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Convex Bodies and Multiplicities of Ideals

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Abstract—We associate convex regions in $\mathbb{R}^n$ to $\mathfrak{m}$-primary graded sequences of subspaces, in particular $\mathfrak{m}$-primary graded sequences of ideals, in a large class of local algebras (including analytically irreducible local domains). These convex regions encode information about Samuel multiplicities. This is in the spirit of the theory of Gröbner bases and Newton polyhedra on the one hand, and the theory of Newton–Okounkov bodies for linear systems on the other hand. We use this to give a new proof as well as a generalization of a Brunn–Minkowski inequality for multiplicities due to Teissier and Rees–Sharp.

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1. INTRODUCTION

The purpose of this note is to employ, in the local case, techniques from the theory of semigroups of integral points and Newton–Okounkov bodies (for the global case) and to obtain new results as well as new proofs of some previously known results about multiplicities of ideals in local rings.

Let $R = \mathcal{O}_{X,p}$ be the local ring of a point p on an n-dimensional irreducible algebraic variety X over an algebraically closed field $\mathbf{k}$. Let $\mathfrak{m}$ denote the maximal ideal of R and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal, i.e. $\mathfrak{a}$ is an ideal containing a power of the maximal ideal $\mathfrak{m}$. Geometrically speaking, $\mathfrak{a}$ is $\mathfrak{m}$-primary if its zero set (around p) is the single point p itself. Let $f_1, \ldots, f_n$ be n generic elements in $\mathfrak{a}$. The *multiplicity* $e(\mathfrak{a})$ of the ideal $\mathfrak{a}$ is the intersection multiplicity, at the origin, of the hypersurfaces $H_i = \{x \mid f(x) = 0\}$, $i = 1, \ldots, n$ (it can be shown that this number is independent of the choice of the f_i). According to the Hilbert–Samuel theorem, the multiplicity $e(\mathfrak{a})$ is equal to

$$n! \lim_{k\to\infty} \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}^k)}{k^n}.$$

(This result is analogous to Hilbert's theorem on the Hilbert function and degree of a projective variety.) More generally, let R be an n-dimensional Noetherian local domain over $\mathbf{k}$ (where $\mathbf{k}$ is isomorphic to the residue field $R/\mathfrak{m}$ and $\mathfrak{m}$ is the maximal ideal). Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R. Since $\mathfrak{a}$ contains a power of the maximal ideal $\mathfrak{m}$, $R/\mathfrak{a}$ is finite dimensional regarded as a vector space over $\mathbf{k}$. The *Hilbert–Samuel function* of the $\mathfrak{m}$-primary ideal $\mathfrak{a}$ is defined by

$$H_{\mathfrak{a}}(k) = \dim_{\mathbf{k}}(R/\mathfrak{a}^k).$$

For large values of k, $H_{\mathfrak{a}}(k)$ coincides with a polynomial of degree n called the *Hilbert–Samuel polynomial* of $\mathfrak{a}$. The *Samuel multiplicity* $e(\mathfrak{a})$ of $\mathfrak{a}$ is defined to be the leading coefficient of $H_{\mathfrak{a}}(k)$ multiplied by $n!$.

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It is well-known that the Samuel multiplicity satisfies a Brunn–Minkowski inequality [23, 22]. That is, for any two $\mathfrak{m}$-primary ideals $\mathfrak{a}, \mathfrak{b} \in R$ we have

$$e(\mathfrak{a})^{1/n} + e(\mathfrak{b})^{1/n} \geq e(\mathfrak{a}\mathfrak{b})^{1/n}. \tag{1.1}$$

More generally we define multiplicity for $\mathfrak{m}$-*primary graded sequences of subspaces*. That is, a sequence $\mathfrak{a}_1, \mathfrak{a}_2, \dots$ of $\mathbf{k}$-subspaces in R such that for all k and m we have $\mathfrak{a}_k\mathfrak{a}_m \subset \mathfrak{a}_{k+m}$ and $\mathfrak{a}_1$ contains a power of the maximal ideal $\mathfrak{m}$ (Definition 6.2). We recall that if $\mathfrak{a}$ and $\mathfrak{b}$ are two $\mathbf{k}$-subspaces of R, $\mathfrak{a}\mathfrak{b}$ denotes the $\mathbf{k}$-span of all the xy where $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. In particular, a graded sequence $\mathfrak{a}_\bullet$ where each $\mathfrak{a}_k$ is an $\mathfrak{m}$-primary ideal is an $\mathfrak{m}$-primary graded sequence of subspaces. We call such $\mathfrak{a}_\bullet$ an $\mathfrak{m}$-*primary graded sequence of ideals*.

For an $\mathfrak{m}$-primary graded sequence of subspaces we define the multiplicity $e(\mathfrak{a}_\bullet)$ to be

$$e(\mathfrak{a}_\bullet) = n! \lim_{k\to\infty} \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}_k)}{k^n}. \tag{1.2}$$

(It is not a priori clear that the limit exists.)

We will use convex geometric arguments to prove the existence of the limit in (1.2) and a generalization of (1.1) to $\mathfrak{m}$-primary graded sequences of subspaces, for a large class of local domains R.

Let us briefly discuss the convex geometry part of the story. Let $\mathcal{C}$ be a closed strongly convex cone with apex at the origin (i.e. $\mathcal{C}$ is a convex cone and does not contain any line). We call a closed convex set $\Gamma \subset \mathcal{C}$ a $\mathcal{C}$-*convex region* if for any $x \in \Gamma$ we have $x + \mathcal{C} \subset \Gamma$. We say that Γ is *cobounded* if $\mathcal{C} \setminus \Gamma$ is bounded. It is easy to verify that the set of cobounded $\mathcal{C}$-convex regions is closed under addition (Minkowski sum of convex sets) and multiplication with a positive real number. For a cobounded $\mathcal{C}$-convex region Γ we call the volume of the bounded region $\mathcal{C} \setminus \Gamma$ the *covolume* of Γ and denote it by $\operatorname{covol}(\Gamma)$. Also we refer to $\mathcal{C} \setminus \Gamma$ as a $\mathcal{C}$-*coconvex body.* (Instead of working with convex regions one can alternatively work with coconvex bodies.) In [14, 15], similar to convex bodies and their volumes (and mixed volumes), the authors develop a theory of convex regions and their covolumes (and mixed covolumes). Moreover, they prove an analogue of the Alexandrov–Fenchel inequality for mixed covolumes (see Theorem 3.3). The usual Alexandrov–Fenchel inequality is an important inequality about mixed volumes of convex bodies in $\mathbb{R}^n$ and generalizes the classical isoperimetric inequality and the Brunn–Minkowski inequality. In a similar way, the result in [14] implies a Brunn–Minkowski inequality for covolumes, that is, for any two cobounded $\mathcal{C}$-convex regions Γ_1 and Γ_2 where $\mathcal{C}$ is an n-dimensional cone, we have

$$\operatorname{covol}(\Gamma_1)^{1/n} + \operatorname{covol}(\Gamma_2)^{1/n} \geq \operatorname{covol}(\Gamma_1 + \Gamma_2)^{1/n}. \tag{1.3}$$

We associate convex regions to $\mathfrak{m}$-primary graded sequences of subspaces (in particular, $\mathfrak{m}$-primary ideals) and use inequality (1.3) to prove the Brunn–Minkowski inequality for multiplicities. To associate a convex region to a graded sequence of subspaces, we need a valuation on the ring R. We will assume that there is a valuation v on R with values in $\mathbb{Z}^n$ (with respect to a total order on $\mathbb{Z}^n$ respecting addition) such that the residue field of v is $\mathbf{k}$ and, moreover, the following conditions (i) and (ii) hold.[1] We call such v a *good valuation* on R (Definition 8.3):

(i) Let $\mathcal{S} = v(R \setminus \{0\}) \cup \{0\}$ be the value semigroup of (R, v). Let $\mathcal{C} = C(\mathcal{S})$ be the closure of the convex hull of $\mathcal{S}$. It is a closed convex cone with apex at the origin. We assume that $\mathcal{C}$ is a strongly convex cone.

Let $\ell\colon \mathbb{R}^n \to \mathbb{R}$ be a linear function. For any $a \in \mathbb{R}$ let $\ell_{\geq a}$ denote the half-space $\{x \mid \ell(x) \geq a\}$. Since the cone $\mathcal{C} = C(\mathcal{S})$ associated to the semigroup $\mathcal{S}$ is assumed to be strongly convex, we can

[1] In [10] a valuation v with values in $\mathbb{Z}^n$ and residue field $\mathbf{k}$ is called a *valuation with one-dimensional leaves* (see Definition 8.1).

find a linear function ℓ such that the cone $\mathcal{C}$ lies in $\ell_{\geq 0}$ and intersects the hyperplane $\ell^{-1}(0)$ only at the origin.

(ii) We assume there exists $r_0 > 0$ and a linear function ℓ as above such that for any $f \in R$, if $\ell(v(f)) \geq kr_0$ for some $k > 0$, then $f \in \mathfrak{m}^k$.

Let $\mathcal{M}_k = v(\mathfrak{m}^k \setminus \{0\})$ denote the image of $\mathfrak{m}^k$ under the valuation v. Condition (ii) in particular implies that for any $k > 0$ we have $\mathcal{M}_k \cap \ell_{\geq kr_0} = \mathcal{S} \cap \ell_{\geq kr_0}$.

As an example, let $R = \mathbf{k}[x_1, \dots, x_n]_{(0)}$ be the algebra of polynomials localized at the maximal ideal $(x_1, \dots, x_n)$. Then the map v which associates to a polynomial its lowest exponent (with respect to some term order) defines a good valuation on R, and the value semigroup $\mathcal{S}$ coincides with the semigroup $\mathbb{Z}^n_{\geq 0}$, that is, the semigroup of all the integral points in the positive orthant $\mathcal{C} = \mathbb{R}^n_{\geq 0}$. In the same fashion any regular local ring has a good valuation, as well as the local ring of a toroidal singularity (Example 8.5 and Theorem 8.6). More generally, in Section 8 we see that an analytically irreducible local domain R has a good valuation (Theorems 8.7 and 8.8; see also [3, Theorem 4.2, Lemma 4.3]). A local ring R is said to be analytically irreducible if its completion is an integral domain. Regular local rings and local rings of toroidal singularities are analytically irreducible. (We should point out that in the first version of the paper we had addressed only the case where R is a regular local ring or the local ring of a toroidal singularity.)

Given a good $\mathbb{Z}^n$-valued valuation v on the domain R, we associate the (strongly) convex cone $\mathcal{C} = \mathcal{C}(R) \subset \mathbb{R}^n$ to the domain R which is the closure of the convex hull of the value semigroup $\mathcal{S}$. Then to each $\mathfrak{m}$-primary graded sequence of subspaces $\mathfrak{a}_\bullet$ in R we associate a convex region $\Gamma(\mathfrak{a}_\bullet) \subset \mathcal{C}$ such that the set $\mathcal{C} \setminus \Gamma(\mathfrak{a}_\bullet)$ is bounded (Definition 8.11). The main result of the paper (Theorem 8.12) is that the limit in (1.2) exists and

$$e(\mathfrak{a}_\bullet) = n! \operatorname{covol}(\Gamma(\mathfrak{a}_\bullet)). \tag{1.4}$$

Equality (1.4) and the Brunn–Minkowski inequality for covolumes (see (1.3) or Corollary 3.4) are the main ingredients in proving a generalization of inequality (1.1) to $\mathfrak{m}$-primary graded sequences of subspaces (Corollary 8.14).

We would like to point out that the construction of $\Gamma(\mathfrak{a}_\bullet)$ is an analogue of the construction of the Newton–Okounkov body of a linear system on an algebraic variety (see [21, 20, 10, 17]). In fact, the approach and results in the present paper are analogous to the approach and results in [10] regarding the asymptotic behavior of Hilbert functions of a general class of graded algebras. In the present paper we also deal with certain graded algebras (i.e. $\mathfrak{m}$-primary graded sequences of subspaces), but instead of the dimension of graded pieces we are interested in the codimension (i.e. dimension of $R/\mathfrak{a}_k$); that is why in our main theorem (Theorem 8.12) the covolume of a convex region appears as opposed to the volume of a convex body [10, Theorem 2.31]. Also our Theorem 8.12 generalizes [10, Corollary 3.2], which gives a formula for the degree of a projective variety X in terms of the volume of its corresponding Newton–Okounkov body, because the Hilbert function of a projective variety X can be regarded as the difference derivative of the Hilbert–Samuel function of the affine cone over X at the origin and hence has the same leading coefficient.

On the other hand, the construction of $\Gamma(\mathfrak{a})$ generalizes the notion of the Newton diagram of a power series (see [16; 1, Sect. 12.7]). To a monomial ideal in a polynomial ring (or a power series ring), i.e. an ideal generated by monomials, one can associate its (unbounded) *Newton polyhedron*. It is the convex hull of the exponents of the monomials appearing in the ideal. The *Newton diagram* of a monomial ideal is the union of the bounded faces of the Newton polyhedron. One can see that for a monomial ideal $\mathfrak{a}$, the convex region $\Gamma(\mathfrak{a})$ coincides with its Newton polyhedron (Theorem 7.5). The main theorem in this paper (Theorem 8.12) for the case of monomial ideals recovers the local case of the well-known Bernstein–Kushnirenko theorem about computing the multiplicity at the origin of a system $f_1(x) = \ldots = f_n(x) = 0$ where the f_i are generic functions from $\mathfrak{m}$-primary monomial ideals (see Section 7 and [1, Sect. 12.7]).

Another immediate corollary of (1.4) is the following: let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal in $R = \mathbf{k}[x_1, \ldots, x_n]_{(0)}$. Fix a term order on $\mathbb{Z}^n$ and for each $k > 0$ let $\mathbf{in}(\mathfrak{a}^k)$ denote the initial ideal of the ideal $\mathfrak{a}^k$ (generated by the lowest terms of elements of $\mathfrak{a}^k$). Then the sequence of numbers

$$\frac{e(\mathbf{in}(\mathfrak{a}^k))}{k^n}$$

is decreasing and converges to $e(\mathfrak{a})$ as $k \to \infty$ (Corollary 8.16).

The Brunn–Minkowski inequality proved in this paper is closely related to the more general Alexandrov–Fenchel inequality for mixed multiplicities. Take $\mathfrak{m}$-primary ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in a local ring $R = \mathcal{O}_{X,p}$ of a point p on an n-dimensional algebraic variety X. The *mixed multiplicity* $e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ is equal to the intersection multiplicity, at the origin, of the hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \ldots, n$, where each f_i is a generic function from $\mathfrak{a}_i$. Alternatively one can define the mixed multiplicity as the polarization of the Hilbert–Samuel multiplicity $e(\mathfrak{a})$; i.e. it is the unique function $e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ which is invariant under permuting the arguments, is multi-additive with respect to product of ideals, and for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$ the mixed multiplicity $e(\mathfrak{a}, \ldots, \mathfrak{a})$ coincides with $e(\mathfrak{a})$. In fact, in the above the $\mathfrak{a}_i$ need not be ideals and it suffices for them to be $\mathfrak{m}$-primary subspaces.

The Alexandrov–Fenchel inequality is the following inequality among the mixed multiplicities of the $\mathfrak{a}_i$:

$$e(\mathfrak{a}_1, \mathfrak{a}_1, \mathfrak{a}_3, \ldots, \mathfrak{a}_n)\, e(\mathfrak{a}_2, \mathfrak{a}_2, \mathfrak{a}_3, \ldots, \mathfrak{a}_n) \geq e(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots, \mathfrak{a}_n)^2. \tag{1.5}$$

When $n = \dim R = 2$, it is easy to see that the Brunn–Minkowski inequality (1.1) and the Alexandrov–Fenchel inequality (1.5) are equivalent. By a reduction-of-dimension theorem for mixed multiplicities one can get a proof of the Alexandrov–Fenchel inequality (1.5) from the Brunn–Minkowski inequality (1.1) for $\dim(R) = 2$. The Brunn–Minkowski inequality (1.1) was originally proved in [23, 22].

In a recent paper [11] we give a simple proof of the Alexandrov–Fenchel inequality for mixed multiplicities of ideals using arguments similar to but different from those of this paper. This then implies an Alexandrov–Fenchel inequality for covolumes of convex regions (in a similar way that in [10, 12] the authors obtain an alternative proof of the usual Alexandrov–Fenchel inequality for volumes of convex bodies from similar inequalities for intersection numbers of divisors on algebraic varieties).

We would like to point out that the Alexandrov–Fenchel inequality in [14] for covolumes of coconvex bodies is related to an analogue of this inequality for convex bodies in a higher dimensional hyperbolic space (or higher dimensional Minkowski space–time). From this point of view, the Alexandrov–Fenchel inequality has been proved for certain coconvex bodies in [7].

After the first version of this note was completed, we learned about the recent papers [3, 4, 8], which establish the existence of limit (1.2) in more general settings. We would also like to mention the paper of Teissier [24], which discusses the Newton polyhedron of a power series and notes the relationship/analogy between notions from local commutative algebra and convex geometry. Also we were notified that, for ideals in a polynomial ring, ideas similar to the construction of $\Gamma(\mathfrak{a}_\bullet)$ (see Definition 8.11) appear in [19], where the highest term of polynomials is used instead of a valuation. Moreover, in [19, Corollary 1.9] the Brunn–Minkowski inequality for multiplicities of graded sequences of $\mathfrak{m}$-primary ideals is proved for regular local rings using Teissier's Brunn–Minkowski inequality (1.1).

Finally, as the final version of this paper was being prepared for publication, the preprint of D. Cutkosky [5] appeared, in which the author uses similar methods to prove the Brunn–Minkowski inequality for graded sequences of $\mathfrak{m}$-primary ideals in local domains.

And few words about the organization of the paper: Section 2 recalls the basic background material about volumes/mixed volumes of convex bodies. Section 3 is about convex regions and their covolumes/mixed covolumes, which we can think of as a local version of the theory of mixed volumes of convex bodies. In Sections 4 and 5 we associate a convex region to a primary sequence of subsets in a semigroup and prove the main combinatorial result required later (Definition 5.7 and Theorem 5.10). In Section 6 we recall some basic definitions and facts from commutative algebra about multiplicities of $\mathfrak{m}$-primary ideals (and subspaces) in local rings. The next section (Section 7) discusses the case of monomial ideals and the Bernstein–Kushnirenko theorem. Finally, in Section 8, using a valuation on the ring R, we associate a convex region $\Gamma(\mathfrak{a}_\bullet)$ to an $\mathfrak{m}$-primary graded sequence of subspaces $\mathfrak{a}_\bullet$ and prove the main results of this note (Theorem 8.12 and Corollary 8.14).

2. MIXED VOLUME OF CONVEX BODIES

The collection of all convex bodies in $\mathbb{R}^n$ is a cone, that is, we can add convex bodies and multiply a convex body with a positive number. For two convex bodies $\Delta_1, \Delta_2 \subset \mathbb{R}^n$, their (Minkowski) sum $\Delta_1 + \Delta_2$ is $\{x + y \mid x \in \Delta_1,\, y \in \Delta_2\}$. Let vol denote the n-dimensional volume in $\mathbb{R}^n$ with respect to the standard Euclidean metric. The function vol is a homogeneous polynomial of degree n on the cone of convex bodies, i.e. its restriction to each finite dimensional section of the cone is a homogeneous polynomial of degree n. In other words, for any collection of convex bodies $\Delta_1, \dots, \Delta_r$, the function

$$P_{\Delta_1,\dots,\Delta_r}(\lambda_1, \dots, \lambda_r) = \mathrm{vol}(\lambda_1\Delta_1 + \ldots + \lambda_r\Delta_r)$$

is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_r$. By definition the *mixed volume* $V(\Delta_1, \dots, \Delta_n)$ of an n-tuple $(\Delta_1, \dots, \Delta_n)$ of convex bodies is the coefficient of the monomial $\lambda_1 \dots \lambda_n$ in the polynomial $P_{\Delta_1,\dots,\Delta_n}(\lambda_1, \dots, \lambda_n)$ divided by $n!$. This definition implies that mixed volume is the *polarization* of the volume polynomial, that is, it is the unique function on the n-tuples of convex bodies satisfying the following:

(i) (symmetry). V is symmetric with respect to permuting the bodies $\Delta_1, \dots, \Delta_n$.

(ii) (multi-linearity). It is linear in each argument with respect to the Minkowski sum. The linearity in the first argument means that for convex bodies $\Delta_1', \Delta_1'', \Delta_2, \dots, \Delta_n$ and real numbers $\lambda', \lambda'' \geq 0$ we have

$$V(\lambda'\Delta_1' + \lambda''\Delta_1'', \Delta_2, \dots, \Delta_n) = \lambda' V(\Delta_1', \Delta_2, \dots, \Delta_n) + \lambda'' V(\Delta_1'', \Delta_2, \dots, \Delta_n).$$

(iii) (relation to volume). On the diagonal it coincides with the volume, i.e. if $\Delta_1 = \ldots = \Delta_n = \Delta$, then $V(\Delta_1, \dots, \Delta_n) = \mathrm{vol}(\Delta)$.

The following inequality attributed to Alexandrov and Fenchel is important and very useful in convex geometry (see [2]):

Theorem 2.1 (Alexandrov–Fenchel)**.** *Let $\Delta_1, \dots, \Delta_n$ be convex bodies in $\mathbb{R}^n$. Then*

$$V(\Delta_1, \Delta_1, \Delta_3, \dots, \Delta_n)\, V(\Delta_2, \Delta_2, \Delta_3, \dots, \Delta_n) \leq V(\Delta_1, \Delta_2, \dots, \Delta_n)^2.$$

In dimension 2, this inequality is elementary. We call it the *generalized isoperimetric inequality*, because when Δ_2 is the unit ball, it coincides with the classical isoperimetric inequality. The celebrated *Brunn–Minkowski inequality* concerns volume of convex bodies in $\mathbb{R}^n$. It is an easy corollary of the Alexandrov–Fenchel inequality. (For $n = 2$ it is equivalent to the Alexandrov–Fenchel inequality.)

Theorem 2.2 (Brunn–Minkowski)**.** *Let Δ_1 and Δ_2 be convex bodies in $\mathbb{R}^n$. Then*

$$\mathrm{vol}(\Delta_1)^{1/n} + \mathrm{vol}(\Delta_2)^{1/n} \leq \mathrm{vol}(\Delta_1 + \Delta_2)^{1/n}.$$

3. MIXED COVOLUME OF CONVEX REGIONS

Let $\mathcal{C}$ be a strongly convex closed n-dimensional cone in $\mathbb{R}^n$ with apex at the origin. (A convex cone is strongly convex if it does not contain any lines through the origin.) We are interested in closed convex subsets of $\mathcal{C}$ which have bounded complement.

Definition 3.1. We call a closed convex subset $\Gamma \subset \mathcal{C}$ a *$\mathcal{C}$-convex region* (or simply a convex region when the cone $\mathcal{C}$ is understood from the context) if for any $x \in \Gamma$ and $y \in \mathcal{C}$ we have $x + y \in \Gamma$. Moreover, we say that a convex region Γ is *cobounded* if the complement $\mathcal{C} \setminus \Gamma$ is bounded. In this case the volume of $\mathcal{C} \setminus \Gamma$ is finite, which we call the *covolume of* Γ and denote by $\operatorname{covol}(\Gamma)$. One also refers to $\mathcal{C} \setminus \Gamma$ as a *$\mathcal{C}$-coconvex body*.

The collection of $\mathcal{C}$-convex regions (respectively, cobounded regions) is closed under the Minkowski sum and multiplication by positive scalars. Similar to the volume of convex bodies, one proves that the covolume of convex regions is a homogeneous polynomial [14]. More precisely, one has

Theorem 3.2. *Let $\Gamma_1, \dots, \Gamma_r$ be cobounded $\mathcal{C}$-convex regions in the cone $\mathcal{C}$. Then the function*

$$P_{\Gamma_1,\dots,\Gamma_r}(\lambda_1, \dots, \lambda_r) = \operatorname{covol}(\lambda_1\Gamma_1 + \dots + \lambda_r\Gamma_r)$$

is a homogeneous polynomial of degree n in the λ_i.

As in the case of convex bodies, one uses this theorem to define mixed covolume of cobounded regions. By definition the *mixed covolume* $CV(\Gamma_1, \dots, \Gamma_n)$ of an n-tuple $(\Gamma_1, \dots, \Gamma_n)$ of cobounded convex regions is the coefficient of the monomial $\lambda_1 \dots \lambda_n$ in the polynomial $P_{\Gamma_1,\dots,\Gamma_n}(\lambda_1, \dots, \lambda_n)$ divided by $n!$. That is, mixed covolume is the unique function on the n-tuples of cobounded regions satisfying the following:

(i) (symmetry). CV is symmetric with respect to permuting the regions $\Gamma_1, \dots, \Gamma_n$.

(ii) (multi-linearity). It is linear in each argument with respect to the Minkowski sum.

(iii) (relation to covolume). For any cobounded region $\Gamma \subset \mathcal{C}$,

$$CV(\Gamma, \dots, \Gamma) = \operatorname{covol}(\Gamma).$$

The mixed covolume satisfies an Alexandrov–Fenchel inequality [14]. Note that the inequality is reversed compared to the Alexandrov–Fenchel inequality for mixed volumes of convex bodies.

Theorem 3.3 (Alexandrov–Fenchel inequality for mixed covolume)**.** *Let $\Gamma_1, \dots, \Gamma_n$ be cobounded $\mathcal{C}$-convex regions. Then*

$$CV(\Gamma_1, \Gamma_1, \Gamma_3, \dots, \Gamma_n)\, CV(\Gamma_2, \Gamma_2, \Gamma_3, \dots, \Gamma_n) \ge CV(\Gamma_1, \Gamma_2, \Gamma_3, \dots, \Gamma_n)^2.$$

The (reversed) Alexandrov–Fenchel inequality implies a (reversed) Brunn–Minkowski inequality. (For $n = 2$ it is equivalent to the Alexandrov–Fenchel inequality.)

Corollary 3.4 (Brunn–Minkowski inequality for covolume)**.** *Let Γ_1 and Γ_2 be cobounded $\mathcal{C}$-convex regions. Then*

$$\operatorname{covol}(\Gamma_1)^{1/n} + \operatorname{covol}(\Gamma_2)^{1/n} \ge \operatorname{covol}(\Gamma_1 + \Gamma_2)^{1/n}.$$

4. SEMIGROUPS OF INTEGRAL POINTS

In this section we recall some general facts from [10] about the asymptotic behavior of semigroups of integral points. Let $S \subset \mathbb{Z}^n \times \mathbb{Z}_{\ge 0}$ be an additive semigroup. Let $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ denote the projection onto the second factor, and let $S_k = S \cap \pi^{-1}(k)$ be the set of points in S at level k. For simplicity, assume that $S_1 \neq \varnothing$ and that S generates the whole lattice $\mathbb{Z}^{n+1}$. Define the function H_S by

$$H_S(k) = \#S_k.$$

We call H_S the *Hilbert function of the semigroup* S. We wish to describe the asymptotic behavior of H_S as $k \to \infty$.

Let $C(S)$ be the closure of the convex hull of $S \cup \{0\}$, that is, the smallest closed convex cone (with apex at the origin) containing S. We call the projection of the convex set $C(S) \cap \pi^{-1}(1)$ to $\mathbb{R}^n$ (under the projection onto the first factor $(x,1) \mapsto x$) the *Newton–Okounkov convex set of the semigroup* S and denote it by $\Delta(S)$. In other words,

$$\Delta(S) = \overline{\bigcup_{k>0} \left\{ \frac{x}{k} \;\middle|\; (x,k) \in S_k \right\}}.$$

(Using the fact that S is a semigroup, one can show that $\Delta(S)$ is convex.) If $C(S) \cap \pi^{-1}(0) = \{0\}$, then $\Delta(S)$ is compact and hence is a convex body.

The Newton–Okounkov convex set $\Delta(S)$ is responsible for the asymptotic behavior of the Hilbert function of S (see [10, Corollary 1.16]):

Theorem 4.1. *The limit*

$$\lim_{k\to\infty} \frac{H_S(k)}{k^n}$$

exists and is equal to $\operatorname{vol}(\Delta(S))$.

5. PRIMARY SEQUENCES IN A SEMIGROUP AND CONVEX REGIONS

In this section we discuss the notion of a primary graded sequence of subsets in a semigroup and describe its asymptotic behavior using Theorem 4.1. In Section 8 we will employ this to describe the asymptotic behavior of the Hilbert–Samuel function of a graded sequence of $\mathfrak{m}$-primary ideals in a local domain.

Let $\mathcal{S} \subset \mathbb{Z}^n$ be an additive semigroup containing the origin. Without loss of generality we assume that $\mathcal{S}$ generates the whole $\mathbb{Z}^n$. Let as above $\mathcal{C} = C(\mathcal{S})$ denote the cone of $\mathcal{S}$, i.e. the closure of the convex hull of $\mathcal{S} = \mathcal{S} \cup \{0\}$. We also assume that $\mathcal{C}$ is a strongly convex cone, i.e. it does not contain any lines through the origin.

For two subsets $\mathcal{I}, \mathcal{J} \subset \mathcal{S}$, the sum $\mathcal{I} + \mathcal{J}$ is the set $\{x + y \mid x \in \mathcal{I},\ y \in \mathcal{J}\}$. For any integer $k > 0$, by the product $k * \mathcal{I}$ we mean $\mathcal{I} + \ldots + \mathcal{I}$ (k times).

Definition 5.1. A *graded sequence of subsets* in $\mathcal{S}$ is a sequence $\mathcal{I}_\bullet = (\mathcal{I}_1, \mathcal{I}_2, \ldots)$ of subsets such that for any $k, m > 0$ we have $\mathcal{I}_k + \mathcal{I}_m \subset \mathcal{I}_{k+m}$.

Example 5.2. Let $\mathcal{I} \subset \mathcal{S}$. Then the sequence $\mathcal{I}_\bullet$ defined by $\mathcal{I}_k = k * \mathcal{I}$ is clearly a graded sequence of subsets.

Let $\mathcal{I}'_\bullet$ and $\mathcal{I}''_\bullet$ be graded sequences of subsets. Then the sequence $\mathcal{I}_\bullet = \mathcal{I}'_\bullet + \mathcal{I}''_\bullet$ defined by

$$\mathcal{I}_k = \mathcal{I}'_k + \mathcal{I}''_k$$

is also a graded sequence of subsets, which we call the *sum of the sequences* $\mathcal{I}'_\bullet$ *and* $\mathcal{I}''_\bullet$.

Let $\ell\colon \mathbb{R}^n \to \mathbb{R}$ be a linear function. For any $a \in \mathbb{R}$ let $\ell_{\geq a}$ (respectively, $\ell_{>a}$) denote the half-space $\{x \mid \ell(x) \geq a\}$ (respectively, $\{x \mid \ell(x) > a\}$), and similarly for $\ell_{\leq a}$ and $\ell_{<a}$. By assumption the cone $\mathcal{C} = C(\mathcal{S})$ associated to the semigroup $\mathcal{S}$ is strongly convex. Thus we can find a linear function ℓ such that the cone $\mathcal{C} = C(\mathcal{S})$ lies in $\ell_{\geq 0}$ and intersects the hyperplane $\ell^{-1}(0)$ only at the origin. Let us fix such a linear function ℓ.

We will be interested in graded sequences of subsets $\mathcal{I}_\bullet$ satisfying the following condition:

Definition 5.3. We say that a graded sequence of subsets $\mathcal{I}_\bullet$ is *primary* if there exists an integer $t_0 > 0$ such that for any integer $k > 0$ we have

$$\mathcal{I}_k \cap \ell_{\geq kt_0} = \mathcal{S} \cap \ell_{\geq kt_0}. \tag{5.1}$$

Remark 5.4. One verifies that if ℓ' is another linear function such that $\mathcal{C}$ lies in $\ell'_{\geq 0}$ and it intersects the hyperplane $\ell'^{-1}(0)$ only at the origin, then it automatically satisfies (5.1) with perhaps a different constant $t'_0 > 0$. Hence the condition of being a primary graded sequence does not depend on the linear function ℓ. Nevertheless, when we refer to a primary graded sequence $\mathcal{I}_\bullet$, choices of a linear function ℓ and an integer $t_0 > 0$ are implied.

Proposition 5.5. *Let $\mathcal{I}_\bullet$ be a primary graded sequence. Then for all $k > 0$ the set $\mathcal{S} \setminus \mathcal{I}_k$ is finite.*

Proof. Since $\mathcal{C}$ intersects $\ell^{-1}(0)$ only at the origin, it follows that for any $k > 0$ the set $\mathcal{C} \cap \ell_{<kt_0}$ is bounded, which implies that $\mathcal{S} \cap \ell_{<kt_0}$ is finite. But by (5.1), $\mathcal{S} \setminus \mathcal{I}_k \subset \mathcal{S} \cap \ell_{<kt_0}$ and hence is finite. $\square$

Definition 5.6. Let $\mathcal{I}_\bullet$ be a primary graded sequence. Define the function $H_{\mathcal{I}_\bullet}$ by

$$H_{\mathcal{I}_\bullet}(k) = \#(\mathcal{S} \setminus \mathcal{I}_k).$$

(Note that by Proposition 5.5 this number is finite for all $k > 0$.) We call it the *Hilbert–Samuel function of $\mathcal{I}_\bullet$*.

To a primary graded sequence of subsets $\mathcal{I}_\bullet$ we can associate a $\mathcal{C}$-convex region $\Gamma(\mathcal{I}_\bullet)$ (see Definition 3.1). This convex set encodes information about the asymptotic behavior of the Hilbert–Samuel function of $\mathcal{I}_\bullet$.

Definition 5.7. Let $\mathcal{I}_\bullet$ be a primary graded sequence of subsets. Define the convex set $\Gamma(\mathcal{I}_\bullet)$ by

$$\Gamma(\mathcal{I}_\bullet) = \overline{\bigcup_{k>0} \left\{ \frac{x}{k} \;\middle|\; x \in \mathcal{I}_k \right\}}.$$

(One can show that $\Gamma(\mathcal{I}_\bullet)$ is an unbounded convex set in $\mathcal{C}$.)

Proposition 5.8. *Let $\mathcal{I}_\bullet$ be a primary graded sequence. Then $\Gamma = \Gamma(\mathcal{I}_\bullet)$ is a $\mathcal{C}$-convex region in the cone $\mathcal{C}$, i.e. for any $x \in \Gamma$, $x + \mathcal{C} \subset \Gamma$. Moreover, the region Γ is cobounded, i.e. $\mathcal{C} \setminus \Gamma$ is bounded.*

Proof. Let ℓ and t_0 be as in Definition 5.3. From the definitions it follows that the region Γ contains $\mathcal{C} \cap \ell_{\geq t_0}$. Thus $\mathcal{C} \setminus \Gamma \subset \mathcal{C} \cap \ell_{<t_0}$ and hence is bounded. Next let $x \in \Gamma$. Since $x + \mathcal{C} \subset \mathcal{C}$, Γ contains the set $(x + \mathcal{C}) \cap \ell_{\geq t_0}$. But the convex hull of x and $(x + \mathcal{C}) \cap \ell_{\geq t_0}$ is $x + \mathcal{C}$. Thus $x + \mathcal{C} \subset \Gamma$ because Γ is convex. $\square$

The following is an important example of a primary graded sequence in a semigroup $\mathcal{S}$.

Proposition 5.9. *Let $\mathcal{C}$ be an n-dimensional strongly convex rational polyhedral cone in $\mathbb{R}^n$ and let $\mathcal{S} = \mathcal{C} \cap \mathbb{Z}^n$. Also let $\mathcal{I} \subset \mathcal{S}$ be a subset such that $\mathcal{S} \setminus \mathcal{I}$ is finite. Then the sequence $\mathcal{I}_\bullet$ defined by $\mathcal{I}_k := k * \mathcal{I}$ is a primary graded sequence and $\Gamma(\mathcal{I}_\bullet) = \mathrm{conv}(\mathcal{I})$.*

Proof. Since $\mathcal{S} \setminus \mathcal{I}$ is finite, there exists $t_1 > 0$ such that $\mathcal{S} \cap \ell_{\geq t_1} \subset \mathcal{I}$. Put $\mathcal{M}_1 = \mathcal{S} \cap \ell_{\geq t_1}$. Because $\mathcal{C}$ is a rational polyhedral cone, $\mathcal{M}_1$ is a finitely generated semigroup. Let $v_1, \dots, v_s$ be semigroup generators for $\mathcal{M}_1$. Let $t_0 > 0$ be greater than all the $\ell(v_i)$. For $k > 0$ take $x \in \mathcal{S} \cap \ell_{\geq kt_0} \subset \mathcal{M}_1$. Then $x = \sum_{i=1}^s c_i v_i$ for $c_i \in \mathbb{Z}_{\geq 0}$. Thus $kt_0 \leq \ell(x) = \sum_i c_i \ell(v_i) \leq \bigl(\sum_i c_i\bigr) t_0$. This implies that $k \leq \sum_i c_i$ and hence $\bigl(\sum_i c_i\bigr) * \mathcal{M}_1 \subset k * \mathcal{M}_1$. It follows that $x \in k * \mathcal{M}_1$. That is, $\mathcal{S} \cap \ell_{\geq kt_0} = (k * \mathcal{M}_1) \cap \ell_{\geq kt_0} \subset (k * \mathcal{I}) \cap \ell_{\geq kt_0}$ and hence $\mathcal{S} \cap \ell_{\geq kt_0} = (k * \mathcal{I}) \cap \ell_{\geq kt_0}$ as required. The assertion $\Gamma(\mathcal{I}_\bullet) = \mathrm{conv}(\mathcal{I}_\bullet)$ follows from the observation that $\mathrm{conv}(k * \mathcal{I}) = k\,\mathrm{conv}(\mathcal{I})$. $\square$

The following is our main result about the asymptotic behavior of a primary graded sequence.

Theorem 5.10. *Let $\mathcal{I}_\bullet$ be a primary graded sequence. Then*

$$\lim_{k\to\infty} \frac{H_{\mathcal{I}_\bullet}(k)}{k^n}$$

exists and is equal to $\mathrm{covol}(\Gamma(\mathcal{I}_\bullet))$.

Proof. Let $t_0 > 0$ be as in Definition 5.3. Then for all $k > 0$ we have $\mathcal{S} \cap \ell_{\geq kt_0} = \mathcal{I}_k \cap \ell_{\geq kt_0}$. Moreover, take t_0 to be large enough so that the finite set $\mathcal{I}_1 \cap \ell_{<t_0}$ generates the lattice $\mathbb{Z}^n$ (this is possible because $\mathcal{S}$ and hence $\mathcal{I}_1$ generate $\mathbb{Z}^n$). Consider

$$\widetilde{S} = \{(x,k) \mid x \in \mathcal{I}_k \cap \ell_{<kt_0}\}, \qquad \widetilde{T} = \{(x,k) \mid x \in \mathcal{S} \cap \ell_{<kt_0}\}.$$

$\widetilde{S}$ and $\widetilde{T}$ are semigroups in $\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$ and we have $\widetilde{S} \subset \widetilde{T}$. From the definition it follows that both of the groups generated by $\widetilde{S}$ and $\widetilde{T}$ are $\mathbb{Z}^{n+1}$. Also the Newton–Okounkov bodies of $\widetilde{S}$ and $\widetilde{T}$ are

$$\Delta(\widetilde{S}) = \Gamma(\mathcal{I}_\bullet) \cap \Delta(t_0), \qquad \Delta(\widetilde{T}) = \Delta(t_0),$$

where $\Delta(t_0) = \mathcal{C} \cap \ell_{\leq t_0}$. Since $\mathcal{S} \cap \ell_{\geq kt_0} = \mathcal{I}_k \cap \ell_{\geq kt_0}$, we have

$$\mathcal{S} \setminus \mathcal{I}_k = \widetilde{T}_k \setminus \widetilde{S}_k.$$

Here as usual $\widetilde{S}_k = \{(x,k) \mid (x,k) \in \widetilde{S}\}$ (respectively, $\widetilde{T}_k$) denotes the set of points in $\widetilde{S}$ (respectively, in $\widetilde{T}$) at level k. Hence

$$H_{\mathcal{I}_\bullet}(k) = \#\widetilde{T}_k - \#\widetilde{S}_k.$$

By Theorem 4.1 we have

$$\lim_{k\to\infty} \frac{\#\widetilde{S}_k}{k^n} = \mathrm{vol}(\Delta(\widetilde{S})), \qquad \lim_{k\to\infty} \frac{\#\widetilde{T}_k}{k^n} = \mathrm{vol}(\Delta(t_0)).$$

Thus

$$\lim_{k\to\infty} \frac{\#(\mathcal{S} \setminus \mathcal{I}_k)}{k^n} = \mathrm{vol}(\Delta(t_0)) - \mathrm{vol}(\Delta(\widetilde{S})).$$

On the other hand, we have

$$\Delta(t_0) \setminus \Delta(\widetilde{S}) = \mathcal{C} \setminus \Gamma(\mathcal{I}_\bullet),$$

and hence $\mathrm{vol}(\Delta(t_0)) - \mathrm{vol}(\Delta(\widetilde{S})) = \mathrm{covol}(\Gamma(\mathcal{I}_\bullet))$. This finishes the proof. □

Definition 5.11. For a primary graded sequence $\mathcal{I}_\bullet$ we denote $n! \lim_{k\to\infty} H_{\mathcal{I}_\bullet}(k)/k^n$ by $e(\mathcal{I}_\bullet)$. Motivated by commutative algebra, we call it the *multiplicity of* $\mathcal{I}_\bullet$. We have just proved that $e(\mathcal{I}_\bullet) = n!\,\mathrm{covol}(\Gamma(I_\bullet))$. Note that it is possible for a primary graded sequence to have multiplicity equal to zero.

The following additivity property is straightforward from the definition.

Proposition 5.12. *Let $\mathcal{I}'_\bullet$ and $\mathcal{I}''_\bullet$ be primary graded sequences. We have*

$$\Gamma(\mathcal{I}'_\bullet) + \Gamma(\mathcal{I}''_\bullet) = \Gamma(\mathcal{I}'_\bullet + \mathcal{I}''_\bullet).$$

Proof. From the definition it is clear that $\Gamma(\mathcal{I}'_\bullet + \mathcal{I}''_\bullet) \subset \Gamma(\mathcal{I}'_\bullet) + \Gamma(\mathcal{I}''_\bullet)$. We need to show the reverse inclusion. Let $\mathcal{I}_\bullet$ denote $\mathcal{I}'_\bullet + \mathcal{I}''_\bullet$. For $k, m > 0$ take $x \in \mathcal{I}'_k$ and $y \in \mathcal{I}''_m$. Then $(x/k) + (y/m) = (mx + ky)/km \in (1/km)\mathcal{I}_{km}$. This shows that $(x/k) + (y/m) \in \Gamma(\mathcal{I}_\bullet)$, which finishes the proof. □

Let $\mathcal{I}_{1,\bullet}, \dots, \mathcal{I}_{n,\bullet}$ be n primary graded sequences. Define the function $P_{\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet}} \colon \mathbb{N}^n \to \mathbb{N}$ by

$$P_{\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet}}(k_1, \dots, k_n) = e(k_1 * \mathcal{I}_{1,\bullet} + \dots + k_n * \mathcal{I}_{n,\bullet}).$$

Theorem 5.13. *The function $P_{\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet}}$ is a homogeneous polynomial of degree n in $k_1, \dots, k_n$.*

Proof. Follows immediately from Proposition 5.12 and Theorems 5.10 and 3.2. □

Definition 5.14. Let $\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet}$ be primary graded sequences. Define the *mixed multiplicity* $e(\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet})$ to be the coefficient of $k_1\dots k_n$ in the polynomial $P_{\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet}}$ divided by $n!$.

From Theorem 5.10 and Proposition 5.12 we have the following corollary:

Corollary 5.15.

$$e(\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet}) = n!\, CV(\Gamma(\mathcal{I}_{1,\bullet}),\dots,\Gamma(\mathcal{I}_{n,\bullet})),$$

where as before CV denotes the mixed covolume of cobounded regions.

From Theorem 3.3 and Corollary 3.4 we then obtain

Corollary 5.16 (Alexandrov–Fenchel inequality for mixed multiplicity in semigroups). *For primary graded sequences $\mathcal{I}_{1,\bullet},\dots,\mathcal{I}_{n,\bullet}$ in the semigroup $\mathcal{S}$ we have*

$$e(\mathcal{I}_{1,\bullet},\mathcal{I}_{1,\bullet},\mathcal{I}_{3,\bullet},\dots,\mathcal{I}_{n,\bullet})\, e(\mathcal{I}_{2,\bullet},\mathcal{I}_{2,\bullet},\mathcal{I}_{3,\bullet},\dots,\mathcal{I}_{n,\bullet}) \ge e(\mathcal{I}_{1,\bullet},\mathcal{I}_{2,\bullet},\mathcal{I}_{3,\bullet},\dots,\mathcal{I}_{n,\bullet})^2.$$

Corollary 5.17 (Brunn–Minkowski inequality for multiplicities in semigroups). *Let $\mathcal{I}_\bullet$ and $\mathcal{J}_\bullet$ be primary graded sequences in the semigroup $\mathcal{S}$. We have*

$$e(\mathcal{I}_\bullet)^{1/n} + e(\mathcal{J}_\bullet)^{1/n} \ge e(\mathcal{I}_\bullet + \mathcal{J}_\bullet)^{1/n}.$$

6. MULTIPLICITIES OF IDEALS AND SUBSPACES IN LOCAL RINGS

Let R be a Noetherian local domain of Krull dimension n over a field $\mathbf{k}$, and with maximal ideal $\mathfrak{m}$. We also assume that the residue field $R/\mathfrak{m}$ is $\mathbf{k}$.

Example 6.1. Let X be an irreducible variety of dimension n over $\mathbf{k}$, and let p be a point in X. Then the local ring $R = \mathcal{O}_{X,p}$ (consisting of rational functions on X which are regular in a neighborhood of p) is a Noetherian local domain of Krull dimension n over $\mathbf{k}$. The ideal $\mathfrak{m}$ consists of functions which vanish at p.

If $\mathfrak{a},\mathfrak{b} \subset R$ are two $\mathbf{k}$-subspaces, then by $\mathfrak{a}\mathfrak{b}$ we denote the $\mathbf{k}$-span of all the xy where $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. Note that if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R, then $\mathfrak{a}\mathfrak{b}$ coincides with the product of $\mathfrak{a}$ and $\mathfrak{b}$ as ideals.

Definition 6.2. (i) A $\mathbf{k}$-subspace $\mathfrak{a}$ in R is called $\mathfrak{m}$-*primary* if it contains a power of the maximal ideal $\mathfrak{m}$.

(ii) A *graded sequence of subspaces* is a sequence $\mathfrak{a}_\bullet = (\mathfrak{a}_1,\mathfrak{a}_2,\dots)$ of $\mathbf{k}$-subspaces in R such that for all $k,m>0$ we have $\mathfrak{a}_k\mathfrak{a}_m \subset \mathfrak{a}_{k+m}$. We call $\mathfrak{a}_\bullet$ an $\mathfrak{m}$-*primary sequence* if moreover $\mathfrak{a}_1$ is $\mathfrak{m}$-primary. It then follows that every $\mathfrak{a}_k$ is $\mathfrak{m}$-primary and hence $\dim_{\mathbf{k}}(R/\mathfrak{a}_k)$ is finite. (If each $\mathfrak{a}_k$ is an $\mathfrak{m}$-primary ideal in R, we call $\mathfrak{a}_\bullet$ an $\mathfrak{m}$-*primary graded sequence of ideals.*)

When $\mathbf{k}$ is algebraically closed, an ideal $\mathfrak{a}$ in $R = \mathcal{O}_{X,p}$ is $\mathfrak{m}$-primary if the subvariety it defines around p coincides with the single point p itself.

Example 6.3. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary subspace. Then the sequence $\mathfrak{a}_\bullet$ defined by $\mathfrak{a}_k = \mathfrak{a}^k$ is an $\mathfrak{m}$-primary graded sequence of subspaces.

Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $\mathfrak{m}$-primary graded sequences of subspaces. Then the sequence $\mathfrak{c}_\bullet = \mathfrak{a}_\bullet\mathfrak{b}_\bullet$ defined by

$$\mathfrak{c}_k = \mathfrak{a}_k\mathfrak{b}_k$$

is also an $\mathfrak{m}$-primary graded sequence of subspaces, which we call the *product of* $\mathfrak{a}_\bullet$ *and* $\mathfrak{b}_\bullet$.

Definition 6.4. Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$-primary graded sequence of subspaces. Define the function $H_{\mathfrak{a}_\bullet}$ by

$$H_{\mathfrak{a}_\bullet}(k) = \dim_{\mathbf{k}}(R/\mathfrak{a}_k).$$

We call it the *Hilbert–Samuel function of* $\mathfrak{a}_\bullet$. The *Hilbert–Samuel function* $H_\mathfrak{a}(k)$ of an $\mathfrak{m}$-primary subspace $\mathfrak{a}$ is the Hilbert–Samuel function of the sequence $\mathfrak{a}_\bullet = (\mathfrak{a}, \mathfrak{a}^2, \ldots)$. That is, $H_\mathfrak{a}(k) = \dim_\mathbf{k}(R/\mathfrak{a}^k)$.

Remark 6.5. For an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ it is well-known that, for sufficiently large values of k, the Hilbert–Samuel function $H_\mathfrak{a}$ coincides with a polynomial of degree n called the *Hilbert–Samuel polynomial of* $\mathfrak{a}$ [25].

Definition 6.6. Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$-primary graded sequence of subspaces. We define the *multiplicity* $e(\mathfrak{a}_\bullet)$ to be

$$e(\mathfrak{a}_\bullet) = n! \lim_{k\to\infty} \frac{H_{\mathfrak{a}_\bullet}(k)}{k^n}.$$

(It is not a priori clear that the limit exists.) The multiplicity $e(\mathfrak{a})$ of an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ is the multiplicity of its associated sequence $(\mathfrak{a}, \mathfrak{a}^2, \ldots)$. That is,

$$e(\mathfrak{a}) = n! \lim_{k\to\infty} \frac{H_{\mathfrak{a}}(k)}{k^n}.$$

(Note that by Remark 6.5 the limit exists in this case.)

The notion of multiplicity comes from the following basic example:

Example 6.7. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary subspace in the local ring $R = \mathcal{O}_{X,p}$ of a point p in an irreducible variety X over an algebraically closed filed $\mathbf{k}$. Let $f_1, \ldots, f_n$ be generic elements in $\mathfrak{a}$. Then the multiplicity $e(\mathfrak{a})$ is equal to the intersection multiplicity at p of the hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \ldots, n$.

In Section 8 we use the material in Section 5 to give a formula for $e(\mathfrak{a}_\bullet)$ in terms of the covolume of a convex region.

One can also define the notion of mixed multiplicity for $\mathfrak{m}$-primary ideals as the polarization of the Hilbert–Samuel multiplicity $e(\mathfrak{a})$; i.e. it is the unique function $e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ which is invariant under permuting the arguments, is multi-additive with respect to product, and for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$ the mixed multiplicity $e(\mathfrak{a}, \ldots, \mathfrak{a})$ coincides with $e(\mathfrak{a})$. In fact one can show that in the above definition of mixed multiplicity the $\mathfrak{a}_i$ need not be ideals and it suffices for them to be $\mathfrak{m}$-primary subspaces.

Similar to multiplicity we have the following geometric meaning for the notion of mixed multiplicity when $R = \mathcal{O}_{X,p}$ is the local ring of a point p on an n-dimensional algebraic variety X. Take $\mathfrak{m}$-primary subspaces $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in R. The mixed multiplicity $e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ is equal to the intersection multiplicity, at the origin, of the hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \ldots, n$, where each f_i is a generic function from the $\mathfrak{a}_i$.

7. CASE OF MONOMIAL IDEALS AND NEWTON POLYHEDRA

In this section we discuss the case of monomial ideals. It is related to the classical notion of Newton polyhedron of a power series in n variables. We will see that our Theorem 5.10 in this case immediately recovers (and generalizes) the local version of the celebrated Bernstein–Kushnirenko theorem [16; 1, Sect. 12.7]).

Let R be the local ring of an affine toric variety at its torus fixed point. The algebra R can be realized as follows: Let $\mathcal{C} \subset \mathbb{R}^n$ be an n-dimensional strongly convex rational polyhedral cone with apex at the origin, that is, $\mathcal{C}$ is an n-dimensional convex cone generated by a finite number of rational vectors and it does not contain any lines through the origin. Consider the semigroup algebra over $\mathbf{k}$ of the semigroup of integral points $\mathcal{S} = \mathcal{C} \cap \mathbb{Z}^n$. In other words, consider the algebra of Laurent polynomials consisting of all the f of the form $f = \sum_{\alpha\in\mathcal{C}\cap\mathbb{Z}^n} c_\alpha x^\alpha$, where we have used the shorthand notation $x = (x_1, \ldots, x_n)$, $\alpha = (a_1, \ldots, a_n)$ and $x^\alpha = x_1^{a_1} \ldots x_n^{a_n}$. Let R be the localization of this Laurent polynomial algebra at the maximal ideal $\mathfrak{m}$ generated by nonconstant monomials. (Similarly, instead of R we can take its completion at the maximal ideal $\mathfrak{m}$, which is an algebra of power series.)

Definition 7.1. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary monomial ideal in R, that is, an $\mathfrak{m}$-primary ideal generated by monomials. To $\mathfrak{a}$ we can associate a subset $\mathcal{I}(\mathfrak{a}) \subset \mathcal{C} \cap \mathbb{Z}^n$ by

$$\mathcal{I}(\mathfrak{a}) = \{\alpha \mid x^\alpha \in \mathfrak{a}\}.$$

The convex hull $\Gamma(\mathfrak{a})$ of $\mathcal{I}(\mathfrak{a})$ is usually called the *Newton polyhedron* of the monomial ideal $\mathfrak{a}$. It is a convex unbounded polyhedron in $\mathcal{C}$; moreover, it is a $\mathcal{C}$-convex region. The *Newton diagram of* $\mathfrak{a}$ is the union of bounded faces of its Newton polyhedron.

Remark 7.2. It is easy to see that if $\mathfrak{a}$ is an ideal in R, then $\mathcal{I} = \mathcal{I}(\mathfrak{a})$ is a semigroup ideal in $\mathcal{S} = \mathcal{C} \cap \mathbb{Z}^n$; that is, if $x \in \mathcal{I}$ and $y \in \mathcal{S}$, then $x + y \in \mathcal{I}$.

Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary monomial ideal. Then for any $k > 0$ we have $\mathcal{I}(\mathfrak{a}^k) = k * \mathcal{I}(\mathfrak{a})$. It follows from Proposition 5.9 that $\mathcal{I}_\bullet$ defined by $\mathcal{I}_k = k * \mathcal{I}(\mathfrak{a})$ is a primary graded sequence in $\mathcal{S} = \mathcal{C} \cap \mathbb{Z}^n$ and the convex region $\Gamma(\mathcal{I}_\bullet)$ associated to $\mathcal{I}_\bullet$ coincides with the Newton polyhedron $\Gamma(\mathfrak{a}) = \operatorname{conv}(\mathcal{I}(\mathfrak{a}))$ defined above.

More generally, let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$-primary graded sequence of monomial ideals in R. Associate a graded sequence $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$ in $\mathcal{S}$ to $\mathfrak{a}_\bullet$ by

$$\mathcal{I}_k = \mathcal{I}(\mathfrak{a}_k) = \{\alpha \mid x^\alpha \in \mathfrak{a}_k\}.$$

Clearly, for any k we have $k * \mathcal{I}(\mathfrak{a}_1) = \mathcal{I}(\mathfrak{a}_1^k) \subset \mathcal{I}(\mathfrak{a}_k)$. From the above we then conclude the following:

Proposition 7.3. *The graded sequence $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$ is a primary graded sequence in the sense of Definition* 5.3.

Let $\Gamma(\mathfrak{a}_\bullet)$ denote the convex region associated to the primary graded sequence $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$ (Definition 5.7). We make the following important observation that $\mathfrak{a}_\bullet \mapsto \Gamma(\mathfrak{a}_\bullet)$ is additive with respect to the product of graded sequences of monomial ideals.

Proposition 7.4. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $\mathfrak{m}$-primary graded sequences of monomial ideals in R. Then $\mathcal{I}(\mathfrak{a}_\bullet\mathfrak{b}_\bullet) = \mathcal{I}(\mathfrak{a}_\bullet) + \mathcal{I}(\mathfrak{b}_\bullet)$. It follows from Proposition* 5.12 *that*

$$\Gamma(\mathfrak{a}_\bullet\mathfrak{b}_\bullet) = \Gamma(\mathfrak{a}_\bullet) + \Gamma(\mathfrak{b}_\bullet).$$

From Propositions 7.3 and 7.4 and Theorem 5.10 we readily obtain the following.

Theorem 7.5. *Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$-primary graded sequence of monomial ideals in R. Then*

$$e(\mathfrak{a}_\bullet) = n! \operatorname{covol}(\Gamma(\mathfrak{a}_\bullet)).$$

In particular, if $\mathfrak{a}$ is an $\mathfrak{m}$-primary monomial ideal, then

$$e(\mathfrak{a}) = n! \operatorname{covol}(\Gamma(\mathfrak{a})).$$

Here $\Gamma(\mathfrak{a})$ is the Newton polyhedron of $\mathfrak{a}$, i.e. the convex hull of $\mathcal{I}(\mathfrak{a})$.

Theorem 7.6. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be $\mathfrak{m}$-primary monomial ideals in R. Then the mixed multiplicity $e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ is given by*

$$e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = n!\ CV(\Gamma(\mathfrak{a}_1), \ldots, \Gamma(\mathfrak{a}_n)),$$

where as before CV denotes the mixed covolume of cobounded convex regions.

Remark 7.7. Using Theorem 5.13, one can define the mixed multiplicity of $\mathfrak{m}$-primary graded sequences of monomial ideals. Then Theorem 7.6 can immediately be extended to mixed multiplicities of $\mathfrak{m}$-primary graded sequences of monomial ideals.

One knows that the mixed multiplicity of an n-tuple $(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ of $\mathfrak{m}$-primary subspaces in R gives the intersection multiplicity, at the origin, of hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \dots, n$, where each f_i is a generic element from the subspace $\mathfrak{a}_i$. Theorem 7.6 then gives the following corollary.

Corollary 7.8 (local Bernstein–Kushnirenko theorem)**.** *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be $\mathfrak{m}$-primary monomial ideals in R. Consider a system of equations $f_1(x) = \ldots = f_n(x) = 0$ where each f_i is a generic element from $\mathfrak{a}_i$. Then the intersection multiplicity at the origin of this system is equal to* $n! \operatorname{covol}(\Gamma(\mathfrak{a}_1), \dots, \Gamma(\mathfrak{a}_n))$.

Remark 7.9. (i) When R is the algebra of polynomials $\mathbf{k}[x_1, \dots, x_n]_{(0)}$ localized at the origin (or the algebra of power series localized at the origin), i.e. the case corresponding to the local ring of a smooth affine toric variety, Corollary 7.8 is the local version of the classical Bernstein–Kushnirenko theorem [1, Sect. 12.7].

(ii) As opposed to the proof above, the original proof of the Kushnirenko theorem is quite involved.

(iii) Corollary 7.8 has been known to the second author since the early 1990s (cf. [13]), although, as far as the authors know, it has not been published.

8. MAIN RESULTS

Let R be a domain over a field $\mathbf{k}$. Equip the group $\mathbb{Z}^n$ with a total order respecting addition.

Definition 8.1 (valuation)**.** A *valuation* $v\colon R \setminus \{0\} \to \mathbb{Z}^n$ is a function satisfying the following:

(1) For all $0 \neq f, g \in R$, $v(fg) = v(f) + v(g)$.
(2) For all $0 \neq f, g \in R$ with $f + g \neq 0$ we have $v(f + g) \geq \min(v(f), v(g))$. (One then shows that when $v(f) \neq v(g)$, $v(f + g) = \min(v(f), v(g))$.)
(3) For all $0 \neq \lambda \in \mathbf{k}$, $v(\lambda) = 0$.

We say that v has *one-dimensional leaves* if whenever $v(f) = v(g)$, there exists $\lambda \in \mathbf{k}$ with $v(g + \lambda f) > v(g)$.

By definition $\mathcal{S} = v(R \setminus \{0\}) \cup \{0\}$ is an additive subsemigroup of $\mathbb{Z}^n$, which we call the *value semigroup of* (R, v). Any valuation on R extends to the field of fractions K of R by defining $v(f/g) = v(f) - v(g)$. The set $R_v = \{0 \neq f \in K \mid v(f) \geq 0\} \cup \{0\}$ is a local subring of K called the *valuation ring of* v. Also $\mathfrak{m}_v = \{0 \neq f \in K \mid v(f) > 0\} \cup \{0\}$ is the maximal ideal in R_v. The field $R_v/\mathfrak{m}_v$ is called the *residue field of* v. One can see that v has one-dimensional leaves if and only if the residue field of v is $\mathbf{k}$.

Definition 8.2. For a subspace $\mathfrak{a}$ in R define $\mathcal{I} = \mathcal{I}(\mathfrak{a}) \subset \mathcal{S}$ by

$$\mathcal{I} = \{v(f) \mid f \in \mathfrak{a} \setminus \{0\}\}.$$

Similarly, for a graded sequence of subspaces $\mathfrak{a}_\bullet$ in R, define $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$ by

$$\mathcal{I}_k = \mathcal{I}(\mathfrak{a}_k) = \{v(f) \mid f \in \mathfrak{a}_k \setminus \{0\}\}.$$

For the rest of the paper we assume that R is a Noetherian local domain of dimension n such that R is an algebra over a field $\mathbf{k}$ isomorphic to the residue field $R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. Moreover, we assume that R has a good valuation in the following sense:

Definition 8.3. We say that a $\mathbb{Z}^n$-valued valuation v on R with one-dimensional leaves is *good* if the following hold:

(i) The value semigroup $\mathcal{S} = v(R \setminus \{0\}) \cup \{0\}$ generates the whole lattice $\mathbb{Z}^n$, and its associated cone $C(\mathcal{S})$ is a strongly convex cone (recall that $C(\mathcal{S})$ is the closure of convex hull of $\mathcal{S}$). It implies that there is a linear function $\ell\colon \mathbb{R}^n \to \mathbb{R}$ such that $C(\mathcal{S})$ lies in $\ell_{\geq 0}$ and intersects $\ell^{-1}(0)$ only at the origin.

(ii) We assume there exists $r_0 > 0$ and a linear function ℓ as above such that for any $f \in R$ if $\ell(v(f)) \geq kr_0$ for some $k > 0$, then $f \in \mathfrak{m}^k$.

Condition (ii) in particular implies that for any $k > 0$ we have

$$\mathcal{I}(\mathfrak{m}^k) \cap \ell_{\geq kr_0} = \mathcal{S} \cap \ell_{\geq kr_0}.$$

In other words, the sequence $\mathcal{M}_\bullet$ given by $\mathcal{M}_k = \mathcal{I}(\mathfrak{m}^k)$ is a primary graded sequence in the value semigroup $\mathcal{S}$.

The following is a generalization of Proposition 7.3.

Proposition 8.4. *Let v be a good valuation on R. Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$-primary graded sequence of subspaces in R. Then the associated graded sequence $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$ is a primary graded sequence in the value semigroup $\mathcal{S}$ in the sense of Definition* 5.3.

Proof. Let $m > 0$ be such that $\mathfrak{m}^m \subset \mathfrak{a}_1$. Then for any $k > 0$ we have $\mathfrak{m}^{km} \subset \mathfrak{a}_k$, which then implies that $\mathcal{M}_{km} \subset \mathcal{I}_k$. This proves the claim. □

Example 8.5. As in Section 7, let R be the local ring of an affine toric variety at its torus fixed point: Take $\mathcal{C} \subset \mathbb{R}^n$ to be an n-dimensional strongly convex rational polyhedral cone with apex at the origin. Consider the algebra of Laurent polynomials consisting of all the f of the form $f = \sum_{\alpha \in \mathcal{C} \cap \mathbb{Z}^n} c_\alpha x^\alpha$. Then R is the localization of this algebra at the maximal ideal generated by nonconstant monomials. Take a total order on $\mathbb{Z}^n$ which respects addition and is such that the semigroup $\mathcal{S} = \mathcal{C} \cap \mathbb{Z}^n$ is well-ordered. We also require that if $\ell(\alpha) > \ell(\beta)$, then $\alpha > \beta$, for any $\alpha, \beta \in \mathbb{Z}^n$. Such a total order can be constructed as follows: pick linear functions $\ell_2, \dots, \ell_n$ on $\mathbb{R}^n$ such that $\ell, \ell_2, \dots, \ell_n$ are linearly independent and for each i the cone $\mathcal{C}$ lies in $(\ell_i)_{\geq 0}$. Given $\alpha, \beta \in \mathbb{Z}^n$, set $\alpha > \beta$ if $\ell(\alpha) > \ell(\beta)$, or $\ell(\alpha) = \ell(\beta)$ and $\ell_2(\alpha) > \ell_2(\beta)$, and so on.

Now one defines a (lowest term) valuation v on the algebra R with values in $\mathcal{S} = \mathcal{C} \cap \mathbb{Z}^n$ as follows: For $f = \sum_{\alpha \in \mathcal{S}} c_\alpha x^\alpha$ put

$$v(f) = \min\{\alpha \mid c_\alpha \neq 0\}.$$

Clearly, v extends to the field of fractions of Laurent polynomials and in particular to R. Similarly v can be defined for formal power series and formal Laurent series. It is easy to see that v is a valuation with one-dimensional leaves on R. Let us show that it is moreover a good valuation. Take $0 \neq f \in R$. Without loss of generality we can take f to be a Laurent series $f = \sum_{\alpha \in \mathcal{S}} c_\alpha x^\alpha$. Applying Proposition 5.9 to the sequence $\mathcal{I}_\bullet$, where $\mathcal{I}_k = \{\alpha \mid x^\alpha \in \mathfrak{m}^k\}$, we know that there exists $r_0 > 0$ with the following property: if for some $\alpha \in \mathcal{S}$ we have $\ell(\alpha) \geq kr_0$, then $x^\alpha \in \mathfrak{m}^k$. On the other hand, if $\alpha \leq \beta$, then $\ell(\alpha) \leq \ell(\beta)$. Thus $\ell(\beta) \geq kr_0$ and hence $x^\beta \in \mathfrak{m}^k$. It follows that if $\ell(v(f)) \geq kr_0$, then all the nonzero monomials in f lie in $\mathfrak{m}^k$ and hence $f \in \mathfrak{m}^k$. This proves the claim that v is a good valuation on R.

The arguments in Example 8.5 in particular show the following:

Theorem 8.6. *If R is a regular local ring, then R has a good valuation.*

Proof. The completion $\overline{R}$ of R is isomorphic to an algebra of formal power series over the residue field $\mathbf{k}$. The above construction gives a good valuation on $\overline{R}$. One verifies that the restriction of this valuation to R is still a good valuation. □

More generally, one has

Theorem 8.7. *Suppose R is an analytically irreducible local domain (i.e. the completion of R has no zero divisors). Moreover, suppose that there exists a regular local ring S containing R such that S is essentially of finite type over R, R and S have the same quotient field $\mathbf{k}$ and the residue field map $R/\mathfrak{m}_R \to S/\mathfrak{m}_S$ is an isomorphism. Then R has a good valuation.*

Proof. By Theorem 8.6, S has a good valuation. By the linear Zariski subspace theorem in [9, Theorem 1] or [3, Lemma 4.3], $v_{|R}$ is a good valuation for R too. $\square$

Using Theorem 8.7, as in [3, Theorem 5.2] we have the following:

Theorem 8.8. *Let R be an analytically irreducible local domain over $\mathbf{k}$. Then R has a good valuation.*

Proposition 8.9. *Let $\mathcal{I} = \mathcal{I}(\mathfrak{a})$ be the subset of integral points associated to an $\mathfrak{m}$-primary subspace $\mathfrak{a}$ in R. Then we have*

$$\dim_{\mathbf{k}}(R/\mathfrak{a}) = \#(\mathcal{S} \setminus \mathcal{I}).$$

Proof. Take $m > 0$ with $\mathfrak{m}^m \subset \mathfrak{a}$ and let r_0 and ℓ be as in Definition 8.3. If $\ell(v(f)) > mr_0$, then $f \in \mathfrak{m}^m \subset \mathfrak{a}$. Thus the set of valuations of elements in $R \setminus \mathfrak{a}$ is bounded. In particular $\mathcal{S} \setminus \mathcal{I}$ is finite. Let $\{v_1, \dots, v_r\} = \mathcal{S} \setminus \mathcal{I}$. Let $B = \{b_1, \dots, b_r\} \subset R$ be such that $v(b_i) = v_i$, $i = 1, \dots, r$. We claim that no linear combination of $b_1, \dots, b_r$ lies in $\mathfrak{a}$. By contradiction suppose $\sum_i c_i b_i = a \in \mathfrak{a}$. Then $v\bigl(\sum_i c_i b_i\bigr)$ is equal to $v(b_j)$ for some j. This implies that $v(b_j)$ should lie in $\mathcal{I}$, which contradicts the choice of the v_i. Thus the image of B in $R/\mathfrak{a}$ is a linearly independent set. Among the set of elements in R that are not in the span of $\mathfrak{a}$ and B take f with maximum $v(f)$. If $v(f) = v(b)$ for some $b \in B$, then we can subtract a multiple of b from f getting an element g with $v(g) > v(f)$, which contradicts the choice of f. Similarly $v(f)$ cannot lie in $\mathcal{I}$; otherwise we can subtract an element of $\mathfrak{a}$ from f to arrive at a similar contradiction. This shows that the set of images of elements of B in $R/\mathfrak{a}$ is a $\mathbf{k}$-vector space basis for $R/\mathfrak{a}$, which proves the proposition. $\square$

Corollary 8.10. *Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$-primary graded sequence of subspaces in R and put $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$. We then have*

$$e(\mathfrak{a}_\bullet) = e(\mathcal{I}_\bullet).$$

Definition 8.11. To the sequence of subspaces $\mathfrak{a}_\bullet$ we associate a $\mathcal{C}$-convex region $\Gamma(\mathfrak{a}_\bullet)$, which is the convex region $\Gamma(\mathcal{I}_\bullet)$ associated to the primary sequence $\mathcal{I}_\bullet = \mathcal{I}(\mathfrak{a}_\bullet)$. The convex region $\Gamma(\mathfrak{a}_\bullet)$ depends on the choice of the valuation v. By definition the convex region $\Gamma(\mathfrak{a})$ associated to an $\mathfrak{m}$-primary subspace $\mathfrak{a}$ is the convex region associated to the sequence of subspaces $(\mathfrak{a}, \mathfrak{a}^2, \mathfrak{a}^3, \dots)$.

Theorem 8.12. *Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$-primary graded sequence of subspaces in R. Then*

$$e(\mathfrak{a}_\bullet) = n! \lim_{k\to\infty} \frac{H_{\mathfrak{a}_\bullet}(k)}{k^n} = n! \operatorname{covol}(\Gamma(\mathfrak{a}_\bullet)).$$

In particular, if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, we have $e(\mathfrak{a}) = n! \operatorname{covol}(\Gamma(\mathfrak{a}))$.

The following superadditivity follows from Proposition 5.12. Note that $\mathcal{I}(\mathfrak{a}_\bullet) + \mathcal{I}(\mathfrak{b}_\bullet) \subset \mathcal{I}(\mathfrak{a}_\bullet\mathfrak{b}_\bullet)$ (cf. Proposition 7.4).

Proposition 8.13. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be two $\mathfrak{m}$-primary graded sequences of subspaces in R. We have*

$$\Gamma(\mathfrak{a}_\bullet) + \Gamma(\mathfrak{b}_\bullet) \subset \Gamma(\mathfrak{a}_\bullet\mathfrak{b}_\bullet).$$

From Theorem 8.12, Proposition 8.13 and Corollary 3.4 we readily obtain

Corollary 8.14 (Brunn–Minkowski inequality for multiplicities). *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be two $\mathfrak{m}$-primary graded sequences of subspaces in R. Then*

$$e(\mathfrak{a}_\bullet)^{1/n} + e(\mathfrak{b}_\bullet)^{1/n} \geq e(\mathfrak{a}_\bullet\mathfrak{b}_\bullet)^{1/n}. \tag{8.1}$$

Remark 8.15. By Theorem 8.8 and Corollary 8.14 we obtain the Brunn–Minkowski inequality (8.1) for an analytically irreducible local domain R. But in fact the assumption that R is analytically irreducible is not necessary: Suppose R is not necessarily analytically irreducible. First by a reduction theorem the statement can be reduced to $\dim R = n = 2$. In dimension 2, inequality (8.1) implies that the mixed multiplicity of ideals $e(\cdot,\cdot)$, regarded as a bilinear form on the (multiplicative) semigroup of $\mathfrak{m}$-primary graded sequences of ideals, is positive semidefinite restricted to each local analytic irreducible component. But the sum of positive semidefinite forms is again positive semidefinite, which implies that Corollary 8.14 should hold for R itself.

As another corollary of Theorem 8.12 we can immediately obtain inequalities between the multiplicity of an $\mathfrak{m}$-primary ideal, multiplicity of its associated initial ideal and its length. Let R be a regular local ring of dimension n with a good valuation (as in Example 8.5 and Theorem 8.6).

Corollary 8.16 (multiplicity of an ideal versus multiplicity of its initial ideal)**.** *Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal in R and let $\mathbf{in}(\mathfrak{a})$ denote the initial ideal of $\mathfrak{a}$, that is, the monomial ideal in the polynomial algebra localized at the origin $\mathbf{k}[x_1,\dots,x_n]_{(0)}$ corresponding to the semigroup ideal $\mathcal{I}(\mathfrak{a})$. We have*

$$e(\mathfrak{a}) \le e(\mathbf{in}(\mathfrak{a})) \le n!\,\dim_{\mathbf{k}}(R/\mathfrak{a}).$$

More generally, if $\mathbf{in}(\mathfrak{a}^k)$ denotes the monomial ideal in $\mathbf{k}[x_1,\dots,x_n]_{(0)}$ corresponding to the semigroup ideal $\mathcal{I}(\mathfrak{a}^k)$, then the sequence of numbers

$$\frac{e(\mathbf{in}(\mathfrak{a}^k))}{k^n}$$

is decreasing and converges to $e(\mathfrak{a})$ as $k\to\infty$.

Proof. From the definition one sees that $\Gamma(\mathbf{in}(\mathfrak{a}))$ is the convex hull of $\mathcal{I}(\mathfrak{a})$ (see Theorem 7.5). It easily follows that

$$\mathcal{I}(\mathfrak{a}) \subset \Gamma(\mathbf{in}(\mathfrak{a})) \subset \Gamma(\mathfrak{a}).$$

We now notice that $\dim_{\mathbf{k}}(R/\mathfrak{a})$ is the number of integral points in $\mathcal{S}\setminus\mathcal{I}(\mathfrak{a})$, which in turn is greater than or equal to the volume of $\mathbb{R}^n_{\ge 0}\setminus\Gamma(\mathbf{in}(\mathfrak{a}))$ and hence the volume of $\mathbb{R}^n_{\ge 0}\setminus\Gamma(\mathfrak{a})$. More generally, from the definition of $\Gamma(\mathfrak{a})$ we have an increasing sequence of convex regions:

$$\Gamma(\mathbf{in}(\mathfrak{a})) \subset \frac{1}{2}\Gamma(\mathbf{in}(\mathfrak{a}^2)) \subset \ldots \subset \Gamma(\mathfrak{a}) = \overline{\bigcup_{k=1}^{\infty}\frac{1}{k}\Gamma(\mathbf{in}(\mathfrak{a}^k))}.$$

Now from Theorem 8.12 we have $e(\mathfrak{a}) = n!\operatorname{covol}(\Gamma(\mathfrak{a}))$ and $e(\mathbf{in}(\mathfrak{a}^k)) = n!\operatorname{covol}(\Gamma(\mathbf{in}(\mathfrak{a}^k)))$ for each k. This finishes the proof. $\square$

The inequality $e(\mathfrak{a}) \le n!\,\dim_{\mathbf{k}}(R/\mathfrak{a})$ is a special case of an inequality of Lech [18, Theorem 3] (see also [6, Lemma 1.3]).

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