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An Analog of Determinant Related to Parshin–Kato Theory and Integer Polytopes*

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Abstract. Parshin–Kato theory involves a multilinear function of $n+1$ vectors in the n-dimensional vector space over the field $\mathbb{Z}/2\mathbb{Z}$. The same function arises in the computation of the product in the group $(\mathbb{C}^*)^n$ of all roots of several polynomial equations with sufficiently generic Newton polytopes. We discuss this remarkable function and related geometry of integer polytopes.

Key words: analog of determinant, Parshin–Kato theory, integer polytope.

The determinant of n vectors in the n-dimensional vector space L^n over the field $\mathbb{Z}/2\mathbb{Z}$ is the only nonzero multilinear function of n vectors that ranges in the field $\mathbb{Z}/2\mathbb{Z}$, is invariant under all linear transformations, and is zero whenever the rank of the n vectors is less than n. There exists a unique function of $n+1$ vectors in L^n with exactly the same properties. This function arises in the computation of the product in the group $(\mathbb{C}^*)^n$ of all roots of a system of n polynomial equations with sufficiently generic Newton polytopes [1] as well as in Parshin–Kato theory [2, 3]. In this article, we discuss this remarkable function.

The determinant of a matrix A over the field of real numbers is the volume of the oriented parallelepiped spanned by the columns of A. Does the analog of determinant for $n+1$ vectors in the n-dimensional space over the field $\mathbb{Z}/2\mathbb{Z}$ compute the volume of some figure? In this paper, the positive answer to this question is given. It is closely related to geometry of integer polytopes.**

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1. An Analog of Determinant for $n+1$ Vectors in n-Dimensional Space over the Field $\mathbb{Z}/2\mathbb{Z}$

In this section, we describe linear algebra related to an analog of determinant. We start from a definition.

Definition. Let D be a function of $n+1$ vectors in the n-dimensional vector space over the field $\mathbb{Z}/2\mathbb{Z}$ ranging in the field $\mathbb{Z}/2\mathbb{Z}$ and determined by the following properties: the value $D(k_1,\dots,k_{n+1})$ of the function D is equal to

(a) zero if the rank of the vectors $k_1,\dots,k_{n+1}$ is less than n;

(b) $\lambda_1+\dots+\lambda_{n+1}+1$ if the vectors $k_1,\dots,k_{n+1}$ satisfy the single relation $\lambda_1k_1+\dots+\lambda_{n+1}k_{n+1}=0$.

Lemma 1. *The function D has the following properties*:

(1) *It is $GL_n(\mathbb{Z}/2\mathbb{Z})$-invariant; i.e., $D(k_1,\dots,k_{n+1})=D(Ak_1,\dots,Ak_{n+1})$ for every linear transformation $A\in GL_n(\mathbb{Z}/2\mathbb{Z})$.*

(2) *It vanishes on $(n+1)$-tuples of vectors $k_1,\dots,k_{n+1}$ whose rank is less than n.*

(3) *It is multilinear.*

Proof. Properties (1) and (2) follow readily from the definition of D. To prove property (3), it suffices to show that for any given vectors $k_1,\dots,k_n$ the function $\varphi(k)=D(k_1,\dots,k_n,k)$ is linear.

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**As A. N. Parshin told me, he had not been aware of the simple definition of the function D discussed below and geometry of integer polytopes related to this function, although the same function D appears in the symbols invented by him.

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The rank of the system $k_1, \dots, k_n$ can be n, $n-1$, or less than $n-1$. Consider these three cases separately.

1. The rank is n. In this case, the vectors $k_1, \dots, k_n$ form a basis in L^n, and each vector k is uniquely represented as a linear combination $k = \lambda_1 k_1 + \dots + \lambda_n k_n$. Then $\varphi(k) = D(k_1, \dots, k_n, k) = \lambda_1 + \dots + \lambda_n$ is a linear function of k.

2. The rank is $n-1$. In this case, the function φ vanishes on the hyperplane Λ spanned by the vectors $k_1, \dots, k_n$ and is constant on the complement of this hyperplane. Indeed, if $k \in \Lambda$, then the rank of the system $k_1, \dots, k_n$, k is less than n and $D(k_1, \dots, k_n, k) = 0$. If $k \notin \Lambda$, then the vectors satisfy a single relation, which is independent of k. Hence the function φ is constant on the complement of Λ. Clearly, a function on L^n with this property is linear.

3. The rank is less than $n-1$. In this case, φ is zero identically and hence is linear.

Lemma 2. *There exists a unique nonzero function D satisfying properties* (1)–(3) *in Lemma* 1.

Proof. To define a multilinear function, it suffices to specify its values on all tuples $e_{i_1}, \dots, e_{i_{n+1}}$ of vectors in the standard basis $e_1, \dots, e_n$. It follows from property (2) that the function can be nonzero only if all but two vectors in the tuple are distinct. It follows from property (1) that D takes the same value on all such tuples. If this value is zero, then D is zero. The only other possibility is that the value is equal to one. This corresponds to the function D defined above, which is indeed $GL_n(\mathbb{Z}/2\mathbb{Z})$-invariant by Lemma 1.

Lemma 3. *The function D is given in coordinates by the formula*

$$D(k_1, \dots, k_{n+1}) = \sum_{j>i} \det_{ij},$$

where $\det_{ij}$ is the determinant of the $n \times n$-matrix whose first $n-1$ rows are formed by the sequence of vectors $k_1, \dots, k_{n+1}$ with the ith and jth vectors removed and whose last row is the coordinatewise product of k_i and k_j.

Proof. The function $\sum_{j>i} \det_{ij}$ is a multilinear function of the vectors $k_1, \dots, k_{n+1}$. It obviously coincides with D on tuples of standard basis vectors.

Let us give yet another formula for D. Let $\widetilde{A}$ be the $n \times (n+1)$ matrix whose rows are the vectors $k_1, \dots, k_{n+1}$, and let $\det_i$ be the determinant of the matrix obtained from $\widetilde{A}$ by deleting the ith row.

Lemma 4. *The vectors $k_1, \dots, k_{n+1}$ satisfy the relation*

$$\sum (-1)^{i-1} \det_i k_i = 0.$$

Lemma 4 is a simple fact of linear algebra.

Lemma 5. *The function $\prod_{1 \leqslant i \leqslant n+1} (1 + \det_i)$ is zero if the vectors $k_1, \dots, k_{n+1}$ span L^n. Otherwise, it is equal to one.*

Proof. The vectors $k_1, \dots, k_{n+1}$ span L^n if and only if at least one of the subdeterminants $\det_i$ is nonzero (and hence is equal to one).

Theorem 1.

$$D(k_1, \dots, k_{n+1}) = 1 + \det_1 + \dots + \det_{n+1} + \prod_{1 \leqslant i \leqslant n+1} (1 + \det_i).$$

Proof. If the vectors $k_1, \dots, k_{n+1}$ do not span L^n, then all subdeterminants $\det_i$ are zero and $D = 1 + 1 = 0$. If $k_1, \dots, k_{n+1}$ span the entire space, then they satisfy the single relation $\sum \det_i k_i = 0$ (Lemma 4). In this case, $\prod (1 + \det_i) = 0$ (Lemma 5). By the definition of D

$$D(k_1, \dots k_{n+1}) = 1 + \det_1 + \dots + \det_{n+1}.$$

The proof is complete.

2. The Volume of a Chain in an $(n+1)$-Dimensional Torus whose Boundary is a Sum of Rational n-Dimensional Tori

2.1. In Sec. 2, we discuss geometry related to the function D. Let us say a couple of words about the results.

Let $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$ be the integer lattice in $\mathbb{R}^{n+1}$, and let $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ be the real $(n+1)$-dimensional torus. We give a geometric definition of a real-valued function J on the space of n-dimensional cycles homologous to zero in T^{n+1} (see Sec. 2.2). This function is defined up to adding an integer. (It can be viewed as the antiderivative of the volume form on the torus.) With each n-tuple $\boldsymbol{a} = (a_1, \dots, a_n)$ of integer vectors $a_i \in \mathbb{Z}^{n+1}$, we associate an n-dimensional cycle $C(\boldsymbol{a})$ homologous to zero in T^{n+1} (see Sec. 2.2).

Consider an $n \times (n+1)$-matrix A whose columns are integer vectors $a_1, \dots, a_n$. Let $k_1, \dots, k_{n+1}$ be the rows of the matrix $\widetilde{A}$ over $\mathbb{Z}/2\mathbb{Z}$ obtained from A by reduction modulo 2. The value of the function $2J$ on $C(\boldsymbol{a})$ lies in $\mathbb{Z}/2\mathbb{Z}$ (see Theorem 2) and coincides with the value of the function D on the tuple $k_1, \dots, k_{n+1}$ (see Theorem 4). This statement gives a clear geometric meaning to the function D. As a by-product, we obtain some new results in geometry of integer polytopes (see Theorems 3 and 3′).

Note that a similar geometric construction not only explains the geometric meaning of the signs arising in Parshin–Kato theory but also provides a geometric definition for the "cohomology classes" (in the sense of this theory) of analytic varieties with coefficients in $\mathbb{C}^*$ and for some generalizations of these classes (see [4]).

2.2. Notation and statements of theorems. Let $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ be the standard $(n+1)$-dimensional torus equipped with an orientation and the standard volume form ω, $\int_{T^{n+1}} \omega = 1$. We define a function J on the space of piecewise smooth n-dimensional cycles in T^{n+1} homologous to zero.

Definition. The value $J(C)$ of the function J on an n-dimensional cycle C homologous to zero is the volume $\int_{\sigma_{n+1}} \omega$ of any chain σ_{n+1} such that $\partial\sigma_{n+1} = C$. The chain σ_n is defined modulo integer multiples of the fundamental cycle of T^{n+1}; hence $J(C)$ is a well-defined element of the group $\mathbb{R}/\mathbb{Z}$.

An affine subspace in $\mathbb{R}^{n+1}$ is said to be *rational* if it is the affine hull of some subset of the integer lattice $\mathbb{Z}^{n+1}$. In each k-dimensional rational vector subspace $K \subseteq \mathbb{R}^{n+1}$, one defines the *integer k-dimensional volume* V_k. This is a translation-invariant volume function normalized by the condition $V_k(\Delta) = 1$, where Δ is a k-dimensional parallelepiped whose edges form a basis of the lattice $K \cap \mathbb{Z}^{n+1}$. The canonical projection $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}/\mathbb{Z}^{n+1} = T^{n+1}$ takes K to a rational k-dimensional torus. On this torus, V_k induces a translation-invariant volume function such that the volume of the entire torus is equal to one. Parallel translations in $\mathbb{R}^{n+1}$ allow one to extend the definition of integer volume to rational affine subspaces.

With an n-tuple $\boldsymbol{a} = (a_1, \dots, a_n)$ of integer vectors $a_i \in \mathbb{Z}^{n+1}$, we associate the following objects: the *oriented parallelepiped* $\Pi[\boldsymbol{a}] \subset \mathbb{R}^{n+1}$ spanned by vectors $a_1, \dots, a_n$; the integer n-dimensional *volume* $V_n[\boldsymbol{a}]$ *of the parallelepiped* $\Pi[\boldsymbol{a}]$; *the chain* $T[\boldsymbol{a}]$ in T^{n+1} that is the image of $\Pi[\boldsymbol{a}]$ under the projection $\mathbb{R}^{n+1} \to T^{n+1}$; the *multivector* $v[\boldsymbol{a}] = a_1 \wedge \dots \wedge a_n$.

The chain $T[\boldsymbol{a}]$ is a cycle. If $v[\boldsymbol{a}]$ is nonzero, then this chain coincides with $V_n[\boldsymbol{a}]$ times the fundamental cycle of the n-dimensional torus, the image under the projection $\mathbb{R}^{n+1} \to T^{n+1}$ of the oriented n-dimensional subspace containing $\Pi[\boldsymbol{a}]$. If $v[\boldsymbol{a}] = 0$, then $T[\boldsymbol{a}]$ is a cycle homologous to zero and hence the boundary of a chain whose dimension is less than $n+1$.

Let $\boldsymbol{a}_j$, $j = 1, \dots, m$, be n-tuples of integer vectors. Clearly, *the sum $v[\boldsymbol{a}_1] + \dots + v[\boldsymbol{a}_m]$ of multivectors $v[\boldsymbol{a}_j]$ is zero if and only if the sum $T[\boldsymbol{a}_1] + \dots + T[\boldsymbol{a}_m]$ of the cycles $T[\boldsymbol{a}_j]$ is zero in the group $H_n(T^{n+1}, \mathbb{Z})$.*

The central result of this section is the following.

Theorem 2. *Suppose that* $v[\boldsymbol{a}_1] + \cdots + v[\boldsymbol{a}_m] = 0$*. Then the value* $J(C)$ *of the function* J *on the cycle* $C = T[\boldsymbol{a}_1] + \cdots + T[\boldsymbol{a}_m]$ *is a half-integer. The integer* $2J(C)$ (*which is only defined modulo* 2) *satisfies the congruence* $2J(C) \equiv (V_n[\boldsymbol{a}_1] + \cdots + V_n[\boldsymbol{a}_m]) \mod 2$.

We shall prove Theorem 2 simultaneously with Theorem 3 below, which belongs in geometry of integer polytopes. We say that an n-dimensional face Γ of a convex $(n+1)$-dimensional polytope is *cancelable* if this polytope has another face equal to Γ up to a parallel translation.

Definition. A convex $(n+1)$-dimensional polytope is said to be of *parallelepipedal type* if each of its noncancelable n-dimensional faces is a parallelepiped.

Theorem 3. *Let a convex* $(n+1)$*-dimensional integer polytope* Δ *be of parallelepipedal type. Then its doubled integer* $(n+1)$*-dimensional volume is an integer of the same parity as the sum of integer* n*-dimensional volumes of all faces of* Δ *that are parallelepipeds.*

The integer volume of an integer parallelepiped is an integer. If a pair of cancelable faces of a polytope is a pair of parallelepipeds, then the sum of their volumes is even. Hence the volumes of cancelable pairs of parallelepipeds in Theorem 3 can be disregarded. Theorem 3 has a generalization that applies to a wider class of polytopes (see Theorem 3′ in Sec. 2.6).

Let $e_1, \dots, e_{n+1}$ be the standard basis in $\mathbb{R}^{n+1}$, and let $v[\boldsymbol{e}_i]$ be the wedge product of all vectors $e_1, \dots, e_{n+1}$ except for e_i. Let $\boldsymbol{a}$ be an ordered n-tuple $(a_1, \dots, a_n)$ of integer vectors $a_i \in \mathbb{Z}^{n+1}$, and let $v[\boldsymbol{a}] = \sum M_i v[\boldsymbol{e}_i]$ be the expansion of the multivector $v[\boldsymbol{a}]$ in the standard basis of multivectors. Denote by $C(\boldsymbol{a})$ the cycle $C(\boldsymbol{a}) = T[\boldsymbol{a}] - \sum M_i T[\boldsymbol{e}_i]$, which is homologous to zero.

Theorem 4. *The value of the function* $2J$ *on the cycle* $C(\boldsymbol{a})$ *is a well-defined element of the field* $\mathbb{Z}/2\mathbb{Z}$ *and coincides with the function* D *applied to the vectors* $k_1, \dots, k_{n+1}$ *that are the rows of the* $n \times (n+1)$*-matrix* $\widetilde{A}$ *obtained by reduction modulo* 2 *of the matrix* A *whose columns are the vectors* $a_1, \dots, a_n$.

2.3. One-dimensional case. Recall classical Pick's formula.

Pick's formula. *The area* $V_2(\Delta)$ *of an integer polygon* Δ, *the number* $B(\Delta)$ *of its interior integer points, and the integer length of its boundary* $\sum_j V_1(\Delta_j)$ (*here the sum is over all edges* Δ_j *of the polygon* Δ) *are related by*

$$V_2(\Delta) = B(\Delta) + \frac{1}{2}\sum_j V_1(\Delta_j) - 1.$$

Lemma 6. *Theorem* 2 *holds for* $n = 1$.

Proof. For $n = 1$, the multivectors $\boldsymbol{a}_j$ are vectors in the plane. Since the sum of $\boldsymbol{a}_j$ is zero, we can assume that the vectors $\boldsymbol{a}_j$ are the edges Δ_j of some convex polygon Δ that are oriented counterclockwise. One can see from Pick's formula that the doubled area of Δ is an integer that differs from the integer length $\sum_j V_1(\Delta_j)$ of its boundary by the even number $2(B(\Delta) - 1)$. The image of the oriented polygon Δ under the projection $\mathbb{R}^2 \to T^2$ is a two-dimensional chain whose boundary C is equal to $T[\boldsymbol{a}_1] + \cdots + T[\boldsymbol{a}_m]$. Hence $2J(C) \equiv 2V(\Delta) \mod 2$. Clearly, $V_1[\boldsymbol{a}_j] = V_1(\Delta_j)$. The proof of the lemma is complete.

Thus Theorem 2 follows for $n = 1$ from Pick's formula. The volumes of higher-dimensional integer polytopes are not half-integers in general. But an analog of Pick's formula over the field $\mathbb{Z}/2\mathbb{Z}$ still holds for very special higher-dimensional polytopes (see Theorems 3 and 3′ below).

Any convex polygon is of parallelepipedal type. Theorem 3 for polygons also follows from Pick's formula.

2.4. Induction step. In this section, we state and prove the main lemma needed in the proof of Theorems 2 and 3. Let $\Delta_1 + \Delta_2$ denote the Minkowski sum of convex polytopes Δ_1 and Δ_2.

Lemma 7. *Let a segment* I *in an affine space be transversal to a convex* $(n+1)$*-dimensional polytope* Δ. *Then each noncancelable facet of* $\Delta + I$ *has the form* $\Gamma_i + I$, *where* Γ_i *is a noncancelable facet of* Δ. *In particular, if* Δ *is of parallelepipedal type, then so is* $\Delta + I$.

Proof. The polytope $\Delta + I$ contains the following $(n+1)$-dimensional faces: (a) the pair $\Delta + a$, $\Delta + b$ of cancelable faces, where a and b are the endpoints of I; (b) the pairs $\Gamma_1 + I$, $\Gamma_2 + I$ of cancelable faces for each pair Γ_1, Γ_2 of cancelable n-dimensional faces of Δ. All remaining $(n+1)$-dimensional faces of $\Delta + I$ are noncancelable and have the form $\Gamma_i + I$, where the Γ_i are noncancelable n-dimensional faces of Δ.

Main Lemma. *If Theorem* 3 *holds for every integer polytope affinely equivalent to a given* $(n+1)$*-dimensional polytope* Δ*, then it holds for every integer polytope affinely equivalent to the* $(n+2)$*-dimensional polytope* $\Delta + I$*, where* I *is a segment transversal to* Δ*.*

Proof. Let $\mathscr{L}$ and l be the $(n+1)$- and one-dimensional affine rational spaces containing Δ and I, respectively. Without loss of generality, we can assume that $\mathscr{L}$ and l are vector spaces. Their sum $L = \mathscr{L} + l$ is an $(n+2)$-dimensional rational space containing $\Delta + I$.

Step 1. Assume additionally that the lattice in L is the sum of lattices in the spaces $\mathscr{L}$ and l. In this case, the lemma is obvious. Indeed, by the assumptions of the lemma, the integer $(n+1)$-dimensional volume of Δ is a half-integer, and the following congruence holds:

$$2V_{n+1}(\Delta) \equiv \sum_i V_n(\Gamma_i) \mod 2, \tag{$*$}$$

where the sum is over all n-dimensional faces Γ_i of Δ that are parallelepipeds. Under the conditions of step 1, for each rational subspace $K \subset \mathscr{L}$ the lattice in $K + l$ coincides with the sum of lattices in K and l. The integer volume in $K + l$ is the product of integer volumes in K and l. A similar relation between volumes holds for rational affine spaces parallel to K, l, and $K + l$. Multiplying Eq. $(*)$ by the integer length of I, we obtain the assertion of step 1.

Step 2. To reduce the general case to that of step 1, we need the transformation of fiberwise translation. We say that a continuous map $F\colon L \to L$ is a *fiberwise translation in the direction of a line* l if it takes each line parallel to l to itself and if the restriction of F to any such line is a translation (by a vector depending on the choice of the line). Clearly, every fiberwise translation preserves the translation-invariant volume form on L. Suppose that an affine rational subspace $K \subset L$ contains a line parallel to the rational line l. Then K is invariant under the fiberwise translation F, and the restriction of F to K preserves the integer volume in K.

Step 3. In the space $L = \mathscr{L} + l$, take the line l. Choose an integer vector subspace $\mathscr{L}_1 \subset L$ such that $L = \mathscr{L}_1 + l$ and $\Lambda_{n+2} = \Lambda_{n+1} + \Lambda_1$, where Λ_{n+2}, Λ_{n+1}, and Λ_1 are the integer lattices in L, $\mathscr{L}_1$, and l, respectively. There exists a space $\mathscr{L}_1$ with these properties. To construct $\mathscr{L}_1$, consider a primitive vector e_1 of the lattice Λ_1 and include it in a basis $e_1, \dots, e_{n+2}$ of the lattice Λ_{n+2}. For $\mathscr{L}_1$ it suffices to take the vector space spanned by the vectors $e_2, \dots, e_{n+2}$. Consider the projection $\pi\colon \mathscr{L} \to \mathscr{L}_1$ of $\mathscr{L}$ onto $\mathscr{L}_1$ along l. The integer $(n+1)$-dimensional polytope $\pi(\Delta)$ is affinely equivalent to Δ; hence the integer $(n+1)$-dimensional volume of $\pi(\Delta)$ is a half-integer, and the following congruence holds:

$$2V_{n+1}(\pi(\Delta)) \equiv \sum_i V_n(\pi(\Gamma_i)) \mod 2,$$

where the sum is over all n-dimensional faces $\pi(\Gamma_i)$ of $\pi(\Delta)$ that are n-dimensional parallelepipeds. According to step 1,

$$2V_{n+2}(\pi(\Delta) + I) \equiv \sum_i V_{n+1}(\pi(\Gamma_i) + I) \mod 2.$$

We can now define an affine fiberwise translation $F\colon L \to L$ along l taking $\Delta + I$ to $\pi(\Delta) + I$. To this end, on each line λ parallel to l we mark the points $x(\lambda) = \lambda \cap \mathscr{L}$ and $y(\lambda) = \lambda \cap \mathscr{L}_1$. Define F as the map whose restriction to the line λ is the translation by the vector $y(\lambda) - x(\lambda)$. It is clear that $F(\Delta) = \pi(\Delta)$, $F(\Delta + I) = \pi(\Delta) + I$, $F(\Gamma_i) = \pi(\Gamma_i)$, and $F(\Gamma_i + I) = \pi(\Gamma_i) + I$. According to step 2, the general case follows from the case considered at step 1.

2.5. Proofs of Theorems 2 and 3. To conclude the proof of Theorem 2, we use the main lemma in the form of the following corollary.

Corollary. *Theorem 3 holds for the polytopes $\Delta_{n+1} = \Delta_2 + \Pi$, where Δ_2 is an integer polygon and Π is an $(n-1)$-dimensional integer parallelepiped transversal to Δ_2. All n-dimensional faces of the polytope Δ_{n+1}, except for some pairs of cancelable faces, have the form $a_i + \Pi$, where a_i is a side of Δ_2.*

Proof of Theorem 2. 1) Let us define an elementary relation between multivectors as a relation of the form $a \wedge a_2 \wedge \cdots \wedge a_n + b \wedge a_2 \wedge \cdots \wedge a_n + c \wedge a_2 \wedge \cdots \wedge a_n = 0$, where $a, b, c, a_2, \dots a_n$ are any integer vectors such that $a+b+c = 0$. It suffices to verify Theorem 2 for elementary relations. Indeed, if a relation between multivectors involves an integer vector $a \in \mathbb{Z}^{n+1}$ such that the sum of absolute values of its coordinates is greater than one, then this vector can be represented in the form $a = -b - c$, where $b, c \in \mathbb{Z}^{n+1}$ are vectors such that the sum of absolute values of coordinates for each of these vectors is less than that of the vector a. Continuing this process, we reduce the original relation to relations containing only the vectors $\pm e_i$. For such relations, Theorem 2 is obvious.

2) It remains to verify Theorem 2 for an elementary relation defined by three n-tuples $\boldsymbol{a} = (a, a_2, \dots, a_n)$, $\boldsymbol{b} = (b, a_2, \dots, a_n)$, and $\boldsymbol{c} = (c, a_2, \dots, a_n)$ of vectors such that $a+b+c = 0$. First, we perform this under the condition that the vectors $a, b, c, a_2, \dots a_n$ lie in some n-dimensional subspace K. In this case, the cycle $C = T[\boldsymbol{a}]+T[\boldsymbol{b}]+T[\boldsymbol{c}]$ homologous to zero lies in the n-dimensional torus $T(K)$, the image of K under the canonical projection $\mathbb{R}^{n+1} \to T^{n+1}$. We can take a chain σ_{n+1} spanning C in the torus $T(K)$; hence $J(C) = 0$. On the other hand, $V_n[\boldsymbol{a}] + V_n[\boldsymbol{b}] + V_n[\boldsymbol{c}] = 0$. Indeed, the computation of the volume of an n-dimensional parallelepiped in K is reduced to the computation of the determinant, and the determinant is multilinear.

3) Let us proceed to the case in which the vectors $a, b, c, a_2, \dots, a_n$ do not belong to any n-dimensional subspace. In this case, the triangle Δ_2 with sides a, b, c is transversal to the $(n-1)$-dimensional parallelepiped Π with sides $a_2, \dots, a_n$. The corollary applies to the polytope $\Delta_{n+1} = \Delta_2 + \Pi$. It follows that first, the boundary $\partial\sigma_{n+1}$ of the chain σ_{n+1} that is the image under the projection $\mathbb{R}^{n+1} \to T^{n+1}$ of the oriented polytope Δ_{n+1} is equal to the cycle $C = T[\boldsymbol{a}]+T[\boldsymbol{b}]+T[\boldsymbol{c}]$. Indeed, all pairs of cancelable faces of Δ_{n+1} are killed by the projection and hence occur with zero coefficient in the boundary of the chain σ_{n+1}. Second, by the corollary, Theorem 3 holds for the polytope Δ_{n+1}. Therefore, we have the relation $2J(C) \equiv (V_n[a]+V_n[b]+V_n[c]) \mod 2$, as desired.

One can readily derive Theorem 3 from Theorem 2.

Proof of Theorem 3. Let σ_{n+1} be a chain in T^{n+1} obtained as the image under the projection $\mathbb{R}^{n+1} \to T^{n+1}$ of an integer polytope Δ of parallelepipedal type. The boundary of σ_{n+1} is equal to the sum of images under the canonical projection of all oriented faces of Δ that are parallelepipeds. Indeed, pairs of cancelable faces of Δ are killed by the projection and hence occur with zero coefficient in the boundary of σ_{n+1}. Now Theorem 3 follows from Theorem 2.

2.6. A stronger version of Theorem 3. Theorem 3 has a stronger version. With every hyperplane $K \subset \mathbb{R}^{n+1}$ and an $(n+1)$-dimensional convex polytope $\Delta \subset \mathbb{R}^{n+1}$, we associate two faces $\Gamma_1(K)$ and $\Gamma_2(K)$ of Δ, for which the hyperplane parallel to K and containing any of these faces is a support hyperplane of the polytope. Let $\pi\colon \mathbb{R}^{n+1} \to T^{n+1}$ be the canonical projection. We say that a polytope Δ is *balanced* with respect to a rational hyperplane K *of weight* $m(K) \geqslant 0$ if the relation $\pi(\Gamma_1(K))+\pi(\Gamma_2(K)) = \pm m(K)T(K)$ holds in the group of n-dimensional chains of the n-dimensional torus $\pi(K) = T(K)$. In this relation, $\Gamma_1(K)$ and $\Gamma_2(K)$ are viewed as n-dimensional chains equipped with the orientation of the boundary of Δ and $T(K)$ is treated as the fundamental cycle of the torus with an arbitrary orientation. The volumes of the faces $\Gamma_1(K)$ and $\Gamma_2(K)$ of a polytope balanced with respect to a rational hyperplane K with weight $m(K) \geqslant 0$ satisfy the relation $|V_n(\Gamma_1(K)) - V_n(\Gamma_2(K))| = m(K)$. We say that an integer polytope Δ is of *generalized parallelepipedal type* if it is balanced with respect to each rational hyperplane K. (This condition is only meaningful for finitely many hyperplanes K parallel to n-dimensional faces of the polytope Δ.)

Theorem 3′. *Let Δ be a convex integer $(n+1)$-dimensional polytope of generalized parallelepipedal type. Then the doubled integer $(n+1)$-dimensional volume of Δ is an integer of the* *same parity as the sum of weights $m(K)$ over all rational hyperplanes K. (It suffices to sum over finitely many hyperplanes K parallel to n-dimensional faces of the polytope Δ.)*

Theorem 3′ can be derived from Theorem 2 in the same way as Theorem 3.

Note that in three-dimensional space there are many polytopes of generalized parallelepipedal type. One can readily verify the following assertion.

Lemma 8. *The Minkowski sum of any number of integer segments and polygons is of generalized parallelepipedal type regardless of their location in $\mathbb{R}^3$.*

2.7. A geometric meaning of the function D. In this section, we prove Theorem 4 stated in Sec. 2.2.

Recall the definition of the cross product of n vectors in the $(n+1)$-dimensional space $\mathbb{R}^{n+1}$ equipped with the standard volume form ω. The cross product of ordered n-tuple of vectors $a_1, \dots, a_n$ is the covector $\boldsymbol{a}^\perp \in (\mathbb{R}^{n+1})^*$ such that $\langle \boldsymbol{a}^\perp, a\rangle = \omega(a \wedge a_1 \wedge \dots \wedge a_n)$ for all $a \in \mathbb{R}^{n+1}$. Let A be the $n \times (n+1)$-matrix whose columns are vectors $a_1, \dots, a_n$ written in the standard basis of the space $\mathbb{R}^{n+1}$, and let $\det_i(\boldsymbol{a})$ be the determinant of the matrix obtained from A by deleting the ith row.

Lemma 9. 1) *In the standard basis of $(\mathbb{R}^{n+1})^*$, the cross product $\boldsymbol{a}^\perp$ of the vectors $a_1, \dots, a_n$ is given by the formula $\boldsymbol{a}^\perp = (\det_1(\boldsymbol{a}), \dots, (-1)^n \det_{n+1}(\boldsymbol{a}))$.*

2) *The covector $\boldsymbol{a}^\perp$ is orthogonal to each of the vectors $a_1, \dots, a_n$; i.e., $\langle \boldsymbol{a}^\perp, a_i\rangle = 0$, $i = 1, \dots, n$.*

3) *If the vectors $a_1, \dots, a_n$ are integer, then the covector $\boldsymbol{a}^\perp$ is also integer (i.e., belongs to the lattice $(\mathbb{Z}^{n+1})^*$ dual to the lattice $\mathbb{Z}^{n+1}$). Furthermore, the integer length $V_1(\boldsymbol{a}^\perp)$ of $\boldsymbol{a}^\perp$ is equal to the integer volume $V_n[\boldsymbol{a}]$ of the parallelepiped $\Pi[\boldsymbol{a}]$.*

Lemma 9 is a simple fact from linear algebra. We omit the proof.

Proof of Theorem 4. The multivector $v[\boldsymbol{a}] = a_1 \wedge \dots \wedge a_n$ has the expansion $v[\boldsymbol{a}] = \det_1(\boldsymbol{a}) e_2 \wedge \dots \wedge e_{n+1} + \dots + \det_{n+1}(\boldsymbol{a}) e_1 \wedge \dots \wedge e_n$ in the basis of coordinate multivectors. By Theorem 2, $2J(C(A))$ is well defined as a residue modulo 2, and

$$2J(C[\boldsymbol{a}]) \equiv [V_n[\boldsymbol{a}] + \det_1(\boldsymbol{a}) + \dots + \det_{n+1}(\boldsymbol{a})] \mod 2.$$

By definition, $V_n[\boldsymbol{a}]$ is equal to the integer volume of the parallelepiped $\Pi[\boldsymbol{a}]$ spanned by $a_1, \dots, a_n$. By Lemma 9, this volume is equal to the integer length $V_1(\boldsymbol{a}^\perp)$ of the covector $\boldsymbol{a}^\perp = (\det_1(\boldsymbol{a}), \dots, (-1)^n \det_{n+1}(\boldsymbol{a}))$. The following congruence holds:

$$V_1(\boldsymbol{a}^\perp) \equiv \Bigl[1 + \prod_{1 \leqslant i \leqslant n+1} (1 + \det_i(\boldsymbol{a}))\Bigr] \mod 2.$$

Indeed, both left- and right-hand sides of this congruence are even if and only if all numbers $\det_1(\boldsymbol{a}), \dots, \det_{n+1}(\boldsymbol{a})$ are simultaneously even. Thus

$$2J(C[\boldsymbol{a}]) \equiv \Bigl[1 + \det_1(\boldsymbol{a}) + \dots + \det_{n+1}(\boldsymbol{a}) + \prod_{1 \leqslant i \leqslant n+1} (1 + \det_i(\boldsymbol{a}))\Bigr] \mod 2.$$

To conclude the proof of Theorem 4, it remains to use the formula for D in Theorem 1.

End note. Quite recently, the function D discussed in this paper received another application. Jointly with A. I. Esterov, we computed the ratio of the coefficients of any two extreme monomials of a multidimensional resultant. Multidimensional resultants are special Laurent polynomials of several variables defined up to sign. They were intensively studied by Gelfand, Kapranov, Zelevinsky, Sturmfels, and others. Their Newton polytopes are known. It is also known that the coefficients of the extreme monomials (i.e., monomials corresponding to the vertices of the Newton polytope) in any such resultant are equal to ± 1. Our formula for the ratio of such coefficients uses the combinatorics of integer polytopes and the function D.

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