(F)$. For the splitting formula [4, Proposition 9.1] asserts that

$$I_M(\gamma, f) = \sum_{M_1, M_2 \in \mathcal{L}(M)} d_M^G(M_1, M_2) I_M^{M_1}(\gamma, f_{1,P_1}) I_M^{M_2}(\gamma, f_{2,P_2}), \tag{2.7}$$

where $d_M^G(M_1, M_2)$ is a constant, and

$$f_{P_i}(m) = \delta_{P_i}(m)^{\frac{1}{2}} \int_K \int_{N_{P_i}(F)} f_i(k^{-1} mnk)\, dn\, dk, \qquad m \in M_{P_i}(F),$$

for any group $P_i \in \mathcal{P}(M_i)$. We point out that this constant $d_M^G(M_1, M_2)$ also reflects the geometry of the polytopes $\Pi_M(x)$. It equals 0 unless the canonical map

$$(\mathfrak{a}_{M_1}/\mathfrak{a}_G) \oplus (\mathfrak{a}_{M_2}/\mathfrak{a}_G) \longrightarrow \mathfrak{a}_M/\mathfrak{a}_G$$

between Euclidean spaces in an isomorphism, in which case it is the Jacobian determinant of the map. In other words $d_M^G(M_1, M_2)$ is the volume of the parallelepiped determined by orthonormal bases of the complementary subspaces of $\mathfrak{a}_M/\mathfrak{a}_G$ attached to M_1 and M_2.

The constituents of the spectral expansion (2.6) are defined in terms of the decomposition of a certain distribution I_{disc} into irreducible characters. By definition, $I_{\mathrm{disc}}(f)$ equals

$$\sum_M \sum_s \sum_\sigma |W_0^M| \, |W_0^G|^{-1} | \det(s-1)_{\mathfrak{a}_M^G}|^{-1} \varepsilon_\sigma(s) \mathrm{tr}\big(R(s, \sigma^\vee \otimes \sigma) \mathcal{I}_P(\sigma^\vee \otimes \sigma, f^1)\big),$$

where M is summed over $\mathcal{L}(M_0)$, s is summed over the regular elements

$$\{s \in W(\mathfrak{a}_M) : \det(s-1)_{\mathfrak{a}_M^G} \neq 0\}$$

in the Weyl group of (G, A_M), and σ is summed over the set

$$\{\sigma \in \Pi_2(M(F)) : s\sigma \cong \sigma\} / i\mathfrak{a}_G^*$$

of orbits of $i\mathfrak{a}_G^*$ in $\Pi_2(M(F))$. For each orbit σ, $\mathcal{I}_P(\sigma^\vee \otimes \sigma)$ is a well defined induced representation of the subgroup

$$(G(F) \times G(F))^1 = \{(y_1, y_2) : H_G(y_1) = H_G(y_2)\}$$

of $G(F) \times G(F)$. It can therefore be evaluated at the restriction f^1 of f to $\big(G(F) \times G(F)\big)^1$. The only other constituent of $I_{\mathrm{disc}}(f)$ that requires comment is the function $\varepsilon_\sigma(s)$. By definition, ε_σ is the sign character on the group

$$W_\sigma = W'_\sigma \rtimes R_\sigma = \{s \in W(\mathfrak{a}_M) : s\sigma \cong \sigma\}$$

which is 1 on the R-group R_σ and is the usual sign character on the complementary subgroup W'_σ. Having thus defined $I_{\mathrm{disc}}(f)$, we take $\Pi_{\mathrm{disc}}(G)$ to be a union of orbits of $i\mathfrak{a}^*_G$ in $\Pi_{\mathrm{temp}}\big(G(F) \times G(F)\big)$ and $\{a^G_{\mathrm{disc}}(\pi)\}$ to be a corresponding set of coefficients, such that

$$\begin{aligned} I_{\mathrm{disc}}(f) &= \sum_{\pi \in \Pi_{\mathrm{disc}}(G)/i\mathfrak{a}^*_G} a^G_{\mathrm{disc}}(\pi)\mathrm{tr}\big(\pi(f^1)\big) \\ &= \int_{\Pi_{\mathrm{disc}}(G)} a^G_{\mathrm{disc}}(\pi)\mathrm{tr}\big(\pi(f)\big)d\pi . \end{aligned}$$

This accounts for all the terms in (2.6) except for the distribution $i_M(\pi, f)$. By definition,

$$i_M(\pi, f) = r_M(\pi)\mathrm{tr}\big(\mathcal{I}_P(\pi, f)\big) ,$$

where

$$r_M(\pi) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} r_{Q|\overline{Q}}(\pi_1)^{-1} r_{Q|\overline{Q}}(\pi_{1,\Lambda})\theta_Q(\Lambda)^{-1} ,$$

for any representation $\pi = \pi_1 \otimes \pi_2$ in $\Pi_{\mathrm{disc}}(M)$. Here,

$$r_{Q|\overline{Q}}(\pi_1) = r_{Q|\overline{Q}}(\pi_2)$$

is the local normalizing factor for either of the intertwining operators $R_{Q|\overline{Q}}(\pi_1)$ or
$R_{Q|\overline{Q}}(\pi_2)$. (Since π_1 and π_2 are implicitly constituents of the same induced, tempered representation, it is immaterial whether we take π_1 or π_2.) One shows that the limit $r_M(\pi)$ exists and is a nonsingular function of π.

The local trace formula, then, is the identity of (2.5) with (2.6). To get some feeling for it, one could experiment with the simple case of $G = GL(2)$. Take f_1 and f_2 to be p-adic spherical functions, and consider (2.5) and (2.6) as bilinear forms in the Satake transforms

$$\sum_{\nu \in \mathbb{Z}^2} a_i(\nu) z^\nu, \qquad z \in (\mathbb{C}^*)^2, \ \ i = 1, 2,$$

of f_1 and f_2. The spectral side (2.6) consists of two innocuous terms, but the geometric side (2.5) is more interesting. The term with $M = G$ in (2.5) is a bilinear form in the elliptic orbital integrals of f_1 and f_2, while the term with $M = M_0$ reduces to an expression involving weighted orbital integrals. In [20, §5], the orbital integrals and weighted orbital integrals on $GL(2)$ were computed explicitly in terms of Satake transforms. The identity of (2.5) and (2.6) provides a new relationship between these objects.

3. Some applications

The local trace formula is capable of yielding nontrivial information on local harmonic analysis, although it is not clear at this point how far it will lead. We shall conclude by describing three applications to the three general problems of §1. These applications are only modest advances on what is presently known, but they give some idea of how the local trace formula can be used.

We begin by sketching a local proof of the following result, which completes the induction argument of §1.

Theorem A. *The distributions*

$$I_M(\gamma_1, f_1), \qquad \gamma_1 \in M(F) \cap G_{\mathrm{reg}}(F), \quad f_1 \in \mathcal{H}(G),$$

are supported on characters.

Proof. Let $f_1 \in \mathcal{H}(G)$ be a function such that $\mathrm{tr}\big(\pi_1(f_1)\big) = 0$ for every $\pi_1 \in \Pi_{\mathrm{temp}}\big(G(F)\big)$. We must show that $I_M(\gamma_1, f_1)$ vanishes for every $M \in \mathcal{L}(M_0)$ and $\gamma_1 \in M(F) \cap G_{\mathrm{reg}}(F)$. We have been carrying the induction hypothesis of §1, which asserts that the theorem holds if G is replaced by a proper Levi subgroup. Combined with a descent formula [4, Corollary 8.3], it tells us that $I_M(\gamma_1, f_1) = 0$ if γ_1 belongs to a proper Levi subgroup of M. We may therefore assume that γ_1 belongs to $\Gamma_{ell}\big(M(F)\big)$.

Apply the local trace formula, with f_1 as given and f_2 an arbitrary function in $\mathcal{H}(G)$. The spectral side (2.6) vanishes, in view of the condition on f_1. On the geometric side (2.5), we use the splitting formula (2.7) for $I_M(\gamma, f)$. Our induction hypothesis insures the vanishing of all terms in (2.7) with $M_1 \neq G$. Since $d_M^G(G, M_2)$ equals 0 unless $M_2 = M$, we obtain

$$I_M(\gamma, f) \;=\; I_M(\gamma, f_1) I_M^M(\gamma, f_{2,P}) \,, \qquad P \in \mathcal{P}(M).$$

The geometric side therefore equals

$$\sum_M |W_0^M| \, |W_0^G|^{-1} \, (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{ell}(M(F))} I_M(\gamma, f_1) I_M^M(\gamma, f_{2,P}) d\gamma \,.$$

Now fix M and $\gamma_1 \in M_{ell}(F)$. Choose f_2 to be supported on the set of $G(F)$-conjugates of $\Gamma_{ell}\big(M(F)\big)$, and such that, as a function of $\gamma \in \Gamma_{ell}\big(M(F)\big)$, $I_M^M(\gamma, f_{2,P})$ approaches the sum of Dirac measures at the $W(\mathfrak{a}_M)$-translates of γ_1. The geometric side then approaches

$$(-1)^{\dim(A_M/A_G)} \, I_M(\gamma_1, f_1) \,.$$

It follows that $I_M(\gamma_1, f_1) = 0$, as required. $\square$

The theorem was proved by global means in [5, Theorem 5.1]. The idea of using the global trace formula goes back to Kazhdan, who established [18] the special case that $M = G$. The local argument we have given here applies equally well to the twisted case, in which G is an arbitrary component. In particular, it could be used to prove that twisted p-adic orbital integrals are supported on (twisted) characters. This has been proved by global means in [19].

The next application concerns the "discrete part" of $\hat{I}_M(\gamma_1)$. Assume for simplicity that G is semisimple and that $\pi_1 \in \Pi_2(G(F))$ is a fixed square integrable representation. Let $f_1 \in \mathcal{H}(G)$ be a pseudo-coefficient for π_1, in the sense that

$$\mathrm{tr}\big(\pi_1'(f_1)\big) = \begin{cases} 1, & \text{if } \pi_1' \cong \pi_1^\vee \\ 0, & \text{otherwise.} \end{cases}$$

Theorem B. *Suppose that $\gamma_1 \in \Gamma_{ell}(M(F))$ is G-regular. Then*

$$I_M(\gamma_1, f_1) = (-1)^{\dim(A_M/A_G)} \, |D(\gamma_1)|^{\frac{1}{2}} \, \Theta_{\pi_1}(\gamma_1),$$

where Θ_{π_1} is the character of π_1.

Proof. This result was discussed in [9, §9], so we shall be brief. Choose a function $f_2 \in \mathcal{H}(G)$, supported on the set of $G(F)$-conjugates of $\Gamma_{ell}(M(F))$, such that, as a function of $\gamma \in \Gamma_{ell}(M(F))$, $|D(\gamma_1)|^{\frac{1}{2}} I_M^M(\gamma, f_{2,P})$ approaches the sum of Dirac measures at the $W(\mathfrak{a}_M)$-translates of γ_1. One checks that $\mathrm{tr}\big(\pi_1(f_2)\big)$ approaches $\Theta_{\pi_1}(\gamma_1)$. The theorem then follows from the identity of (7.5) and (7.6), and the splitting formula (7.7). □

Theorem B is already known if $F = \mathbb{R}$ [7, Theorem 6.4], or if π_1 is supercuspidal [3]. It is new if F is p-adic and π_1 is not supercuspidal.

The final application is a small contribution to the third problem which is motivated by a comparison of trace formulas. The theory of endoscopy is likely to lead to a parallel family of distributions

$$I_M^{\mathcal{E}}(\gamma_1, f_1) = I_M^{G,\mathcal{E}}(\gamma_1, f_1), \quad M \in \mathcal{L}(M_0), \ \gamma_1 \in \Gamma_{ell}(M(F)), \ f_1 \in \mathcal{H}(G),$$

obtained from stable distributions $\{S\hat{I}_{M_H}(\gamma_{1,H}, f_1^H)\}$ on endoscopic groups. We would expect that the distributions

$$I_M^{\mathcal{E}}(\gamma, f) = \sum_{M_1, M_2 \in \mathcal{L}(M)} d_M^G(M_1, M_2) I_M^{M_1,\mathcal{E}}(\gamma, f_{1,P_1}) I_M^{M_2,\mathcal{E}}(\gamma, f_{2,P_2}), \quad (2.7)^{\mathcal{E}}$$

defined for $\gamma \in \Gamma_{ell}(M(F))$ and $f \in \mathcal{H}(G \times G)$ by the analogue of (2.7), should satisfy their own version of the local trace formula. That is, the geometric expansion

$$\sum_M |W_0^M| \, |W_0^G|^{-1} \, (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{ell}(M)} I_M^{\mathcal{E}}(\gamma, f) d\gamma \qquad (2.5)^{\mathcal{E}}$$

equals some form of spectral expansion (2.6). This is what happens in the comparison of global trace formulas that is required for base change for $GL(n)$ [11, Chapter II]. In general, one would like to show that the two families are in fact the same. In the special case of $GL(n)$ (and with F replaced by a certain finite product of local fields), it is shown that the two distributions differ by a multiple of the invariant orbital integral [11, (II.17.5)]. An argument that is special to $GL(n)$ [11, p. 195-196] then establishes that this multiple is actually 0. The local trace formula can be applied to this point in the general situation.

Theorem C. *Suppose that we are given a family $\{I_M^{\mathcal{E}}(\gamma_1, f_1)\}$ of distributions with the property that $(2.5)^{\mathcal{E}}$ equals (2.6). Assume also that there are functions*

$$c_M^L(\gamma_1), \qquad L \in \mathcal{L}(M), \ \ \gamma_1 \in \Gamma_{ell}(M(F)),$$

with $c_M^M(\gamma_1) = 0$, such that

$$I_M^{L,\mathcal{E}}(\gamma_1, g_1) - I_M^L(\gamma_1, g_1) \ = \ c_M^L(\gamma_1) I_M^M(\gamma_1, g_{1,P})$$

for all M, L, γ_1, and all $g_1 \in \mathcal{H}(L)$. Then $c_M^L(\gamma_1) = 0$ for all $L \in \mathcal{L}(M)$. In other words, the two families of distributions are the same.

Proof. Assume inductively that $c_M^L(\gamma_1) = 0$ whenever $L \neq G$. The expressions (2.5) and $(2.5)^{\mathcal{E}}$ are both equal to (2.6), so they are equal to each other. Therefore

$$\sum_M |W_0^M| \, |W_0^G|^{-1} \, (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{ell}(M(F))} \big(I_M^{\mathcal{E}}(\gamma, f) - I_M(\gamma, f)\big) d\gamma \ = 0 \, .$$

Combining the splitting formulas (2.7) and $(2.7)^{\mathcal{E}}$ with the induction assumption, we obtain

$$I_M^{\mathcal{E}}(\gamma f) - I_M(\gamma, f) \ = \ 2 c_M^G(\gamma) I_M^M(\gamma, f_{1,P}) I_M^M(\gamma, f_{2,P}) \, .$$

Since f_1 and f_2 are arbitrary functions in $\mathcal{H}(G)$, we see without difficulty that $c_M^G(\gamma_1) = 0$ for every $\gamma_1 \in \Gamma_{ell}(M(F))$. □

Something akin to Theorem C might be required as a replacement for the argument in [11, p. 195-196], if the techniques of [11] are to be extended to general groups. One can also use the theorem, or rather its version for G a component in the nonconnected group $\mathrm{Res}_{E/F}(GL(n)) \rtimes \mathrm{Gal}(E/F)$, to strengthen one of the peripheral results of [11]. For in this case it was possible only to determine the distributions $I_M^{\mathcal{E}}(\gamma_1, f_1)$ as a linear combination of distributions $I_M^{M_1}(\gamma_1, f_{1,P_1})$ [11, Theorem 6.1]. If one combines Theorem C with the results obtained in [11] by global methods, one can prove that the distributions $I_M^{\mathcal{E}}(\gamma_1)$ and $I_M(\gamma_1)$ in [11, Theorem 6.1] are actually equal.

The local trace formula has other possible applications. For example, it seems to be a natural tool for investigating questions posed in §1 on the singular support of $\hat{I}_M(\gamma)$. However, I think that it will be necessary to work with the Schwartz space rather than the Hecke algebra.

Problem D. *Show that identity of expansions (2.5) and (2.6) remains valid if f_1 and f_2 are Schwartz functions on $G(F)$.*

We are of course especially interested in the p-adic case. The tool for handling p-adic orbital integrals of Schwartz functions is the Howe conjecture, which has been proved by Clozel [13]. Recall that if

$$\mathcal{H}(G(F)/\!\!/K_0) = C_c^\infty(K_0\backslash G(F)/K_0)$$

is the Hecke algebra of bi-invariant functions under an open compact subgroup K_0 of $G(F)$, the Howe conjecture asserts that the vector space of linear functionals on $\mathcal{H}(G(F)/\!\!/K_0)$, obtained by restricting all invariant distributions with support on the $G(F)$-conjugates of a given compact set, is finite dimensional. In particular, if ω is a compact subset of $G(F)$, the space

$$\{I_G(\gamma_1, f_1) : \gamma_1 \in \omega \cap G_{\mathrm{reg}}(F), f \in \mathcal{H}(G(F)/\!\!/K_0)\},$$

of linear functionals on $\mathcal{H}(G(F)/\!\!/K_0)$, is finite dimensional. However, if $M \neq G$, the invariant distribution $I_M(\gamma_1, f_1)$ is not supported on the $G(F)$-conjugates of a compact set. One would need the following analogue of the Howe conjecture.

Problem E. *Suppose that K_0 is an open compact subgroup of $G(F)$ and that ω is a compact subset of $M(F)$. Show that the space*

$$\{I_M(\gamma_1, f_1) : \gamma_1 \in \omega \cap G_{\mathrm{reg}}(F), f \in \mathcal{H}(G(F)/\!\!/K_0)\}$$

of linear functionals on $\mathcal{H}(G(F)/\!\!/K_0)$ is finite dimensional.

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