GALOIS GROUPS, STOKES OPERATORS AND A THEOREM OF RAMIS

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INTRODUCTION

The basic result of this article is

THEOREM 1. Stokes and monodromy operators belong to the Galois group of a linear differential equation with a non-Fuchsian nonresonant singular point.

This formulation requires explanation. The definition of a Galois group is presented in Sec. 1, and of Stokes operators in Sec. 2. Theorem 1 was formulated by Ramis, who also offered an outline of a proof [1, 2], which will be published in a book under preparation.* An independent proof is presented here, based on what is called a "calculus of functional chains," developed first by the authors. Functional cochains arise in a natural manner in the local theory of resonant analytic vector fields and mappings. Normalizing series in the resonant cases, as a rule, diverge. However, they are asymptotic for "normalizing cochains"—piecewise-continuous functions or vector-functions of a single complex variable, holomorphic outside the discontinuity lines, a jump of which on the discontinuity line quickly decreases in the approach of the argument to the singular point; for a more detailed definition, see Sec. 3. The cochains are given by their asymptotic series uniquely, and can be considered as sums of these (divergent) series. The functional cochains play a basic role in proving the finiteness theorem for limit cycles, the first part of which was published in [3]. This paper is the simplest application of functional cochains. A parallel technique, developed in the West, is the resummation of Ramis. This same technique is mentioned in the outline of the proof of the finiteness theorem for limit cycles announced by J. Écalle, J. Martinet, R. Moussu, and J. P. Ramis [4].

1. Galois Group of a Linear Differential Equation

Let us consider in a neighborhood of zero, a linear nonautonomous differential equation with a holomorphic right part:

$$t^{s+1}\dot{z} = A(t)\,z, \quad z \in \mathbf{C}^n, \quad s \in \mathbf{N}, \quad A(0) \neq 0. \tag{1.1}$$

By a change of scale, one can make the operator-valued function A holomorphic in the disk $|t| \le 1$. If the matrix A has distinct eigenvalues λ_j, then the equation is called nonresonant. In this case, the singular point 0 of Eq. (1.1) is always irregular; this means, by definition, that there exists a solution of the equation, which in some sector with vertex 0 approaches infinity more quickly than any power as $t \to 0$. The arguments of Sec. 1 do not depend on whether Eq. (1.1) is resonant or not.

Let S be an arbitrary sector of the unit disk. Let us denote by M the field of functions meromorphic in the unit disk with a unique pole at the origin. Let $\mathscr{K}_S$ be the extension of M obtained by adjoining to M all the components of the solutions of (1.1), the constraints on S.

Definition 1. The Galois group of Eq. (1.1) over M is the group of all automorphisms of the differential field $\mathscr{K}_S$, keeping stationary all the functions from M meromorphic in the unit disk. The notation:

$$G = G[\mathscr{K}_S:\ M].$$

Everywhere, except the conclusion, the refinement "over M" will be omitted for brevity.

*A preprint of J. P. Ramis, concerned with it, was unavailable to the authors in writing the article.

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Remark 1. The choice of another sector S' instead of S replaces the Galois group by the conjugate: the conjugation is implemented by an automorphism of the analytic continuation $\mathcal{K}_S \to \mathcal{K}_{S'}$, nonuniquely determined.

Remark 2. By definition, every automorphism of a group G is a linear operator in an infinite-dimensional space K_S; however, the Galois group admits an exact n-dimensional representation, to the description of which we, in fact, pass.

Proposition 1. The automorphisms of the Galois group induce a transformation of Cartesian degree $\mathcal{K}_S^n \to \mathcal{K}_S^n$, taking the solutions of Eq. (1.1) to solution S.

Let L be the automorphism described in the proposition, and $z \in \mathcal{K}_S^n$ be a solution of Eq. (1.1). Then $(Lz)^{\cdot} = L\dot{z}$ and $L(Az) = ALz$ by the definition of a Galois group. Consequently, Lz is a solution of (1.1) along with z.

COROLLARY. The Galois group of Eq. (1.1) admits an exact linear representation in the space $\mathcal{L}_S$ of constraints of all solutions of Eq. (1.1) on S.

The representation of the Galois group is given by Proposition 1. In order to prove its accuracy, it is sufficient to establish that the automorphism preserving all the components of all the solutions, is identical on $\mathcal{L}_S$ This immediately follows from the identity of the automorphism on M and the definition of the field $\mathcal{K}_S$.

Let us denote by Z some fundamental matrix of solutions of (1.1) and fix it. Let A be an arbitrary, and B be an integral matrix: $A = (a_{ij})$, $B = (b_{ij})$. Let

$$A^B = \prod_{i,j} a_{ij}^{b_{ij}}$$

(the analog of the usual multiindices). In this notation, an arbitrary element of $\mathcal{K}_S$ has the form

$$f = \sum a_K Z^K / \sum b_K Z^K, \tag{1.2}$$

the integral matrix K with nonnegative elements runs over a finite set depending on f. Differentiation is replaced by arithmetic actions due to (1.1):

$$\dot{Z} = A(t) Z/t^S.$$

The columns of Z form a basis in the space $\mathcal{L}_S$ of all solutions of (1.1) bounded on S. To the automorphism $H: \mathcal{K}_S \to \mathcal{K}_S$ corresponds by Proposition 1, the operator $T_H: \mathcal{L}_S \to \mathcal{L}_S$. In the basis Z it is described by a matrix which is also denoted $T_H: Z \to ZT_H$. Moreover

$$Hf = \sum a_K (ZT_H)^K / \sum b_K (ZT_H)^K. \tag{1.3}$$

At first glance, this formula along with (1.1) allows one for each linear operator T_H: $\mathcal{L}_S \to \mathcal{L}_S$ to construct an automorphism $H: \mathcal{K}_S \to \mathcal{K}_S$. However, the representation (1.2) can be nonunique, and the function Hf given by (1.3) will depend on the representation. For the definition (1.3) to be correct, it is necessary that T_H preserve the relations in $\mathcal{K}_S$, that is, the equations

$$\sum a_K Z^K \equiv 0. \tag{1.4}$$

The operator T_H is only continued to an automorphism of $\mathcal{K}_S$, when it follows from (1.3) that

$$\sum a_K (ZT_H)^K \equiv 0.$$

In particular, the automorphism $\mathcal{L}_S \to \mathcal{L}_S$ is generated by an automorphism of the n^2-dimensional space spanned over C by the components of the solutions, only if it preserves all the linear relations in the components. Theorem 1 follows now from the next theorem.

THEOREM 2. The Stokes and monodromy operators of Eq. (1.1) preserve the relations in the field $\mathcal{K}_S$.

For the monodromy operators, Theorem 2 is trivial: analytic continuation along the loop preserves the relations. Theorem 1, hence, follows for a monodromy operator. Below, Theorem 2 is proved for the Stokes operators, to the definition of which we, in fact, pass.

2. Stokes Operators

Starting from here, let us consider the equation to be nonresonant: $\lambda_i \neq \lambda_j$. In this case, there exists a formal change of variable $z = \hat{H}w$, taking Eq. (1.1) to the integrable equation

$$t^{s+1}\dot{w} = B(t)w, \quad B(t) = \operatorname{diag} b(t), \quad b = (b_1, \ldots, b_n), \tag{2.1}$$

b_j are polynomials of degree no higher than s. The simple proof of this theorem can be found in [5]. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix A(0), then $b_j(0) = \lambda_j$. Equation (2.1) is called the formal normal form of Eq. (1.1) or the normalized equation, and the formal series $\hat{H}$ joining these equations — the normalizing series. The normalizing series, normalized by the condition $\hat{H}(0) = E$, is uniquely determined. The fundamental matrix of the solutions of the normalized equation has the form

$$W(t) = \operatorname{diag} w(t), \quad w = (w_1, \ldots, w_n), \quad w_j = \exp q_j,$$
$$q_j' = b_j/t^{s+1}, \quad q_j = -(\lambda_j/st^s) + \ldots + \mu_j \ln t. \tag{2.2}$$

Analytic continuation around the origin in the positive direction multiplies the fundamental matrix W by the monodromy operator matrix

$$W \to WM_W, \quad M_W = \exp 2\pi i \operatorname{diag} \mu, \quad \mu = (\mu_1, \ldots, \mu_n).$$

Let us now dwell on the connection of the normalized and initial equations. Generally speaking, a divergent normalizing series is asymptotic for a holomorphic substitution joining in some sector with vertex 0, the normalized and initial equations. In order to formulate an existence and uniqueness theorem of such substitutions, we require certain definitions.

Definition 1. The ray of a division corresponding to a pair of complex numbers λ, μ and a natural number s is any of the rays given by the equation

$$\operatorname{Re} (\lambda - \mu)/t^s = 0.$$

A ray of the division of equation (1.1) is any of the rays of the division corresponding to the triads s, λ_i, λ_j, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A(0).

Definition 2. A sector with a vertex at the origin is called good for Eq. (1.1) if it has an opening greater than π/s and for any pair λ_i, λ_j its closure does not contain two rays of the division corresponding to this pair and the number s.

THEOREM 3 (on sectorial normalization [5, 6]). Let S be a good sector for Eq. (1.1). Then, if the radius of the sector is sufficiently small, there exists a unique holomorphic mapping H_S possessing the following properties:

1) the substitution $z = H_S w$ unites Eqs. (1.1) and (2.1) in sector S;

2) the mapping H_S is expanded into an asymptotic Taylor series $\hat{H}$ as $t \to 0$, not depending on S and joining formally Eqs. (1.1) and (2.1).

The substitution H_S described in the theorem is called normalizing for Eq. (1.1) in S.

The Stokes phenomenon consists in the fact that normalizing substitutions in intersecting good sectors, generally speaking, do not coincide. Let H_{S_1} and H_{S_2} be the corresponding normalizing substitutions. Then $Z_{S_j} = H_{S_j}W (j = 1, 2)$ are the fundamental matrices of the solutions in the corresponding sectors S_j. In the intersection of the sectors, one of these matrices is translated to another by multiplication on the right by the constant matrix:

$$Z_{S_1}C = Z_{S_2} \text{ if } H_{S_1}WC = H_{S_2}W. \tag{2.2}$$

The corresponding linear operator from the space of solutions $\mathcal{L}_{S_1}$ of Eq. (1.1) in the sector S_1 to itself is called the Stokes operator.

Now all the concepts involved in formulating Theorem 1 are defined. For what follows, an invariant definition of Stokes operators is needed; it is broader than the preceding one and allows one to determine the Stokes operator for any pair of good sectors, not necessarily intersecting. Let S_1 and S_2 be two good sectors, γ be a curve with origin in S_1 and end in S_2, lying in the punctured disk (here and below the punctured disk has the form $0 < |t| < 1$). Let $\mathcal{L}_j$ and $\mathcal{N}_j$ be the spaces of solutions of the initial and normalized equations in sector S_j, respectively. The theorem on sectorial normalization uniquely gives the defined operator $h_j\colon \mathcal{N}_j \to \mathcal{L}_j$. Let us denote by Δ_γ the analytic continuation operator over γ; it acts in the solution spaces of both the normalized and initial equations

$$\Delta_\gamma\colon \mathcal{N}_1 \to \mathcal{N}_2, \quad \Delta_\gamma\colon \mathcal{L}_1 \to \mathcal{L}_2.$$

The Stokes phenomenon consists in the fact that the diagram

$$\begin{array}{ccc} \mathcal{N}_1 & \xrightarrow{h_1} & \mathcal{L}_1 \\ \Delta_\gamma \downarrow & & \downarrow \Delta_\gamma \\ \mathcal{N}_2 & \xrightarrow{h_2} & \mathcal{L}_2 \end{array}$$

is noncommutative. The next diagram is commutative, which in fact determines the Stokes operator $C = C_{S_1,S_2,\gamma}\colon \mathcal{L}_{S_1} \to \mathcal{L}_{S_1}$ ($\mathcal{L}_{S_1} = \mathcal{L}_1$)

$$\begin{array}{ccccc} \mathcal{N}_1 & \xrightarrow{h_1} & \mathcal{L}_1 & \xrightarrow{C} & \mathcal{L}_1 \\ \Delta_\gamma \downarrow & & & & \downarrow \Delta_\gamma. \\ \mathcal{N}_2 & & \xrightarrow{h_2} & & \mathcal{L}_2 \end{array} \tag{2.3}$$

In the solution space of the normalized equation, there is distinguished a special basis determined with a precision up to an order – the set of columns of the matrix W. In the solution space of the initial equation $\mathcal{L}_{S_1}$ is distinguished a basis to which the previous one passes for a uniquely determined H_{S_1}. This is the set of columns of the matrix $Z = H_{S_1}W$. In this basis, the Stokes operator C is given by a matrix denoted by the same letter. Definition (2.3) in matrix form:

$$\Delta_\gamma H_{S_1} WC = H_{S_2} \Delta_\gamma W. \tag{2.4}$$

When the sectors S_1 and S_2 intersect, and the curve γ consists of one point, this definition changes into the preceding one.

The next remark is not used in proving Theorem 1, but it is, in our view, a useful commentary on it. The Stokes operators possess the group property. Namely, let S_1, S_2, S_3 be three good sectors, γ_1 and γ_2 be two curves, the second of which continues the first; γ_1 is started in S_1 and ends in S_2, γ_2 is started in S_2 and ends in S_3. Let $\gamma_3 = \gamma_1\gamma_2$. Then

$$C_{S_1,S_3,\gamma_3} = \Delta_{\gamma_1}^{-1} \circ C_{S_2,S_3,\gamma_2} \circ \Delta_{\gamma_1} \circ C_{S_1,S_2,\gamma_1}.$$

This immediately follows from the previous definition and is clarified by the commutative diagrams in which, for brevity, we denote $C_1 = C_{S_1,S_2,\gamma_1}$, $C_2 = {}_{S_2,S_3,\lambda_2}$, $C_3 = C_{S_1,S_3,\gamma_3}$, $\tilde{C}_2 = \Delta_{\gamma_1}{}^{-1} \circ C_2 \circ \Delta_{\gamma_1}$:

$$\begin{array}{ccccccc} \mathcal{N}_1 & \xrightarrow{h_1} & \mathcal{L}_1 & \xrightarrow{C_1} & \mathcal{L}_1 & \xrightarrow{\tilde{C}_2} & \mathcal{L}_1 \\ \Delta_{\gamma_1} \downarrow & & & & \Delta_{\gamma_1} \downarrow & & \downarrow \Delta_{\gamma_1} \\ \mathcal{N}_2 & & \xrightarrow{h_2} & & \mathcal{L}_2 & \xrightarrow{C_2} & \mathcal{L}_2 \\ \Delta_{\gamma_2} \downarrow & & & & & & \downarrow \Delta_{\gamma_2} \\ \mathcal{N}_3 & & & \xrightarrow{h_3} & & & \mathcal{L}_3 \end{array},$$

$$\begin{array}{ccccc} \mathcal{N}_1 & \xrightarrow{h_1} & \mathcal{L} & \xrightarrow{C_3} & \mathcal{L}_1 \\ \Delta_{\gamma_3} \downarrow & & & & \downarrow \Delta_{\gamma_3}. \\ \mathcal{N}_3 & & \xrightarrow{h_3} & & \mathcal{L}_3 \end{array}$$

Let us conclude this section by a remark on the connection of the Stokes and monodromy operators for the initial and normalized equations.

Definition 3. A covering of the punctured disk by sectors good for Eq. (1.1) is called good for this equation if the union of any two sectors of the covering is not a good sector (a good covering does not contain superfluous sectors).

Let us enumerate the sectors of a good covering in the natural counter-clockwise order. Let C_j be the Stokes operator corresponding to the intersecting sectors S_j and S_{j+1} of this covering, γ_0 be a positively oriented loop with start and end in S_1, circuiting the origin one time. Let us identify the solution spaces in intersecting sectors with sequential numbers, considering that the solutions serving as an analytic continuation of each other, coincide. One can consider after this that all the operators C_j act in the space $\mathcal{L}_{S_1}$. Then $M_Z G_{S_1 S_1, \gamma_0} = M_Z C_N \ldots C_1 = M_W$, where M_W and M_Z are the monodromy operators of the normalized and initial equations, respectively. This follows from the group property of the Stokes operators and diagram (2.3), in which $\gamma = \gamma_0$, $\mathcal{N}_1 = \mathcal{N}_2$, $\mathcal{L}_1 = \mathcal{L}_2$; the analytic continuations along γ_0 give the monodromy operators.

3. Normalizing and Functional Cochains

Let us consider a covering of the punctured disk by the sectors $S_1, \ldots, S_N$ good for Eq. (1.1). Let $H_1, \ldots, H_N$ be the normalizing mappings given by the sectorial normalization theorem. The set $H = (H_1, \ldots, H_N)$ is called a normalizing cochain. In the intersection of the sectors $S_j \cap S_{j+1}$ let

$$\Phi_j = H_j^{-1} \circ H_{j+1}, \ (\delta H)_j = H_{j+1} - H_j.$$

The sets

$$\Phi = (\Phi_1, \ldots, \Phi_N), \ \delta H = ((\delta H)_1, \ldots, (\delta H)_N)$$

are called the superposed and difference coboundaries of the cochain H, respectively.

LEMMA 1. The difference coboundary of the normalizing cochain satisfies the upper bound

$$|\delta H| < \exp(-C |t|^s)$$

for some $C > 0$ depending on the cochain. The bound of the set signifies a simultaneous bound of all functions of the set.

Let us prove at first that the correction of the superposed coboundary of the normalizing cochain satisfies the bound of Lemma 1. Let us note that it follows from the sectorial normalization theorem that $|\Phi - \mathrm{id}|$ decreases faster than any power. By definition (2.2) of the Stokes operator

$$\Phi_j = W C_j W^{-1}.$$

Let $a_{k\ell}$ and $c_{k\ell}$ be the elements situated in the kth row and ℓth column of the matrices Φ_j and C_j, respectively. Then

$$a_{kl}(t) = c_{kl} \exp(q_k(t) - q_l(t)),$$

see formula (2.2). The decrease of $|\Phi - \mathrm{id}|$ as $t \to 0$ imposes the following constraints on the elements $c_{k\ell}$: $c_{kk} = 1$; $c_{k\ell} \neq 0$ $\exp(q_k - q_\ell) \to 0$ as $t \to 0$ in $S_j \cap S_{j+1}$. By hypothesis, Eq. (1.1) is nonresonant, that is $\lambda_k \neq \lambda_\ell$ for $k \neq \ell$. Therefore, the last requirement is equivalent to the fact that in $S_j \cap S_{j+1}$

$$\mathrm{Re}\,(\lambda_k - \lambda_l)/t^s \to -\infty.$$

It is exactly essential here that Eq. (1.1) be nonresonant. Then in $S_j \cap S_{j+1}$

$$\mathrm{Re}\,(q_k - q_l) < -c |t|^{-s}$$

for some $c > 0$. Hence,

$$|\Phi - \mathrm{id}| < \exp(-c|t|^{-s}).$$

The required bound on the superposed coboundary is obtained.

Let us now note that the correction of the superposed coboundary is of the same order of smallness as the difference one. This follows from the fact that for all linear operators $\mathbf{C}^n \to \mathbf{C}^n$ from a sufficiently small neighborhood of the origin, we have

$$\|A - B\| \leqslant 2\|(E + A)\circ(E + B)^{-1} - E\|.$$

This proves the lemma. ▷

Remark. The operator-valued functions of the set, forming a normalizing cochain, serve in the nature of an analytic continuation of each other. This continuation is good in that all the functions of the set have identical asymptotics. An analogous continuation is possible for any functions from $\mathcal{K}_S$. In order to construct it, let us give the following definition, central in this article. Let us fix a covering U of the punctured disk, good for Eq. (1.1).

Definition 1. By the regular functional cochain, corresponding to a covering U, is meant the set of holomorphic functions $F = \{F_1, \ldots, F_N\}$, bijectively corresponding to the sectors of the covering, where:

1) each function of the set is determined in the corresponding sector and admits an analytic continuation in some wider sector containing a closure of the original one without the origin;

2) the cochain coboundary is the set of differences $(\delta F)_j = F_{j+1} - F_j$, $\delta F = ((\delta F)_1, \ldots, (\delta F)_N)$, considered in the sectors $S_j \cap S_{j+1}$ and satisfying there the bound

$$|\delta F| < \exp(-C|t|^{-s}),$$

C depends on F;

3) all the functions of the set F are expanded into the same asymptotic Laurent series at the origin with a pole of finite order.

The preceding lemma shows that the matrix elements of the normalizing cochains are regular functional cochains; it is necessary only to observe that the sectors of a good covering admit the extension described in Sec. 1 of Definition 1, where the covering remains good.

The set of all regular functional cochains corresponding to a covering U is denoted $\mathcal{FC}_U$ ($\mathcal{FC}$ from functional cochains). The arithmetic actions over the cochains from $\mathcal{FC}_U$, as differentiation also, are conducted "componentwise," as the actions over the functions of the sets defined in the same sector.

LEMMA 2. The regular functional cochains corresponding to a single covering, form an algebra. ▷

The proof immediately follows from Definition 1. ▷

Remark. One can prove more: one can replace the "algebra" in Lemma 2 by a "differential field." However, this assertion is not needed in what follows.

Let us note in conclusion that the functions meromorphic in the unit disk with a unique pole at the origin are those regular functional cochains with a trivial (null) coboundary.

4. Relations in a Field Generated by the Components of the Solutions

In this section, the functions of the field $\mathcal{K}_S$ are expressed in terms of regular functional cochains and components of the solutions of the normalized system. This allows one to describe the relations in $\mathcal{K}_S$ and complete the proof of Theorem 2 with the use of the Phragmén-Lindelöf Theorem for cochains proved in Sec. 5.

The relations in $\mathcal{K}_S$ do not depend on the sector S; for different sectors, they are obtained from each other by analytic continuation. Therefore, for subsequent arguments, one can choose and fix S. Let U be a good covering of the punctured disk described in Sec. 2, and S be an arbitrary sector of this covering. As already noted, the sectorial normalization theorem allows one to distinguish a special fundamental matrix of solutions of (1.1) of the form

$$Z_S = H_S W,$$

where W is given by (2.2), and H_S is a normalizing substitution.

Formula (1.4) assumes the form

$$\sum a_K (H_S W)^K |_S \equiv 0. \tag{4.1}$$

The operator-valued function H_S is continued in the punctured disk to the normalizing chain H. By Lemma 2, the left part of relation (4.1) is continued in the punctured disk to an expression of the form

$$\sum a_K (HW)^K = \sum F_k w^k, \ F_k \in \mathcal{F}\mathcal{C}_U. \tag{4.2}$$

Here F_k are regular functional cochains, $w = (w_1, \ldots, w_n)$ is given by (2.2), k is a multi-index: $k = (k_1, \ldots, k_n) \in Z_+^n$, $w^k = w_1^{k_1} \ldots w_n^{k_n} = \exp(q, k)$, where $q = q_1, \ldots, q_n)$, q_j is given by (2.2). Thus, relation (4.1) assumes the form

$$\sum F_k w^k |_S \equiv 0, \ F_k \in \mathcal{F}\mathcal{C}_U.$$

Let us move immediately now to the proof of Theorem 2. Let S_j be the j-th sector of the covering U, $S = S_1$, γ_j, C_{S_1, S_j, γ_j} be the same as at the end of Sec. 2. To prove that the Stokes operator $C = C_{S_1, S_j, \gamma_j}$ preserves (4.1), is to prove the implication

$$(4.1) \Rightarrow \sum a_K (H_{S_1} WC)^K |_{S_1} \equiv 0. \tag{4.3}$$

By the definition of the Stokes operator (2.4)

$$\Delta_{\gamma_j} H_{S_1} WC = H_{S_j} \Delta_{\gamma_j} W.$$

Therefore, (4.3) follows from the implication

$$(4.1) \Rightarrow \sum a_K (HW)^K \equiv 0 \tag{4.4}$$

on the universal covering over the punctured disk, where $H = (H_{S_1}, \ldots, H_{S_N})$ is a normalizing cochain.

Let us prove (4.4). Without restricting generality, one can consider that in (4.2) the monomials w^k for different k do not differ by a factor t^m for integral m; in the opposite case, it is necessary to multiply the coefficient for one of the monomials by t^m and reduce similar terms.

LEMMA 1. Let $J \in Z_+^n$ be an arbitrary finite subset and no two monomials from the set $\{w^k | k \in J\}$ differ by a factor t^m for integral m. Then these monomials are independent over the ring $\mathcal{F}\mathcal{C}_U |_S$, where S is a sector of the covering U.

Let us assume the contrary. Let

$$\sum_{k \in J} F_k w^k |_S \equiv 0, \quad F_k \in \mathcal{F}\mathcal{C}_U, F_k \not\equiv 0. \tag{4.5}$$

In formula (2.2) let

$$q_j(t) = p_j(1/t) + \mu_j \ln t, \ \mu = (\mu_1, \ldots, \mu_n),$$

p_j are polynomials without a free term. Let $P(t) = (p_1(1/t), \ldots, p_n(1/t))$. Then

$$w^k = t^{(\mu, k)} \exp(P, k).$$

Case 1. All the exponents (P, k), $k \in J$ are identical. Let us divide both parts of (4.5) by exp(P, k). We obtain

$$\sum_{k\in J} F_k t^{(\mu, k)}|_S \equiv 0. \tag{4.6}$$

This equation is possible only for $F_k \equiv 0$ for each $k \in J$, as will be proved in considering Case 2.

Case 2. Among the exponents (P, k), $k \in J$, let there be different ones. Let us take a radius ℓ of the sector S, so that on it the difference of the real parts of any two of these different exponents in modulo approach infinity; it is then bounded in modulo below by $\varepsilon/|t|$ for some $\varepsilon > 0$. Let (P, k_0) be the exponent with the fastest growing real part in ℓ as $t \to 0$. Let us divide both parts of (4.5) by (P, k_0) and transfer to the right part all the terms with exponents (P, k), with (P, k_0) different from zero. We obtain

$$\sum_{J'} F_k t^{(\mu, k)}|_l = o(\exp(-\varepsilon/|t|)). \tag{4.7}$$

Here $J' = \{k \in J | (P, k) = (P, k_0)\}$. Equation (4.6) is a particular case of (4.7); let us prove it follows from (4.7) that $F_k \equiv 0$ for $k \in J'$. This will contradict the assumption (4.5).

The asymptotic series of the left part of (4.7) in ℓ equals a linear combination of asymptotic Taylor series $\hat{F}_k$ for the cochains F_k with coefficients $t^{(\mu,k)}$. It follows from (4.7) that this combination is the null series. By the hypothesis of the lemma, the exponents (μ, k) do not have integral differences. Consequently, for all $k \in J'$, $\hat{F}_k \equiv 0$. Hence, it follows from the Phragmén–Lindelöf Theorem proved in Sec. 5 that $F_k \equiv 0$ – a contradiction. ▷

It follows from (4.1) that in Eq. (4.2) $\Sigma F_k w^k|_S \equiv 0$. It follows from Lemma 1 that $\Sigma F_k w^k \equiv 0$ on the entire universal covering over the punctured disk. It hence follows that

$$\sum a_k (HW)^K \equiv 0,$$

that proves the implication (4.4). This concludes the proof of Theorem 2, and with it Theorem 1 also, modulo of the Phragmén–Lindelöf Theorem.

5. The Phragmén–Lindelöf Theorem for Regular Functional Cochains

THEOREM 4. A regular functional cochain decreasing along some radius faster than any power as $t \to 0$, identically equals zero.

Remark. This theorem is a particular case of the Phragmén–Lindelöf Theorem for what are called simple functional cochains, as proved by one of the authors in the article in press, "Finiteness theorems for limit cycles 1." The theorem formulated above is the simplest in the series, and its complete proof is given here, according to the model of which, the remaining Phragmén–Lindelöf Theorems for cochains are proved.

Let F be a cochain satisfying the condition of the theorem. Let us write this cochain in the chart $\zeta = 1/t$. A good covering of the punctured disk passes to a covering of a neighborhood of infinity by sectors of an opening larger than π/s; this is the only information about the covering used in what follows. The cochain $\tilde{F} = F(1/\zeta)$ is a set of functions holomorphic in 1-neighborhoods of the sectors of the covering if the cochain is considered on the exterior of a sufficiently large disk. All the functions of the set are expanded into a general asymptotic Laurent series with a pole of finite order at ∞. The coboundary of a cochain is bounded above by the function $m_c = \exp(-c|\zeta|^s)$ for some $c > 0$. Let us move from the covering to the decomposition. Namely, for any $j \in \{1, \ldots, N\}$ let us take a ray $\ell_j \subset S_j \cap S_{j+1}'$, directed to infinity and such that the rays ℓ_j meet in a circuit of infinity in the order of their numbers. We consider the neighborhood of infinity to be so small that 2-neighborhoods of the rays do not pairwise intersect. Let us denote by L the union $L = \cup \ell_j$. The rays ℓ_j break a neighborhood of infinity into sectors; sector $\tilde{S}_j$ between the rays ℓ_j and ℓ_{j+1} belongs to the good sector S_j.

LEMMA 1 (on Trivialization of a Cocycle). Let F be a regular functional cochain in a neighborhood of infinity, the coboundary of which is bounded above by a function m, and in Ω_{a-1}: $|\zeta| \geq a - 1$ let

$$\max_{\Omega_{a-1}} m \leqslant m_0, \quad \int_{L\cap\Omega_{a-1}} m\,ds \leqslant I.$$

Then there exists a functional cochain Φ with the same coboundary on L, defined in Ω_a and satisfying

$$|\Phi| \leqslant m_0 + I.$$

The trivialization lemma is proved with the help of an explicit formula. On each ray ℓ_j let us take as the function of the set δF the difference $F_{j+1} - F_j$. Let

$$\Phi(\zeta) = \frac{1}{2\pi i}\int \frac{\delta F(\tau)}{\tau - \zeta}\,d\tau.$$

By a theorem of Plemelj [7] $\delta\Phi = \delta F$ on $L \cap \Omega_{a-1}$. Let us bound $|\Phi(\zeta)|$ in Ω_a. Each of the functions of the set δF is continued analytically to a 1-neighborhood of the corresponding ray, and is there bounded above by m. Let us consider two cases.

1. $\mathrm{dist}(\zeta, L) \geq 1$. Then $|\Phi(\zeta)| \leq I/2\pi$.

2. $\mathrm{dist}(\zeta, L) \leq 1$. Let ζ belong to a 1-neighborhood of ℓ_j. The disk D with center ζ and radius 1 belongs to $S_j \cap S_{j+1}$. In the formula for Φ let us replace the integral over the chord $\ell_j \cap D$ by an integral over an arc of dD with the same ends, which is separated in D by the chord from ζ. The integrals over such arcs are bounded above by a constant m_0, the integral over the remaining part of the contour is a constant $I/2\pi$. This proves the lemma. ▷

LEMMA 2 (the Maximum Principle for Functional Cochains). In the conditions of Lemma 1, let the regular functional cochain F' be bounded by some ray ℓ: $\arg\zeta = \mathrm{const}$. Then it is bounded in the neighborhood of infinity and

$$\sup_{\Omega_a}|F| \leqslant \sup_{\partial\Omega_a}|F| + 2(m_0 + I),$$

where m_0 and I are the same as in Lemma 1.

Let Φ be the trivialization of δF given by Lemma 1. Then the difference $F - \Phi$ is a holomorphic function in Ω_a. Since it grows no faster than some power, it is meromorphically continued to infinity. Since it is bounded on the ray ℓ, this continuation is indeed holomorphic. By the maximum principle for holomorphic functions

$$\sup_{\Omega_a}|F - \Phi| = \sup_{\partial\Omega_a}|F - \Phi|.$$

Lemma 2, hence, follows from the bound on $|\Phi|$ given by Lemma 1. ▷

Let us directly move to proving the Phragmén–Lindelöf Theorem for cochains. One can consider the cochain F as bounded: in the opposite case, one can multiply it by a suitable power ζ and make it bounded, not violating the conditions of the theorem. In Ω_a: $|\zeta| \geq a$ let us consider the cochain

$$F_{\lambda,a} = F(\zeta/a)^{\lambda},\ \lambda > 0.$$

Let us verify the conditions of Lemma 2 for the cochain $F_{\lambda,a}$ in order to apply to it the maximum principle. The cochain F_λ is regular and bounded on ℓ by the condition of the Phragmén–Lindelöf Theorem. Its coboundary satisfies the bound

$$|\delta F_{\lambda,a}| < m_{\lambda,a} \overset{\text{def}}{=} m_c (r/a)^{\lambda} = (r/a)^{\lambda}\exp(-cr^s),\ r = |\zeta|.$$

Let us bound the constants $m_0 = \max\limits_{\Omega_{a-1}} m_{\lambda,a}$ and I:

$$I = \int_{L\cap\Omega_{a-1}} (r/a)^{\lambda}\exp(-cr^s) = N\int_{a-1}^{\infty} m_{\lambda,a}\,dr.$$

It is easy to bound this integral if one ensures the inequality

$$m'_{\lambda,a}/m_{\lambda,a} \leqslant -1 \text{ for } r \geqslant a-1. \tag{5.1}$$

It follows from (5.1) that

$$m_{\lambda,a}(r) \leqslant (m_{\lambda,a}(a-1)) \exp(-r+(a-1)),$$
$$m_0 \leqslant 1,\ I \leqslant N\, m_{\lambda,a}(a-1)$$

for a > 1, since $(a - 1/a)^{\lambda} < 1$ and $\exp(-c\ (a-1)^s) < 1$. Let us note that

$$m'_{\lambda,a}/m_{\lambda,a} = (\lambda \ln r - cr^s)' = \frac{\lambda}{r} - csr^{s-1}.$$

Inequality (5.1) is satisfied if

$$\lambda \leqslant cs\,(a-1)^s/2.$$

Then by Lemma 2, considering $|F| \leq 1$, we obtain

$$|F_{\lambda,a}| \leqslant 2(N+1)+1 = C_1$$

Consequently, $|F(\zeta)| < c_1(r/a)^{-\lambda}$ for $\lambda \leq cs(a-1)^s/2$. Let us now take an arbitrary ζ: $|\zeta| > e$ and let $a = |\zeta|/e$, $\lambda = cs(a-1)^s/2$. Then

$$|F(\xi)| < C_1 \exp(-C_2 r^s)$$

for some $C_2 > 0$. But the functions of the set F are determined in sectors of an opening wider than π/s. Consequently, due to the classical Phragmén–Lindelöf Theorem [8] $F \equiv 0$.

CONCLUSION

Let us consider a linear nonautonomous system on the Riemann sphere with a finite number of singular points

$$\dot{z} = A(t)\,z, \tag{*}$$

A is a rational operator-valued function. The points at which the poles of A are simple, are called Fuchsian, and the remaining poles of A – non-Fuchsian singular points; the point ∞ is Fuchsian if $\lim_{t\to\infty} tA(t) = A_\infty \neq 0$. The system (*) with single Fuchsian singular points is called Fuchsian. For a general system (*) for each of the non-Fuchsian singular points, let us fix a sector with a vertex at this point and in the solution space on it define the Stokes operators. Then let us fix an arbitrary nonsingular point t_0 and unite its curves with the points of the fixed sectors. This allows one to determine the Stokes operators in the solution space of (*) considered near t_0. The Galois group of (*) is defined the same as in Sec. 1, only M is the field of rational functions with poles just at the poles of A.

THEOREM 6. The Stokes operators of equation (*), the non-Fuchsian singular points of which are nonresonance, belong to the Galois group of this equation.

This theorem is proved exactly the same as the preceding one; the change of the field M does not play a role. Apparently, the following strengthening of Theorem 1 holds:

THEOREM 1 bis. Let $\tilde{M}$ be the field obtained from the field of functions meromorphic in the unit disk, by the union of the components of all solutions of the normalized system (2.1). Then the Galois group of Eq. (1.1) (formally equivalent to Eq. (2.1) over the field M is the algebraic closure of the subgroup of GL(n, C), generated by the Stokes and monodromy operators of Eq. (1.1).

In conclusion, let us formulate two problems.

Problem 1. How is the solvability in quadratures of equation (*) connected with its Stokes and monodromy operators?

For the Fuchsian system (*), the solvability of the monodromy group is equivalent to the solvability of the equation in quadratures [9].

Problem 2. Generalize Theorem 1 to the resonance case.

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QUANTIZATION OF FINITE-GAP POTENTIALS AND NONLINEAR QUASICLASSICAL APPROXIMATION IN NONPERTURBATIVE STRING THEORY

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In recent articles [1-3], and ending with [4], there has been discovered a remarkable circumstance resulting from the combinatorial Kazakov—Migdal—Kostov approach in the continuous limit. In cases of nonperturbative conformal string theories interacting with a "two-dimensional gravitation" according to Polyakov's scheme, as well as some other ones, when the central charge is $c < 1$, there appears a simple system of equations for the renormgroup, i.e., a set of Laks type equations with certain ordinary differential operators in x:

$$\frac{\partial L}{\partial t_k} = [L_1 A_k]. \tag{1}$$

Equations (1) are studied for the following boundary conditions:

$$[L, A] = \varepsilon \cdot 1, \tag{2}$$

where ε is a quantum constant significant for our method.

Equations of type (1), (2) for $\varepsilon = 0$ have well-known finite-gap and multisoliton solutions; they are completely integrable Hamiltonian systems and can be exactly solved with θ-functions on Riemann surfaces (see [5-7]).

Definition. Equation (2) is called a quantization of finite-gap potentials.

The simplest case is where we have a second-order scalar operator $L = -\partial_x^2 + u$, and A is an operator of odd degree. All such operators A are well-known in the theory of the Korteweg—de Vries (KdV) equation. In absolutely the simplest case

$$L = -\partial_x^2 + u,\ A = -4\partial_x^3 + 6u\partial_x + 3u' \tag{3}$$

the study of Eq. (2) is a rather complicated task, and from the naive point of view it is

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