THE TRACE FORMULA FOR REDUCTIVE GROUPS*

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This paper is a report on the present state of the trace formula for a general reductive group. The trace formula is not so much an end in itself as it is a key to deep results on automorphic representations. However, such applications have only been carried out for groups of low dimension ([5], [7], [8(c)], [3]). We will not try to discuss them here. For reports on progress towards applying the trace formula for general groups, see the papers of Langlands [8(d)] and Shelstad [11] in these proceedings.

Our discussion will be brief and largely confined to a description of the main results. On occasion we will try to give some idea of the proofs, but more often we shall simply refer the reader to papers in the bibliography. Section 1 will be especially sparse, for it contains a review of results which were summarized in more detail in [1(e)].

This report contains no mention of the twisted trace formula. Such a formula is not available at the present, although I do not think that its proof will require any essentially new ideas. Twisted

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trace formulas for special groups were proved in [8(c)] and [3]. As for the untwisted trace formula, we draw attention to Selberg's original papers [10(a)], [10(b)], and also point out, in addition to the papers cited in the text, the articles [5], [1(a)], [4], [12], [9(a)] and [9(b)].

1. THE TRACE FORMULA - FIRST VERSION.

Le G be a reductive algebraic group defined over $\mathbb{Q}$. Let A_G be the split component of the center of G, and set

$$\mathfrak{a}_G = \mathrm{Hom}(X(G)_{\mathbb{Q}}, \mathbb{R}) \quad ,$$

where $X(G)_{\mathbb{Q}}$ is the group of characters of G defined over $\mathbb{Q}$. Then $\mathfrak{a}_G$ is a real vector space whose dimension equals that of A_G. Let $G(\mathbb{A})^1$ be the kernel of the map

$$H_G : G(\mathbb{A}) \to \mathfrak{a}_G \; ,$$

which is defined by

$$e^{\langle H_G(x),\xi\rangle} = |\xi(x)|, \qquad x \in G(\mathbb{A}),\ \xi \in X(G)_{\mathbb{Q}} \; .$$

Then $G(\mathbb{Q})$ embeds diagonally as a discrete subgroup of $G(\mathbb{A})^1$, and the coset space $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ has finite invariant volume. We are interested in the regular representation R of $G(\mathbb{A})^1$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. If $f \in C_c^\infty(G(\mathbb{A})^1)$, $R(f)$ is an integral operator on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. The source of the trace formula is the circumstance that there are two ways to express the integral kernel of $R(f)$.

We shall state the trace formula in its roughest form, recalling briefly how each side is obtained from an expression for the integral kernel. This version of the trace formula depends on a fixed minimal parabolic subgroup P_0, with Levi component M_0 and unipotent radical N_0, and also on an appropriate maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})^1$. In addition, it depends on a point T in

$$\mathfrak{a}_0 = \mathfrak{a}_{M_0}$$

which is suitably regular with respect to P_0, in the sense that $\alpha(T)$ is large for each root α of P_0 on $A_0 = A_{M_0}$. The trace formula is then an identity

$$\text{(1.1)} \qquad \sum_{\mathfrak{o}\in\mathcal{O}} J^T(f) = \sum_{\chi\in\mathcal{X}} J^T_\chi(f), \qquad f \in C_c^\infty\big(G(\mathbb{A})^1\big).$$

We describe the left hand side first. $\mathcal{O}$ denotes the set of equivalence classes in $G(\mathbb{Q})$, in which two elements in $G(\mathbb{Q})$ are deemed equivalent if their semisimple components are $G(\mathbb{Q})$ conjugate. This relation is just $G(\mathbb{Q})$ conjugacy if G is anisotropic, but it is weaker than conjugacy for general G. If P is a parabolic subgroup of G which is standard with respect to P_0, and $\mathfrak{o} \in \mathcal{O}$, set

$$K_{P,\mathfrak{o}}(x,y) = \int_{N_P(\mathbb{A})} \sum_{\gamma\in\mathfrak{o}\cap M_P(\mathbb{Q})} f(x^{-1}\gamma ny)\,dn, \qquad x, y \in G(\mathbb{A})^1,$$

where N_P is the unipotent radical of P and M_P is the unique Levi component of P which contains M_0. Then

$$\sum_{\mathfrak{o}\in\mathcal{O}} K_{P,\mathfrak{o}}(x,y)$$

is the integral kernel of the operator $R_P(f)$ obtained by convolving f on $L^2\big(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1\big)$. If $P = G$, it is just the integral kernel of $R(f)$. Define

$$k_{\mathfrak{o}}^T(x,f) = \sum_{P \supset P_0} (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\mathfrak{o}}(\delta x, \delta x)\hat{\tau}_P\big(H_P(\delta x) - T\big).$$

The function $\hat{\tau}_P\big(H_P(\cdot) - T\big)$, whose definition we will recall in a moment, equals 1 if $P = G$, and vanishes on a large compact neighborhood of 1 in $G(\mathbb{A})^1$ if $P \neq G$. In other words, $k_{\mathfrak{o}}^T(x,f)$ is obtained by modifying $K_{G,\mathfrak{o}}(x,y)$ in some neighborhood of infinity in $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. The function $k_{\mathfrak{o}}^T(x,f)$ turns out to be integrable, and the distributions on the left of the trace formula are defined by

$$J_{\mathfrak{o}}^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_{\mathfrak{o}}^T(x,f)\,dx .$$

If P is a standard parabolic subgroup, $\hat{\tau}_P$ is the characteristic function of a certain chamber. Write $\mathfrak{a}_P = \mathfrak{a}_{M_P}$ and $A_P = A_{M_P}$. If Q is a parabolic subgroup that contains P, there is a natural map from $\mathfrak{a}_P$ onto $\mathfrak{a}_Q$. We shall write

$$\mathfrak{a}_P^Q = \mathfrak{a}_{M_P}^{M_Q} \subset \mathfrak{a}_P$$

for its kernel. Let Δ_P denote the set of simple roots of (P, A_P). It is naturally embedded in the dual space

$$\mathfrak{a}_P^* = X(M_P)_{\mathbb{Q}} \otimes \mathbb{R}$$

of $\mathfrak{a}_P$. There is associated to each $\alpha \in \Delta_P$ a "co-root" $\alpha^\vee$ in $\mathfrak{a}_P^G$. Let $\hat{\Delta}_P$ be the basis of $\mathfrak{a}_P^*/\mathfrak{a}_G^*$ which is dual to $\{\alpha^\vee : \alpha \in \Delta_P\}$. Then $\hat{\tau}_P$ is the characteristic function of

$$\{H \in \mathfrak{a}_P : \varpi(H) > 0,\ \varpi \in \hat{\Delta}_P\} .$$

Let H_P be the continuous function from $G(\mathbb{A})$ to $\mathfrak{a}_P$ defined by

$$H_P(nmk) = H_{M_P}(m), \qquad n \in N_P(\mathbb{A}),\ m \in M_P(\mathbb{A}),\ k \in K .$$

In the formula for $k_{\mathfrak{o}}^T(x,f)$ above, the point T belongs to $\mathfrak{a}_0$, but it projects naturally onto a point in $\mathfrak{a}_P$. It is in this sense that

$$\hat{\tau}_P\bigl(H_P(\delta x) - T\bigr)$$

is defined.

The set X which appears on the right hand side of (1.1) is best motivated by looking at $\mathcal{O}$. If $\mathfrak{o} \in \mathcal{O}$, consider those standard parabolic subgroups B which are minimal with respect to the property that $\mathfrak{o}$ meets M_B. Then $\mathfrak{o} \cap M_B$ is a finite union of $M_B(\mathbb{Q})$ conjugacy classes which are elliptic, in the sense that they meet no proper parabolic subgroup of M_B which is defined over $\mathbb{Q}$. Let W_0 be the restricted Weyl group of

(G, A_0). It is clear that $\mathcal{O}$ is in bijective correspondence with the set of W_0-orbits of pairs (M_B, c_B), where B is a standard parabolic subgroup of G, and c_B is an elliptic conjugacy class in $M_B(\mathbb{Q})$. If we think of the automorphic representations of $G(\mathbb{A})^1$ as being dual in some sense to the conjugacy classes in $G(\mathbb{Q})$, we can imagine that the *cuspidal* automorphic representations might correspond to elliptic conjugacy classes. $\mathcal{X}$ is defined to be the set of W_0-orbits of pairs (M_B, r_B), where B is a standard parabolic subgroup of G and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})^1$. For a given $\chi \in \mathcal{X}$, let $\mathcal{P}_\chi$ be the set of groups B obtained in this way. It is an associated class of standard parabolic subgroups. Similarly, we have an associated class $\mathcal{P}_\mathfrak{o}$ for any $\mathfrak{o} \in \mathcal{O}$.

Suppose that $\chi \in \mathcal{X}$ and $P \supset P_0$ are given. Let $L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1)_\chi$ be the space of functions ϕ in $L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1)$ with the following property. For every standard parabolic subgroup B, with $B \subset P$, and almost all $x \in G(\mathbb{A})^1$, the projection of the function

$$\phi_{B,x}(m) = \int_{N_B(\mathbb{Q})\backslash N_B(\mathbb{A})} \phi(nmx)\,dn, \qquad m \in M_B(\mathbb{A})^1,$$

onto the space of cusp forms in $L^2(M_B(\mathbb{Q})\backslash M_B(\mathbb{A})^1)$ transforms under $M_B(\mathbb{A})^1$ as a sum of representations r_B, in which (M_B, r_B) is a pair in χ. If there is no such pair in χ, $\phi_{B,x}$ will be orthogonal to the space of cusp forms on $M_B(\mathbb{Q})\backslash M_B(\mathbb{A})^1$. It follows from a basic result in Eisenstein series that $L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1)_\chi$ will be zero unless there is a group in P_χ which is contained in P. Moreover, there is an orthogonal decomposition

$$L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1) = \bigoplus_{\chi\in X} L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1)_\chi .$$

Let $K_{P,\chi}(x,y)$ be the integral kernel of the restriction of $R_P(f)$ to $L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1)_\chi$. One can write down a formula for $K_{P,\chi}(x,y)$ in terms of Eisenstein series. We have

$$\sum_{\mathfrak{o}\in\mathcal{O}} K_{P,\mathfrak{o}}(x,y) = \sum_{\chi\in X} K_{P,\chi}(x,y),$$

each side being equal to the integral kernel of $R_P(f)$. If we define the modified functions

$$k_\chi^T(x,f) = \sum_{P\supset P_0} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\chi}(\delta x, \delta x)\hat{\tau}_P(H_P(\delta x) - T),$$

we immediately obtain an identity

$$\sum_{\mathfrak{o}\in\mathcal{O}} k_{\mathfrak{o}}^T(x,f) = \sum_{\chi\in X} k_\chi^T(x,f).$$

It turns out that the functions $k_\chi^T(x,f)$ are also integrable. In fact, the sums on each side of the identity are absolutely integrable. The distributions on the right of the trace formula are defined by

$$J_\chi^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_\chi^T(x,f)dx .$$

The trace formula (1.1) follows.

Most of the work in proving (1.1) comes in establishing the integrability of $k_{\mathfrak{o}}^T(x,f)$ and $k_{\chi}^T(x,f)$. Of these, the second function is the harder one to handle. To prove its integrability it is necessary to introduce a truncation operator on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. Given T as above, the truncation of a continuous function h on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is the function

$$(\Lambda^T h)(x) = \sum_{P\supset P_0} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} h(n\delta x)\hat{\tau}_P\big(H_P(\delta x) - T\big)dn .$$

If $\chi \in X$, let $\Lambda_1^T\Lambda_2^T\ K_\chi(x,y)$ be the function obtained by truncating the function

$$K_{G,\chi}(x,y) = K_\chi(x,y), \qquad\qquad x,\ y \in G(\mathbb{Q})\backslash G(\mathbb{A})^1 ,$$

in each variable separately. From properties of the truncation operator, one shows that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\chi\in X} |\Lambda_1^T\Lambda_2^T\ K_\chi(x,x)|dx$$

is finite (see [1(d), §1]). One can also show, with some effort, that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\chi\in X} |\Lambda_1^T\Lambda_2^T\ K_\chi(x,x) - k_\chi^T(x,f)|dx \quad < \infty ,$$

([1(d), §2]), from which one immediately concludes that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\chi\in X} |k_\chi^T(x,f)|dx$$

is also finite ([1(d), Theorem 2.1]). This was the result that was required for formula (1.1). In the process, one shows that for any $\chi \in X$, the integral of

$$\Lambda_1^T\Lambda_2^T K_\chi(x,x) - k_\chi^T(x,f)$$

is zero for sufficiently regular T ([1(d), Lemma 2.4]). In other words,

$$J_\chi^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda_1^T\Lambda_2^T K_\chi(x,x)dx . \tag{1.2}$$

This second formula for $J_\chi^T(f)$ is an important bonus. We shall see that it is the starting point for obtaining a more explicit formula for $J_\chi^T(f)$.

2. SOME REMARKS.

It is natural to ask how the terms in the trace formula (1.1) depend on the point T. It is shown in Proposition 2.3 of [1(f)] that the distributions $J_{\mathfrak{o}}^T(f)$ and $J_\chi^T(f)$ are polynomial functions of T, and so can be defined for all points T in $\mathfrak{a}_0$. There turns out to be a natural point T_0 in $\mathfrak{a}_0$ such that the distributions

$$J_{\mathfrak{o}}(f) = J_{\mathfrak{o}}^{T_0}(f), \qquad \mathfrak{o} \in \mathcal{O},$$

and

$$J_\chi(f) = J_\chi^{T_0}(f) , \qquad \chi \in X ,$$

are independent of the minimal parabolic subgroup P_0. For a better version of the trace formula, we can set $T = T_0$ to obtain

$$(2.1) \qquad \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in X} J_\chi(f)$$

(incidentally, T_0 is strongly dependent on the maximal compact subgroup K. For example, if $G = GL_n$ and M_0 is the group of diagonal matrices, T_0 will equal zero if K is the standard maximal compact subgroup of $GL_n(\mathbb{A})$. However, if K is a conjugate of this group by $M_0(\mathbb{A})$, T_0 might not be zero). The distributions $J_{\mathfrak{o}}$ and J_χ will still depend on M_0 and K. Moreover, they are not invariant. There are in fact simple formulas to measure how much they fail to be invariant.

Let $\mathcal{L}(M_0)$ be the set of subgroups of G, defined over $\mathbb{Q}$, which contain M_0 and are Levi components of parabolic subgroups of G. Suppose that $M \in \mathcal{L}(M_0)$. Let $\mathcal{L}(M)$ be the set of groups in $\mathcal{L}(M_0)$ which contain M. Let $\mathcal{F}(M)$ be the set of parabolic subgroups of G, now no longer standard, which are defined over $\mathbb{Q}$ and contain M. Then if $Q \in \mathcal{F}(M)$, M_Q also contains M. Let $\mathcal{P}(M)$ be the set of groups Q in $\mathcal{F}(M)$ such that M_Q equals M. Suppose that $L \in \mathcal{L}(M)$. We write $\mathcal{L}^L(M)$, $\mathcal{F}^L(M)$ and $\mathcal{P}^L(M)$ for the analogues of the sets $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$ when G is replaced by L. Now suppose that $\mathfrak{o}$ is a class in $\mathcal{O}$. Then $\mathfrak{o} \cap L(\mathbb{Q})$ is a disjoint union

$$\mathfrak{o}_1 \cup \ldots \cup \mathfrak{o}_n$$

of equivalence classes in the set, $\mathcal{O}_L$, associated to L. We can certainly define the distributions $J^L_{\mathfrak{o}_i}$ on $L(\mathbb{A})^1$. Set

$$J^L_{\mathfrak{o}} = \sum_{i=1}^{n} J^L_{\mathfrak{o}_i}.$$

By definition, $J^L_{\mathfrak{o}}$ is zero unless $\mathfrak{o}$ meets $L(\mathbb{Q})$. If $\chi \bullet \mathcal{X}$, we can define a distribution J^L_χ on $L(\mathbb{A})^1$ in a similar way. Formula (2.1), applied to L, yields

$$\sum_{\mathfrak{o}\in\mathcal{O}} J^L_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} J^L_\chi(f), \tag{2.2}$$

where now f is any function in $C_c^\infty(L(\mathbb{A})^1)$.

The formulas which measure the noninvariance of our distributions depend on a certain family of smooth functions

$$u'_Q(k,y), \qquad k \in K \cap L(\mathbb{A}), \quad y \in L(\mathbb{A}),$$

indexed by the groups $L \in \mathcal{L}(M_0)$ and $Q \in F^L(M_0)$. We will not define these functions here. They are used to define a continuous map

$$f \to f_{Q,y}, \qquad f \in C_c^\infty(L(\mathbb{A})^1),$$

from $C_c^\infty(L(\mathbb{A})^1)$ to $C_c^\infty(M_Q(\mathbb{A})^1)$ for each $y \in L(\mathbb{A})^1$. If $m \in M(\mathbb{A})^1$, $f_{Q,y}(m)$ is given by

$$\delta_Q(m)^{\frac{1}{2}} \int_K \int_{N_Q(\mathbb{A})} f(k^{-1}mnk)\ u'_Q(k,y)dndk ,$$

where δ_Q is the modular function of $Q(\mathbb{A})$. Then the formulas alluded to are

$$J^L_{\mathfrak{o}}(f^y) = \sum_{Q \in \mathcal{F}^L(M_0)} |W_0^{M_Q}||W_0^L|^{-1} J^{M_Q}_{\mathfrak{o}}(f_{Q,y}), \tag{2.3}$$

and

$$J^L_{\chi}(f^y) = \sum_{Q \in \mathcal{F}^L(M_0)} |W_0^{M_Q}||W_0^L|^{-1} J^{M_Q}_{\chi}(f_{Q,y}), \tag{2.3*}$$

where $\mathfrak{o} \in \mathcal{O}$, $\chi \in \mathcal{X}$, $f \in C_c^\infty(L(\mathbb{A})^1)$ and $y \in L(\mathbb{A})^1$. (See [1(f), Theorem 3.2].) Here $|W_0^{M_Q}|$ stands for the number of elements in the Weyl group of (M_Q, A_0). As usual, f^y is the function

$$f(yxy^{-1}) , \qquad x \in L(A)^1$$

If $Q = L$, $f_{Q,y}$ will equal f by definition. We therefore obtain a formula for the value of each distribution at $f^y - f$ as a sum of terms indexed by the groups $Q \in \mathcal{F}^L(M_0)$, with $Q \neq L$. The distribution will be invariant if and only if for each f and y, the sum vanishes. For example, $J^L_{\mathfrak{o}}$ will be invariant if and only if $\mathfrak{o} \cap M(\mathbb{Q})$ is empty for each group $M \in \mathcal{F}^L(M_0)$

with $M \neq L$.

3. THE TRACE FORMULA IN INVARIANT FORM

There is a natural way to modify the distributions $J_{\mathfrak{o}}$ and J_χ so that they are invariant. This was done in the paper [1(f)], under some natural hypotheses on the harmonic analysis of the local groups $G(\mathbb{Q}_v)$. We shall give a brief discussion of this construction.

If H is a locally compact group, let $\Pi(H)$ denote the set of equivalence classes of irreducible unitary representations of H. Suppose that M is any group in $\mathcal{L}(M_0)$. We shall agree to embed $\Pi\big(M(\mathbb{A})^1\big)$ in $\Pi\big(M(\mathbb{A})\big)$; for $M(\mathbb{A})$ is the direct product of $M(\mathbb{A})^1$ and $A_M(\mathbb{R})^0$, so there is a bijection between $\Pi\big(M(\mathbb{A})^1\big)$ and the representations in $\Pi\big(M(\mathbb{A})\big)$ which are trivial on $A_M(\mathbb{R})^0$. Let $\Pi_{\text{temp}}\big(M(\mathbb{A})^1\big)$ be set of *tempered* representations in $\Pi\big(M(\mathbb{A})^1\big)$. From Harish-Chandra's work we know that there is a natural definition for the Schwartz space, $\mathcal{C}\big(M(\mathbb{A})^1\big)$, of functions on $M(\mathbb{A})^1$. There is also a linear map $\mathcal{T}_M$ from $\mathcal{C}\big(M(\mathbb{A})^1\big)$ to the space of complex valued functions on $\Pi_{\text{temp}}\big(M(\mathbb{A})^1\big)$, given by

$$(\mathcal{T}_M f)(\pi) = \operatorname{tr} \pi(f), \qquad f \in \mathcal{C}\big(M(\mathbb{A})^1\big),\ \pi \in \Pi_{\text{temp}}\big(M(\mathbb{A})^1\big).$$

In [1(f), §5] we proposed a candidate, $\mathcal{I}\big(M(\mathbb{A})^1\big)$, for the image of $\mathcal{T}_M$. Roughly speaking, $\mathcal{I}\big(M(\mathbb{A})^1\big)$ is defined to be the space of complex valued functions on $\Pi_{\text{temp}}\big(M(\mathbb{A})^1\big)$ which are Schwartz functions in all possible parameters. It is easy to show that $\mathcal{T}_M$ maps $\mathcal{C}(M(\mathbb{A})^1)$ continuously into $\mathcal{I}(M(\mathbb{A})^1)$. It is also easy to see that the transpose $\mathcal{T}_M'$ of $\mathcal{T}_M$ maps $\mathcal{I}(M(\mathbb{A})^1)'$, the dual space of $\mathcal{I}(M(\mathbb{A})^1)$, *into* the space of tempered invariant distributions on $M(\mathbb{A})^1$.

<u>HYPOTHESIS 3.1</u>: For each $M \in \mathcal{L}(M_0)$, $\mathcal{T}_M$ maps $\mathcal{C}(M(\mathbb{A})^1)$ *onto* $\mathcal{I}(M(\mathbb{A})^1)$. Moreover, the image of the transpose,

$$\mathcal{T}_M' : \mathcal{I}(M(\mathbb{A})^1)' \to \mathcal{C}(M(\mathbb{A})^1)',$$

is the space of *all* tempered invariant distributions on $M(\mathbb{A})^1$.

This hypothesis will be in force for the rest of §3. If I is any tempered invariant distribution on $M(\mathbb{A})^1$, we will let $\hat{I}$ be the unique element in $\mathcal{I}(M(\mathbb{A})^1)'$ such that $\mathcal{T}_M'(\hat{I}) = I$.

Important examples of tempered invariant distributions are the orbital integrals. Suppose that S is a finite set of valuations on $\mathbb{Q}$, and that for each v in S, T_v is a maximal torus of M defined over $\mathbb{Q}_v$. Set

$$T_S^1 = \Big(\prod_{v\in S} T_v(\mathbb{Q}_v)\Big) \cap M(\mathbb{A})^1,$$

and let $T_{S,\mathrm{reg}}^1$ be the set of elements γ in T_S^1 whose centralizer in

$$M_S^1 = \Big(\prod_{v\in S} M(\mathbb{Q}_v)\Big) \cap M(\mathbb{A})^1$$

equals T_S^1. Given $f \in C(M(\mathbb{A})^1)$ and $\gamma \in T^1_{S,\mathrm{reg}}$, the orbital integral can be defined by

$$I_\gamma(f) = |D^M(\gamma)|^{\frac{1}{2}} \int_{T_S^1\backslash M_S^1} f(x^{-1}\gamma x)\,dx\,.$$

($|D^M(\gamma)|^{\frac{1}{2}}$ is the function on T_S^1 which is usually put in as a normalizing factor.) I_γ is a tempered invariant distribution on $C(M(\mathbb{A})^1)$. By Hypothesis 3.1 it corresponds to a distribution $\hat{I}_\gamma$ on $\mathcal{I}(M(\mathbb{A})^1)$. Now suppose that f has compact support. Then the map

$$\gamma \to I_\gamma(f), \qquad \gamma \in T^1_{S,\mathrm{reg}},$$

has bounded support; that is, the support in $T^1_{S,\mathrm{reg}}$ has compact closure in T_S^1. Let $\mathcal{I}_c(M(\mathbb{A})^1)$ be the set of functions $\phi \in \mathcal{I}(M(\mathbb{A})^1)$ such that for every group T_S^1, the function

$$\gamma \to \hat{I}_\gamma(\phi), \qquad \gamma \in T^1_{S,\mathrm{reg}},$$

has bounded support. There is a natural topology on $\mathcal{I}_c(M(\mathbb{A})^1)$ such that $\mathcal{T}_M$ maps $C_c^\infty(M(\mathbb{A})^1)$ continuously into $\mathcal{I}_c(M(\mathbb{A})^1)$.

HYPOTHESIS 3.2: For each group $M \in \mathcal{L}(M_0)$, $\mathcal{T}_M$ maps $C_c^\infty(M(\mathbb{A})^1)$ *onto* $\mathcal{I}_c(M(\mathbb{A})^1)$. Moreover, the image of the transpose,

$$\mathcal{T}_M' : \mathcal{I}_c(M(\mathbb{A})^1)' \to C_c^\infty(M(\mathbb{A})^1)',$$

is the space of *all* invariant distributions on $M(\mathbb{A})^1$.

If I is an arbitrary invariant distribution on $M(\mathbb{A})^1$, let $\hat{I}$ be the unique element in $I_c(M(\mathbb{A})^1)$ such that $T'_M(\hat{I}) = I$.

Much of the paper [1(f)] is devoted to proving the following theorem

<u>THEOREME 3.3</u> : There is a continuous map

$$\phi_M^L : C_c^\infty(L(\mathbb{A})^1) \to I_c(M(\mathbb{A})^1) ,$$

for every pair of groups $M \subset L$ in $L(M_0)$, such that

i) $$\phi_M^L(f^y) = \sum_{Q \in F^L(M)} \phi_M^{M_Q}(f_{Q,y}) , \quad f \in C_c^\infty(L(\mathbb{A})) , \; y \in L(\mathbb{A}) ,$$

and

ii) $$\phi_M^M = T_M . \qquad \square$$

We will not discuss the proof of this theorem, which is quite difficult. Given the theorem, however, it is easy to see how to put the trace formula into invariant form.

<u>PROPOSITION 3.4</u> : Suppose that

$$J^L : C_c^\infty(L(\mathbb{A})^1) \to \mathbb{C} , \qquad L \in L(M_0) ,$$

is a family of distributions such that

$$J^L(f^y) = \sum_{Q \in F^L(M_0)} |W_0^{M_Q}| \, |W_0^L|^{-1} \, J^{M_Q}(f_{Q,y}) \quad ,$$

for each L, $f \in C_c^\infty(L(\mathbb{A})^1)$ and $y \in L(\mathbb{A})^1$. Then there is a unique family

$$I^L : C_c^\infty(L(\mathbb{A})^1) \to \mathbb{C} \ , \qquad L \in \mathcal{L}(M_0) \ ,$$

of *invariant* distributions such that for every L and f,

$$(3.1) \qquad J^L(f) = \sum_{M \in \mathcal{L}^L(M_0)} |W_0^M| \, |W_0^L|^{-1} \, \hat{I}^M(\phi_M^L(f)) \ .$$

PROOF : Assume inductively that I^M has been defined and satisfies (3.1) for all groups $M \in \mathcal{L}(M_0)$ with $M \subsetneq L$. Define

$$I^L(f) = J^L(f) - \sum_{\substack{M \in \mathcal{L}^L(M_0) \\ M \neq L}} |W_0^M| \, |W_0^L|^{-1} \, \hat{I}^M(\phi_M^L(f)) ,$$

for any $f \in C_c^\infty(L(\mathbb{A})^1)$. Then I^L is certainly a distribution on $L(\mathbb{A})^1$. The only thing to prove is its invariance. We must show for any $y \in L(\mathbb{A})^1$, that $I^L(f^y)$ equals $I^L(f)$. This follows from the fomula for $J^L(f^y)$, the formula for $\phi_M^L(f^y)$, and our induction assumption. □

According to (2.3) and (2.3*), we can apply the proposition to the families $\{J_{\mathcal{O}}^L\}$ and $\{J_\chi^L\}$. We obtain invariant distributions $\{I^L_{\mathfrak{o}}\}$ and $\{I^L_\chi\}$ which satisfy the analogues of (3.1).

The trace formula in invariant form is the case that $L = G$ of the following theorem.

<u>THEOREM 3.5</u>: For any group L in $\mathcal{L}(M_0)$,

$$\sum_{\mathfrak{o}\in\mathcal{O}} I^L_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} I^L_\chi(f), \qquad f \in C^\infty_c\left(L(\mathbb{A})^1\right).$$

PROOF: Assume inductively that the theorem holds for all groups $M \in \mathcal{L}(M_0)$ with $M \subsetneq L$. For any such M we also have

$$\sum_{\mathfrak{o}\in\mathcal{O}} \hat{I}^M_{\mathfrak{o}}(\phi) = \sum_{\chi\in\mathcal{X}} \hat{I}^M_\chi(\phi)$$

for any $\phi \in \mathcal{I}_c\left(M(\mathbb{A})^1\right)$. Then

$$\sum_{\mathfrak{o}\in\mathcal{O}} I^L_{\mathfrak{o}}(f)$$

$$= \sum_{\mathfrak{o}\in\mathcal{O}} J^L_{\mathfrak{o}}(f) - \sum_{\mathfrak{o}\in\mathcal{O}} \sum_{\substack{M\in\mathcal{L}^L(M_0)\\ M\neq L}} |W^M_0||W^L_0|^{-1} \hat{I}^M_{\mathfrak{o}}(\phi^L_M(f))$$

$$= \sum_{\chi\in\mathcal{X}} J^L_\chi(f) - \sum_{\chi\in\mathcal{X}} \sum_{\substack{M\in\mathcal{L}^L(M_0)\\ M\neq L}} |W^M_0||W^L_0|^{-1} \hat{I}^M_\chi(\phi^L_M(f))$$

$$= \sum_{\chi\in\mathcal{X}} I^L_\chi(f),$$

by (2.2) and (3.1). □

4. INNER PRODUCT OF TRUNCATED EISENSTEIN SERIES.

An obvious problem is to evaluate the distributions $J_{\mathfrak{o}}$, $I_{\mathfrak{o}}$, J_{χ} and I_{χ} explicitly. How explicitly is not clear, but we would at least like to be able to decompose the distributions as sums of products of distributions on the local groups $G(\mathbb{Q}_v)$.

In [1(c)] we defined the notion of an *unramified* class in $\mathcal{O}$. If $\mathfrak{o}$ is unramified, $J^T_{\mathfrak{o}}(f)$ can be expressed as a weighted orbital integral of f ([1(c), (8.7)]). It is possible to then express $I_{\mathfrak{o}}(f)$ as a certain invariant distribution associated to a weighted orbital integral. (See [1(f), §14] for the case of GL_n.) If $\mathfrak{o}$ is not unramified, we would expect to express $J_{\mathfrak{o}}(f)$ as some kind of limit of weighted orbital integrals. Then $I_{\mathfrak{o}}(f)$ would be a limit of the corresponding invariant distributions. In any case, the lack of explicit formulas for $J_{\mathfrak{o}}(f)$ and $I_{\mathfrak{o}}(f)$, with $\mathfrak{o}$ ramified, should not be an insurmountable impediment to applying the trace formula.

One can also define an *unramified* class in X. For any such class, it is also not hard to give an explicit formula for $J^T_{\chi}(f)$. (See [1(d), p. 119].) Unlike with the classes $\mathfrak{o}$, however, it seems to be essential to have a formula for *all* χ in order to apply the trace formula. We shall devote the rest of this paper to a description of such a formula.

Suppose that $P \in \mathcal{F}(M_0)$ is a parabolic subgroup. Let $A^2(P)$ be the space of square-integrable automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$ whose restriction to $M_P(\mathbb{A})^1$ is square integrable. There is an Eisenstein series for each $\phi \in A^2(P)$ given by

$$E(x,\phi,\lambda) = \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta x)e^{(\lambda+\rho_P)(H_P(\delta x))} .$$

It converges for $\mathrm{Re}(\lambda)$ in a certain chamber, and continues analytically to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. If $\chi \in X$ and $\pi \in \Pi(M_P(\mathbb{A}))$, let $A^2_{\chi,\pi}(P)$ be the space of vectors ϕ in $A^2(P)$ which have the following two properties.

(i) The restriction of ϕ to $G(\mathbb{A})^1$ belongs to

$$L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})^1)_\chi .$$

(ii) For every x in $G(\mathbb{A})$, the function

$$m \to \phi(mx), \qquad m \in M_P(\mathbb{A}),$$

transforms under $M_P(\mathbb{A})$ according to π.

Let $\bar{A}^2_{\chi,\pi}(P)$ be the completion of $A^2_{\chi,\pi}(P)$ with respect to the inner product

$$(\phi,\phi') = \int_K \int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1} \phi(mk)\,\overline{\phi'(mk)}\,dm\,dk .$$

For each $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ there is an induced representation $\rho_{\chi,\pi}(P,\lambda)$ of $G(\mathbb{A})$ on $\bar{A}^2_{\chi,\pi}(P)$, defined by

$$\left(\rho_{\chi,\pi}(P,\lambda,y)\phi\right)(x) = \phi(xy)e^{(\lambda+\rho_P)(H_P(xy))}e^{-(\lambda+\rho_P)(H_P(x))} .$$

The representation is unitary if λ is purely imaginary.

Now, suppose again that a minimal parabolic subgroup

$P_0 \in \mathcal{P}(M_0)$ has been fixed. Let T be a point in $\mathfrak{a}_0$ which is suitably regular with respect to P_0. We shall begin by describing the right hand side of (1.2) more precisely. The kernel $K_\chi(x,y)$ can be expressed in terms of Eisenstein series as

$$\sum_{P \supset P_0} |\mathcal{P}(M_P)|^{-1} \sum_{\pi \in \Pi(M_P(\mathbb{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_\phi E\big(x\,,\,\rho_{\chi,\pi}(P,\lambda,f)\phi,\lambda\big)\,\overline{E(y,\phi,\lambda)}\,d\lambda\,,$$

where ϕ is summed over a suitable ortho-normal basis of $\bar{A}^2_{\chi,\pi}(P)$. To obtain $\Lambda_1^T\Lambda_2^T K_\chi(x,y)$, we just truncate each of the two Eisenstein series in the formula. Then $J_\chi^T(f)$ is given by setting $x=y$ in the resulting expression, and integrating over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. It turns out that the integral over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ may be taken inside all the sums and integrals in the formula for $\Lambda_1^T\Lambda_2^T K_\chi(x,x)$. This provides a slightly more convenient expression for $J_\chi^T(f)$ ([1(d), Theorem 3.2]). Given $P \supset P_0$, $\pi \in \Pi\big(M_P(\mathbb{A})\big)$, and $\lambda \in i\mathfrak{a}_P^*$, define an operator $\Omega^T_{\chi,\pi}(P,\lambda)$ on $A^2_{\chi,\pi}(P)$ by setting

$$(4.1) \qquad \big(\Omega^T_{\chi,\pi}(P,\lambda)\phi'\,,\,\phi\big) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x,\phi',\lambda)\,\overline{\Lambda^T E(x,\phi,\lambda)}\,dx\,,$$

for any pair of vectors ϕ' and ϕ in $A^2_{\chi,\pi}(P)$. Then (1.2) becomes the formula

$$(4.2) \qquad J_\chi^T(f) = \sum_{P \supset P_0}\ \sum_{\pi \in \Pi(M_P(\mathbb{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda,f)\,d\lambda\,,$$

where

$$(4.3) \qquad \Psi_{\pi}^{T}(\lambda,f) = |P(M_P)|^{-1}\mathrm{tr}\big(\Omega_{\chi,\pi}^{T}(P,\lambda)\rho_{\chi,\pi}(P,\lambda,f)\big).$$

We hasten to point out that (4.2) does not represent an explicit formula for $J_{\chi}^{T}(f)$. It does not allow us to see how to decompose J_{χ}^{T} into distributions on the local groups $G(\mathbb{Q}_v)$. Moreover, we know that $J_{\chi}^{T}(f)$ is a polynomial function of T. However, this is certainly not clear from the right hand side of (4.2).

The most immediate weakness of (4.2) is that the definition of the operator $\Omega_{\chi,\pi}^{T}(P,\lambda)$ is not very explicit. However, there is a more concrete expression for $\Omega_{\chi,\pi}^{T}(P,\lambda)$, due to Langlands. (See [8(a), §9].) It is valid in the special case that P belongs to the associated class $\mathcal{P}_{\chi}$; that is, when the Eisenstein series on the right hand side of (4.1) are cuspidal. To describe it, we first recall that if P and P_1 are groups in $\mathcal{F}(M_0)$ and s belongs to $W(\mathfrak{a}_P,\mathfrak{a}_{P_1})$, the set of isomorphisms from $\mathfrak{a}_P$ onto $\mathfrak{a}_{P_1}$ obtained by restricting elements in W_0 to $\mathfrak{a}_P$, then there is an important function $M_{P_1|P}(s,\lambda)$. For any $\phi \in \mathcal{A}^2(P)$,

$$M_{P_1|P}(s,\lambda)\phi\big)(x)$$

is defined to be

$$\int_{N_{P_1}(\mathbb{A})\cap w_s N_P(\mathbb{A})w_s^{-1}\backslash N_{P_1}(\mathbb{A})} \phi(w_s^{-1}nx)e^{(\lambda+\rho_P)(H_P(w_s^{-1}nx))}e^{-(s\lambda+\rho_{P_1})(H_{P_1}(x))}\,dn.$$

The integral converges only for the real part of λ in a certain chamber, but $M_{P_1|P}(s,\lambda)$ can be analytically continued to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with values in the space of linear maps from $A^2(P)$ to $A^2(P_1)$. Suppose that π is a representation in $\Pi(M_P(\mathbb{A}))$. Then $M_{P_1|P}(s,\lambda)$ maps the subspace $A^2_{\chi,\pi}(P)$ to $A^2_{\chi,s\pi}(P_1)$. If $\lambda \in i\mathfrak{a}^*_P$, let $\omega^T_{\chi,\pi}(P,\lambda)$ be the value at $\lambda' = \lambda$ of

$$(4.4)\qquad \sum_{P_1\supset P_0}\ \sum_{t,t'\in W(\mathfrak{a}_P,\mathfrak{a}_{P_1})} M_{P_1|P}(t,\lambda)^{-1}M_{P_1|P}(t',\lambda')e^{(t'\lambda'-t\lambda)(T)}\theta_{P_1}(t'\lambda'-t\lambda)^{-1},$$

where

$$\theta_{P_1}(t'\lambda'-t\lambda) = \mathrm{vol}\left(\mathfrak{a}^G_{P_1}/\mathbb{Z}(\Delta^\vee_{P_1})\right)^{-1}\prod_{\alpha\in\Delta_{P_1}}(t'\lambda'-t\lambda)(\alpha^\vee).$$

Here, $\mathbb{Z}(\Delta^\vee_{P_1})$ is the lattice in $\mathfrak{a}^G_{P_1}$ generated by

$$\{\alpha^\vee:\ \alpha\in\Delta_{P_1}\}.$$

Then $\omega^T_{\chi,\pi}(P,\lambda)$ is an operator on $A^2_{\chi,\pi}(P)$.

Langlands' formula amounts to the assertion that if P belongs to P_χ, the operators $\Omega^T_{\chi,\pi}(P,\lambda)$ and $\omega^T_{\chi,\pi}(P,\lambda)$ are equal. This makes the right hand side of (4.3) considerably more explicit. However, $J^T_\chi(f)$ is given in (4.2) by a sum over all standard parabolic subgroups P. Unfortunately, if P does not belong to $\mathcal{P}_\chi$, the operators $\Omega^T_{\chi,\pi}(P,\lambda)$ and $\omega^T_{\chi,\pi}(P,\lambda)$ may not be equal. The best we can salvage is a formula which is asymptotic with respect to T.

Set

$$d_{P_0}(T) = \min\{\alpha(T) : \alpha \in \Delta_{P_0}\}.$$

We shall say that T approaches infinity *strongly with respect to* P_0 if $\|T\|$ approaches infinity, but T remains within a region

$$\{T \in \mathfrak{a}_0 : d_{P_0}(T) > \delta\|T\|\},$$

for some $\delta > 0$.

<u>THEOREM 4.1</u>: If ϕ and ϕ' are vectors in $\mathcal{A}^2_{\chi,\pi}(P)$, the difference

$$(\Omega^T_{\chi,\pi}(P,\lambda)\phi', \phi) - (\omega^T_{\chi,\pi}(P,\lambda)\phi', \phi)$$

approaches zero as T approaches infinity strongly with respect to P_0. The convergence is uniform for λ in compact subsets of $i\mathfrak{a}^*_P$.

This theorem is the main result of [1(h)]. The proof uses the formula of Langlands as a starting point. $\square$

5. CONSEQUENCES OF A PALEY-WIENER THEOREM.

The asymptotic formula of Theorem 4.1 is only uniform for λ in *compact* sets. However, the formula (4.2) entails integrating λ over the space $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, which is noncompact if $P \neq G$. Therefore Theorem 4.1 apparently cannot be exploited.

Our rescue is provided by a multiplier theorem, which was proved in [1(g)] as a consequence of the Paley-Wiener theorem for real groups. The multiplier theorem concerns $C_c^\infty(G(\mathbb{R}), K_{\mathbb{R}})$, the algebra of smooth, compactly supported functions on $G(\mathbb{R})$ which are left and right finite under the maximal compact subgroup of $G(\mathbb{R})$. Set

$$\mathfrak{h} = i\mathfrak{h}_K \oplus \mathfrak{h}_0 ,$$

where $\mathfrak{h}_0$ is the Lie algebra of some maximal real split torus in $M_0(\mathbb{R})$ and $\mathfrak{h}_K$ is the Lie algebra of a maximal torus in $K_{\mathbb{R}} \cap M_0(\mathbb{R})$. Then $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, the Lie algebra of $G(\mathbb{C})$, and $\mathfrak{h}$ is invariant under the Weyl group, W, of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let $\mathcal{E}(\mathfrak{h})^W$ be the algebra of compactly supported distributions on $\mathfrak{h}$ which are invariant under W. The multiplier theorem states that for any $\gamma \in \mathcal{E}(\mathfrak{h})^W$ and $f_{\mathbb{R}} \in C_c^\infty(G(\mathbb{R}), K_{\mathbb{R}})$, there is a unique function $f_{\mathbb{R},\gamma}$ in $C_c^\infty(G(\mathbb{R}), K_{\mathbb{R}})$ with the following property. If $\Pi_{\mathbb{R}}$ is any representation in $\Pi(G(\mathbb{R}))$, then

$$\Pi_{\mathbb{R}}(f_{\mathbb{R},\gamma}) = \hat{\gamma}(\nu_{\Pi_{\mathbb{R}}})\Pi_{\mathbb{R}}(f_{\mathbb{R}})$$

where $\{\nu_{\Pi_{\mathbb{R}}}\}$ is the W-orbit in $\mathfrak{h}_{\mathbb{C}}^{*}$ associated to the infinitesimal character of $\Pi_{\mathbb{R}}$, and $\hat{\gamma}$ is the Fourier-Laplace transform of γ. The theorem also provides a bound for the support of $f_{\mathbb{R},\gamma}$ in terms of the support of γ and of $f_{\mathbb{R}}$.

We apply the theorem to $C_c^{\infty}(G(\mathbb{A})^1, K)$, the algebra of K finite functions in $C_c^{\infty}(G(\mathbb{A})^1)$. For each group $P \supset P_0$, there is a natural surjective map $h_P : \mathfrak{h} \to \mathfrak{a}_P$. Let $\mathfrak{h}^1$ be the kernel of h_G. Suppose that $\gamma \in \mathcal{E}(\mathfrak{h})^W$ is actually supported on $\mathfrak{h}^1$. Any function $f \in C_c^{\infty}(G(\mathbb{A})^1, K)$ is the restriction to $G(\mathbb{A})^1$ of a finite sum

$$\sum(\otimes_v f_v) = \sum\left(f_{\mathbb{R}} \otimes (\otimes_{v\neq\mathbb{R}} f_v)\right),$$

where each f_v is a K_v finite function in $C_c^{\infty}(G(\mathbb{Q}_v))$. The restriction to $G(\mathbb{A})^1$ of the function

$$\sum\left(f_{\mathbb{R},\gamma} \otimes (\otimes_{v\neq\mathbb{R}} f_v)\right)$$

depends only on f. We denote it by f_{γ}. Suppose that $P \supset P_0$ and $\pi \in \Pi(M_P(\mathbb{A}))$. Then the operator $\rho_{\chi,\pi}(P, \lambda, f_{\gamma})$ will be a scalar multiple of $\rho_{\chi,\pi}(P, \lambda, f)$. For if

$$\pi = \otimes_v \pi_v, \qquad \pi_v \in \Pi(M_P(\mathbb{Q}_v)),$$

there is a Weyl orbit $\{\nu_{\pi}\}$ in $\mathfrak{h}_{\mathbb{C}}^{*}$ associated to the infinitesimal character of $\pi_{\mathbb{R}}$. Then

$$\rho_{\chi,\pi}(P, \lambda, f_\gamma) = \hat{\gamma}(\nu_\pi + \lambda)\rho_{\chi,\pi}(P, \lambda, f).$$

We shall now try to sketch how the multiplier theorem can be applied to the study of J_χ^T. The key is the formula (4.2), and in particular, the fact that the left hand side of (4.2) is a polynomial function of T. Now this formula is only valid for points T which are suitably regular in a sense that depends on f. If $N > 0$, let $C_N^\infty(G(\mathbb{A})^1, K)$ be the space of functions in $C_c^\infty(G(\mathbb{A})^1, K)$ which are supported on

$$\{x \in G(\mathbb{A})^1 : \log\|x\| \le N\}.$$

(Here

$$\|x\| = \prod_v \|x_v\|, \qquad x = \prod_v x_v \in G(\mathbb{A}),$$

is the usual kind of function used to describe estimates on $G(\mathbb{A})$. See [1(c), §1].) Then it turns out that there is a constant C_0 such that for any N, and any $f \in C_N^\infty(G(\mathbb{A})^1, K)$, formula (4.2) holds whenever

$$d_{P_0}(T) > C_0(1+N).$$

(See [1(i), Proposition 2.2].)

If f belongs to $C_N^\infty(G(\mathbb{A})^1, K)$ and $\gamma \in \mathcal{E}(\mathfrak{h}_\gamma)^W$ is supported on

$$\{H \in \mathfrak{h}^1 : \|H\| \le N_\gamma\},$$

then f_γ will belong to $C^\infty_{N+N_\gamma}(G(\mathbb{A})^1, K)$. We substitute f_γ into the right hand side of (4.2). We obtain

$$\sum_{P\supset P_0} \sum_{\pi\in\Pi(M_P(\mathbb{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \hat{\gamma}(\nu_\pi + \lambda)\Psi_\pi^T(\lambda, f)d\lambda$$

$$= \sum_P \sum_\pi \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \int_{\mathfrak{h}^1} \Psi_\pi^T(\lambda, f)e^{(\nu_\pi+\lambda)(H)} \gamma(H)dH\, d\lambda .$$

This equals

$$(5.1) \qquad \int_{\mathfrak{h}^1} \Big(\sum_{P\supset P_0} \sum_{\pi\in\Pi(M_P(\mathbb{A})^1)} \psi_\pi^T(H)e^{\nu_\pi(H)} \Big) \gamma(H)dH ,$$

where

$$\psi_\pi^T(H) = \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda, f)e^{\lambda(H)}\, d\lambda ,$$

for $H \in \mathfrak{h}^1$ and $\pi \in \Pi(M_P(\mathbb{A})^1)$. The function $\psi_\pi^T(H)$ depends only on the projection of H on $\mathfrak{a}_P$. It vanishes for all but finitely many π. It can also be shown to be a smooth bounded function of H.

The expression (5.1) is a polynomial in T whenever

$$d_{P_0}(T) > C_0(1 + N + N_\gamma) .$$

Fix $H \in \mathfrak{h}^1$. Let γ_H be the Dirac measure on $\mathfrak{h}^1$ at the point H, and set

$$\gamma = |W|^{-1} \sum_{s \in W} \gamma_{s^{-1}H}.$$

Then $N_\gamma = \|H\|$. The expression (5.1) equals

$$(5.2) \qquad |W|^{-1} \sum_{s\in W} \sum_{P \supset P_0} \sum_{\pi \in \Pi(M_P(\mathbb{A})^1)} \psi_\pi^T(s^{-1}H) e^{\nu_\pi(s^{-1}H)} .$$

This function is a polynomial in T whenever

$$d_{P_0}(T) > C_0(1 + N + \|H\|).$$

Its value at $H = 0$ is just the right hand side of (4.2), which equals $J_\chi^T(f)$ as long as $d_{P_0}(T)$ is greater than $C_0(1+N)$.

Suppose we could integrate (5.2) against an arbitrary Schwartz function of $H \in \mathfrak{h}^1$. By the Plancherel theorem on $\mathfrak{h}^1$, the resulting inner product could be replaced by an inner product on $i\mathfrak{h}^*/i\mathfrak{a}_G^*$. If the original Schwartz function were taken from the usual Paley-Wiener space on $\mathfrak{h}^1$, we would be able to replace (4.2) by a formula in which all the integrals were over compact sets. Unfortunately this step cannot be taken immediately, because (5.2) may not be a tempered function of H. For each $\pi \in \Pi(M_P(\mathbb{A})^1)$, let

$$\nu_\pi = X_\pi + iY_\pi, \qquad\qquad X_\pi, Y_\pi \in \mathfrak{h}^*,$$

be the decomposition of ν_π into real and imaginary parts. Then any nonzero point X_π will cause (5.2) to be nontempered. Nevertheless, it is still possible to treat (5.2) as if all the points X_π were zero. This can be justified by an elementary but rather complicated lemma on polynomials. We shall forgo the details, and be content to state only the final result.

Let $S(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ be the space of Schwartz functions on $i\mathfrak{h}^*/i\mathfrak{a}_G^*$ which are invariant under W. If $B \in S(i\mathfrak{h}^*/i\mathfrak{a}_G^*)$ and $\pi \in \Pi(M_P(\mathbb{A})^1)$, set

$$B_\pi(\lambda) = B(iY_\pi + \lambda), \qquad \lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*.$$

It is a Schwartz function on $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$.

<u>THEOREM 5.1</u>: (i) For every function $B \in S(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$ there is a unique polynomial $P^T(B)$ in T such that

$$\sum_{P\supset P_0}\ \sum_{\pi\in\Pi(M_P(\mathbb{A})^1)} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \Psi_\pi^T(\lambda,f)B_\pi(\lambda)\,d\lambda \;-\; P^T(B)$$

approaches zero as T approaches infinity strongly with respect to P_0.

(ii) Suppose that $B(0) = 1$. Then

$$\lim_{\varepsilon\to 0} P^T(B^\varepsilon) = J_\chi^T(f),$$

where

$$B^{\varepsilon}(\nu) = B(\varepsilon\nu), \qquad \nu \in i\mathfrak{h}^*/i\mathfrak{a}_G^*, \quad \varepsilon > 0 .$$

See [1(i), Theorem 6.3]. □

If the function B happens to be compactly supported, the same will be true of all the functions B_π. The first statement of Theorem 5.1 can be combined with Theorem 4.1 to give

THEOREM 5.2: Suppose that $B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W$. Then $P^T(B)$ is the unique polynomial which differs from

$$\sum_{P\supset P_0} \sum_{\pi\in\Pi(M_P(\mathbb{A})^1)} |P(M_P)|^{-1} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \mathrm{tr}\big(\omega_{\chi,\pi}^T(P,\lambda)\rho_{\chi,\pi}(P,\lambda,f)B_\pi(\lambda)d\lambda$$

by an expression which approaches zero as T approaches infinity strongly with respect to P_0.

See [1(i), Theorem 7.1]. □

6. AN EXPLICIT FORMULA.

Theorems 5.1 and 5.2 provide a two step procedure for calculating $J_\chi^T(f)$, if f is any function in $C_c^\infty(G(\mathbb{A})^1)$ which is K finite. One first calculates

$$P^T(B), \qquad B \in C_c^\infty(i\mathfrak{h}^*/i\mathfrak{a}_G^*)^W,$$

as the polynomial which is asymptotic to

$$(6.1) \qquad \sum_{P\supset P_0}\sum_{\pi\in\Pi(M_P(\mathbb{A})^1)} |\mathcal{P}(M_P)|^{-1}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \mathrm{tr}\big(\omega^T_{\chi,\pi}(P,\lambda)\rho_{\chi,\pi}(P,\lambda,f)\big)B_\pi(\lambda)d\lambda\,.$$

One then chooses any B such that $B(0) = 1$, and calculates $J^T_\chi(f)$ by

$$J^T_\chi(f) = \lim_{\varepsilon\to 0} P^T(B^\varepsilon).$$

The second step will follow immediately from the first. The first step, however, is more difficult. It gives rise to some combinatorial problems which are best handled with the notion of a (G, M) family, introduced in [1(f)]. Suppose that $M \in \mathcal{L}(M_0)$. A *(G, M) family* is a set of smooth functions

$$c_Q(\Lambda), \qquad\qquad \Lambda \in i\mathfrak{a}_M^*,$$

indexed by the groups Q in $\mathcal{P}(M)$, which satisfy a certain compatibility condition. Namely, if Q and Q' are adjacent groups in $\mathcal{P}(M)$ and Λ lies in the hyperplane spanned by the common wall of the chambers of Q and Q' in $i\mathfrak{a}_M^*$, then $c_Q(\Lambda) = c_{Q'}(\Lambda)$. A basic result (Lemma 6.2 of [1(f)]) asserts that if $\{c_Q(\Lambda)\}$ is a (G, M) family, then

$$(6.2) \qquad c_M(\Lambda) = \sum_{Q\in\mathcal{P}(M)} c_Q(\Lambda)\,\theta_Q(\Lambda)^{-1}$$

extends to a smooth function on $i\mathfrak{a}_M^*$. A second result, which is what is used to deal with the combinatorial problems we mentioned, concerns products of (G, M) families. Suppose that $\{d_Q(\Lambda)\}$ is another (G, M) family. Then the function (6.1) associated to the (G, M) family

$$\{(cd)_Q(\Lambda) = c_Q(\Lambda)d_Q(\Lambda) : Q \in \mathcal{P}(M)\}$$

is given by

$$(cd)_M(\Lambda) = \sum_{S \in \mathcal{F}(M)} c_M^S(\Lambda)d_S'(\Lambda), \tag{6.3}$$

([1(f), Lemma 6.3]). For any $S \in \mathcal{F}(M)$, $c_M^S(\Lambda)$ is the function (6.2) associated to the (M_S, M) family

$$\{c_R^S(\Lambda) = c_{S(R)}(\Lambda) : R \in \mathcal{P}^{M_S}(M)\},$$

and $c_S'(\Lambda)$ is a certain smooth function on $i\mathfrak{a}_M^*$ which depends only on the projection of Λ onto $i\mathfrak{a}_{M_S}^*$.

For any (G, M) family $\{c_Q(\Lambda)\}$ and any $L \in \mathcal{L}(M)$, there is associated a natural (G, L) family. Let Λ be constrained to lie in $i\mathfrak{a}_L^*$ and choose $Q_1 \in \mathcal{P}(L)$. The compatibility condition implies that the function

$$c_Q(\Lambda), \qquad Q \in \mathcal{P}(M),\ Q \subset Q_1,$$

is independent of Q. We denote it by $c_{Q_1}(\Lambda)$. Then

$$c_{Q_1}(\Lambda), \qquad Q_1 \in \mathcal{P}(L),\ \Lambda \in i\mathfrak{a}_L^*,$$

is a (G, L)-family. We write

$$c_L(\Lambda) = \sum_{Q_1 \in P(L)} c_{Q_1}(\Lambda)\theta_{Q_1}(\Lambda)^{-1}, \qquad \Lambda \in i\mathfrak{a}_L^*,$$

for the corresponding function (6.2). We sometimes denote its value at $\Lambda = 0$ simply by c_L.

A typical example of a (G, M) family is given by

$$e^{\Lambda(Y_Q)}, \qquad Q \in P(M),\ \Lambda \in i\mathfrak{a}_M^*,$$

where $\{Y_Q : Q \in P(M)\}$ is a family of points in $\mathfrak{a}_M$. The compatibility condition requires that for adjacent Q and Q',

$$Y_Q - Y_{Q'} = c_\alpha \alpha^\vee, \qquad c_\alpha \in \mathbb{R},$$

where α is the root in Δ_Q which is orthogonal to the common wall of the chambers of Q and Q'. If each c_α is actually positive, the function $c_M(\Lambda)$ admits a geometric interpretation. It is the Fourier transform of the characteristic function in $\mathfrak{a}_M$ of the convex hull of $\{Y_Q : Q \in P(M)\}$. The number $c_M = c_M(0)$ is just the volume of this convex hull. (G, M) families of this sort are needed to describe the distributions $J_\mathfrak{o}$ and $I_\mathfrak{o}$ in the cases where explicit formulas exist. (See [1(c), §7], [1(f), §14] and also [1(b)].)

For another example, fix $P \in F(M_0)$ and let $M = M_P$. Fix also a point λ in $i\mathfrak{a}_M^*$. For any $Q \in P(M)$, put

$$M_{Q|P}(\lambda) = M_{Q|P}(1,\lambda).$$

Then

$$\mathcal{M}_Q(P,\lambda,\Lambda) = M_{Q|P}(\lambda)^{-1}M_{Q|P}(\lambda+\Lambda), \qquad \Lambda \in i\mathfrak{a}_M^*,$$

is a function on $i\mathfrak{a}_M^*$ with values in the space of operators on $\mathcal{A}^2(P)$. It can be shown that

$$\{\mathcal{M}_Q(P,\lambda,\Lambda) : Q \in \mathcal{P}(M)\}$$

is a (G, M) family of (vector valued) functions.

In order to deal with (6.1) we must look back at the definition of $\omega^T_{\chi,\pi}(P,\lambda)$ in §4. The expression (4.4) can be written as the sum over $s \in W(\mathfrak{a}_P, \mathfrak{a}_P)$ of

$$\sum_{P_1 \supset P_0} \sum_{t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} M_{P_1|P}(t,\lambda)^{-1} M_{P_1|P}(ts, \lambda')e^{(t(s\lambda'-\lambda))(T)}\theta_{P_1}(t(s\lambda'-\lambda))^{-1}.$$

Given $P_1 \supset P_0$ and $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$, set $Q = w_t^{-1}P_1w_t$, for any representative w_t of t in $G(\mathbb{Q})$. Then

$$(P_1, t) \longleftrightarrow Q$$

is a bijection between pairs which occur in the sum above and groups $Q \in \mathcal{P}(M)$. Notice that

$$\theta_{P_1}(t(s\lambda' - \lambda)) = \theta_Q(s\lambda' - \lambda).$$

It can also be shown that

$$M_{P_1|P}(t,\lambda)^{-1}M_{P_1|P}(ts\,,\,\lambda')e^{\left(t(s\lambda'-\lambda)\right)(T)}$$

equals

$$M_{Q|P}(\lambda)^{-1}M_{Q|P}(s\,,\,\lambda')e^{(s\lambda'-\lambda)\left(Y_Q(T)\right)}$$

where $Y_Q(T)$ is the projection onto $\mathfrak{a}_M$ of the point

$$t^{-1}(T-T_0)+T_0\,.$$

Therefore,

$$\mathrm{tr}\left(\omega^T_{\chi,\pi}(P,\lambda)\rho_{\chi,\pi}(P,\lambda,f)\right),$$

the function which must be substituted into (6.1), can be obtained by setting $\lambda' = \lambda$ in the sum over $s \in W(\mathfrak{a}_P\,,\mathfrak{a}_P)$ of

$$(6.4)\qquad \sum_{Q\in\mathcal{P}(M_P)} \mathrm{tr}\left(M_{Q|P}(\lambda)^{-1}M_{Q|P}(s\,,\,\lambda')\rho_{\chi,\pi}(P,\lambda,f)\right)e^{(s\lambda'-\lambda)\left(Y_Q(T)\right)}\theta_Q(s\lambda'-\lambda)^{-1}\,.$$

Formula (6.3) suggests a way to handle (6.4). We set

$$\Lambda = s\lambda' - \lambda\,,$$

and define

$$c_Q(\Lambda) = e^{\Lambda\left(Y_Q(T)\right)}$$

and

$$d_Q(\Lambda) = \mathrm{tr}\big(M_{Q|P}(\lambda)^{-1} M_{Q|P}(s\, ,\lambda')\rho_{\chi,\pi}(P,\lambda,f)\big),$$

for any $Q \in P(M_P)$. It is not hard to show that $\{c_Q(\Lambda)\}$ and $\{d_Q(\Lambda)\}$ are $(G\, , M_P)$ families. The function (6.4) equals

$$\sum_{Q\in P(M_P)} c_Q(\Lambda) d_Q(\Lambda)\theta_Q(\Lambda)^{-1},$$

an expression to which we can apply (6.3). The result is a sum of terms indexed by groups $S \in F(M_P)$. The contribution to (6.1) of each such term can be shown to be asymptotic to a polynomial in T. The sum of all these polynomials will be the required polynomial $P^T(B)$. Once again, we will skip the details and state only the final result.

In the notation above, set

$$M_Q^T(P,\lambda,\Lambda) = e^{\Lambda(Y_Q(T))} M_Q(P,\lambda,\Lambda), \qquad Q \in P(M_P).$$

This is a product of two $(G\, , M_P)$ families, so it is itself a $(G\, , M_P)$ family. If L is any group in $L(M_P)$,

$$M_L^T(P,\lambda) = \lim_{\Lambda\to 0} \sum_{Q_1\in F(L)} M_{Q_1}^T(P,\lambda,\Lambda)\theta_{Q_1}(\Lambda)^{-1}$$

is defined. It is a polynomial in T with values in the space of operators on $A^2(P)$.

If $L \supset M$ are *any* two groups in $\mathcal{L}(M_0)$, let $W^L(\mathfrak{a}_M)_{reg}$ be the set of elements in $W(\mathfrak{a}_M, \mathfrak{a}_M)$ for which $\mathfrak{a}_L$ is the space of fixed vectors.

THEOREM 6.1: The polynomial $P^T(B)$ equals the sum over $P \supset P_0$, $\pi \in \Pi(M_P(\mathbb{A})^1)$, $L \in \mathcal{L}(M_P)$ and $s \in W^L(\mathfrak{a}_{M_P})_{reg}$ of the product of

$$|\mathcal{P}(M_P)|^{-1} |\det(s-1)_{\mathfrak{a}_P/\mathfrak{a}_L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \operatorname{tr}\left(\mathcal{M}_L^T(P,\lambda) M_{P|P}(s,0) \rho_{\chi,\pi}(P,\lambda,f)\right) B_\pi(\lambda)\, d\lambda .$$

See [1(j), Theorem 4.1]. □

The theorem provides an explicit formula for $P^T(B)$. From this we can obtain a formula for $J_\chi^T(f)$ and, in particular, for

$$J_\chi(f) = J_\chi^{T_0}(f).$$

It is easy to show that

$$\mathcal{M}_L^{T_0}(P,\lambda) = \mathcal{M}_L(P,\lambda).$$

Then $J_\chi(f)$ can be obtained from the formula of the theorem by simply suppressing T.

The formula for $J_\chi(f)$ will still depend on the test function B. It would be better if we could remove it. Moreover, the function f is still required to be K finite. Our formula ought to apply to an arbitrary function in $C_c^\infty(G(\mathbb{A})^1)$. The dominated convergence theorem will permit these improvements provided that a certain multiple integral can be shown to converge absolutely. The proof of such absolute convergence turns out to rest on the ability to normalize the intertwining operators between induced representations on the local groups $G(\mathbb{Q}_v)$. At first this may seem like a tall order, but it is not necessary to have the precise normalizations proposed in [8(b), Appendix II]. We require only a general kind of normalization of the sort established in [6] for real groups. The analogue for p-adic groups should not be too difficult to prove. In any case, we assume the existence of such normalizations for the following theorem.

THEOREM 6.2. Suppose that $f \in C_c^\infty(G(\mathbb{A})^1)$. Then $J_\chi(f)$ equals the sum over $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$, $\pi \in \Pi(M(\mathbb{A})^1)$ and $s \in W^L(\mathfrak{a}_M)_{reg}$ of the product of

$$|W_0^M||W_0|^{-1}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} |\mathcal{P}(M)|^{-1} \sum_{P\in\mathcal{P}(M)} \operatorname{tr}\left(\mathcal{M}_L(P,\lambda)M_{P|P}(s,0)\rho_{\chi,\pi}(P,\lambda,f)\right)d\lambda .$$

This is Theorem 8.2 of [1(j)]. Implicit in the statement if the absolute convergence of the expression for $J_\chi(f)$. $\square$

Let $D_\chi(f)$ be the sum of the terms in the expression for $J_\chi(f)$ for which $L = G$. It equals the sum over $M \in \mathcal{L}(M_0)$, $\pi \in \Pi(M(\mathbb{A})^1)$, $P \in \mathcal{P}(M)$ and $s \in W^G(\mathfrak{a}_M)_{\mathrm{reg}}$ of

$$|W_0^M|\,|W_0|^{-1}\,|\mathcal{P}(M)|^{-1}|\det(s-1)_{\mathfrak{a}_M^G}|^{-1}\ \mathrm{tr}(M_{P|P}(s,0)\,\rho_{\chi,\pi}(P,0,f)) \ .$$

In particular, the distribution D_χ is invariant. As the "discrete part" of J_χ, it will play a special role in the applications of the trace formula.

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