QUADRATIC ALLEMANDS

The sequence $\{x_n = n\}$ of natural numbers satisfies the two recursions

$$x_{n+1} + x_{n-1} = 2x_n$$

and

$$x_{n-1}x_{n+1} = x_n^2 - 1.$$

Thus x_{n-1} and x_{n+1} are the two roots of the quadratic equation

$$h(x, x_n) = 0$$

where

$$h(x, y) = x^2 - 2xy + (y^2 - 1) = (x - y - 1)(x - y + 1).$$

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This motivates the definition of a **(quadratic) allemand** $\{x_n\}$ which is determined by a quadratic symmetric function

$$\begin{aligned} h(x,y) &= x^2 + y^2 + \beta xy + \gamma(x+y) + \delta \\ &= x^2 + (\beta y + \gamma)x + (y^2 + \gamma y + \delta). \end{aligned}$$

and a seed x_0.

We form a bilateral sequence by solving the equation $h(x, x_0) = 0$, let x_{-1} be one of its roots and x_1 be the other.

We can then continue the sequence in both directions as follows. If x_{n-1} and x_n are both known, then $h(x_{n-1}, x_n) = 0$, and x_{n-2} is the root other than x_n of $h(x, x_{n-1}) = 0$ and x_{n+1} is the root other than x_{n-1} of $h(x, x_n) = 0$.

The allemand sequence satisfies the pair of recursions:

$$x_{n+1} + x_{n-1} = -(\beta x_n + \gamma) \tag{1}$$

and

$$x_{n+1}x_{n-1} = x_n^2 + \gamma x_n + \delta. \tag{2}$$

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and

$$x_{n+1} x_{n-1} = x_n^2 + \gamma x_n + \delta. \tag{2}$$

It turns out that a sequence defined by either one of these recursions individually is an allemand by a suitable choice of parameter.

If (1) holds, then

$$x_{n+1} x_{n-1} - x_n^2 - \gamma x_n = x_n x_{n-2} - x_{n-1}^2 - \gamma x_{n-1}.$$

If we take any sequence satisfying (1), we can define $\delta = x_0 x_2 - x_1^2 - \gamma x_1$ and the sequence becomes an allemand with the function $h(x, y)$.

Likewise, if (2) holds, then

$$\frac{x_{n+1} + x_{n-1} + \gamma}{x_n} = \frac{x_n + x_{n-2} + \gamma}{x_{n-1}},$$

so that any sequence satifying (2) can be made into an allemand by defining $\beta = -\frac{x_0 + x_2 + \gamma}{x_1}$.

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Example 1. Consider the sequence defined by the recursion:

$$x_0 = 1, x_1 = -1, \qquad x_{n+1} = \frac{x_n^2 - x_n + 1}{x_{n-1}}.$$

This sequence is $\{1, -1, 3, -7, 19, -49, 129, \dots\}$. Prove that every term in this sequence is an integer.

We can take $\gamma = -1$ and $\delta = 1$, and we find that

$$\frac{x_{n+1} + x_{n-1} - 1}{x_n} = \frac{x_2 + x_0 - 1}{x_1} = \frac{3 + 1 - 1}{-1} = -3 = -\beta.$$

Thus the sequence is an allemand satisfying

$$x_{n+1} = -3x_n - x_{n-1} + 1.$$

Each term is clearly an integer.

The function implementing the allemand is

$$h(x, y) = x^2 + y^2 + 3xy - (x + y) + 1.$$

If we change the intial conditions slightly to $x_0 = x_1 = 1$ with the same recursion for $x_{n+1}x_{n-1}$ we can again take $\gamma = -1$, $\delta = 1$. But now $x_n = 1$ for all n and $\beta = -1$, and

$$h(x, y) = x^2 + y^2 - xy - (x + y) + 1.$$

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What are some of the problems that can be explored? The locus of the equation $h(x,y) = 0$ is a conic section, a curve in the plane, one of whose axes is the line $y = x$. For an allemand, the mapping $(x_{n-1}, x_n) \to (x_n, x_{n+1})$ represents a dynamical system along this curve. We can investigate when allemands are periodic regardless of seed or whether for particular seed. We can also look at variation in behaviour when some parameters are held fixed and others allowed to vary.

By translation of the sequence and consider $h(x-\kappa, y-\kappa)$ for a suitable value of κ, as long as $\beta \neq -2$, we can tansform the allemand polynomial to the form $x^2+y^2+\beta xy+\delta$. When $\beta = -2$, the locus of $h(x,y) = 0$ is a parabola. By rescaling x and y, we may assume that $\delta = 0, \pm 1$.

When $\beta = 0$, the locus of $h(x,y) = 0$ is a circle (when not degenerate) and each allemand is periodic with period 4. The associated transformation is quarter turn about the centre.

Example 2. Let

$$h(x,y) = (x-y)^2 - (x+y) = x^2 - (2y+1)x + y(y-1).$$

Seeding this with 0 yields the bilateral sequence of triangular numbers

$$\{\dots, 15, 10, 6, 3, 1, 0, 0, 1, 3, 6, 10, 15, \dots\}$$

which satisfy the recursions

$$x_{n+1} = 2x_n - x_{n-1} + 1 \qquad \text{and} \qquad x_{n+1} = \frac{x_n^2 - x_n}{x_{n-1}}.$$

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The points whose coordinates are pairs of consecutive triangular numbers track along the parabola whose equation is $h(x, y) = 0$. The vertex of the parabola is $(0, 0)$ and its focus is $(1/8, 1/8)$.

Example 3. Let
$h(x, y) = x^2+y^2-xy-5(x+y)+6 = x^2-(5+y)x+(y-2)(y-3)$.

The allemand satisfies the two recursions $x_{n+1} + x_{n-1} = x_n + 5$ and $x_{n+1}x_{n-1} = (x_n - 2)(x_n - 3)$. This allemand has period 6, regardless of the seed. When $h(a, b) = 0$, the period is given by

$$(a, b, b - a + 5, 10 - a, 10 - b, 5 + a - b).$$

Two particular examples are $(0, 2, 7, 10, 8, 3)$ and $(1, 3 + \sqrt{7}, 7 + \sqrt{7}, 9, 7 - \sqrt{7}, 3 - \sqrt{7})$.

Note that $h(x + 5, y + 5) = x^2 + y^2 - xy - 19$, which moves the centre of the locus of the conic to the origian. The period we identified above of the allemand is moved to $(-5, -3, 2, 5, 3, -2)$.

Note that this allemand satisfies the recursion $x_{n+1} - x_n + x_{n-1} = 0$, so that $x_{n+3} = -x_n$ and every allemand has period 6. (This is true whenever $\beta = -1$.)

Suppose that we have an allemand with $\gamma = 0$ that is periodic with period m. Then, summing $x_{k+1} + x_{k-1} = \beta x_k$ as k goes from 1 to m, we find that $2(x_1 + \cdots + x_m) = \beta(x_1 + \cdots + x_m)$, whereupon either $\beta = 2$ (and we get an arithmetic progression) or $x_1 + \cdots + x_m = 0$.

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Example 4. Let

$$\begin{aligned} h(x,y) &= x^2 + y^2 + 2xy - 7(x+y) + 12 \\ &= x^2 + (2y-7)x + (y^2 - 7y + 12) \\ &= (x+y-3)(x+y-4), \end{aligned}$$

so that $x_{n+2} + x_{n-1} = -2x_n + 7$. A slice of the allemand is

$$\{\ldots, a-1, 4-a, a, 3-a, a+1, 2-a, \ldots\}.$$

A particular allemand is
$\{\ldots, -2, 5, -1, 4, 0, 3, 1, 2, 2, 1, 3, 0, 4, -1, 5, -2, 6, \ldots\}$.

Example 5. The sequence $\{x_n = an^2 + bn + c\}$ generated by a quadratic corresponds to the allemand

$$\begin{aligned} h(x,y) &= x^2 + y^2 - 2xy - 2a(x+y) + (a^2 - b^2 + 4ac) \\ &= x^2 - 2(y+a)x + (y-a)^2 - (b^2 - 4ac) \\ &= (x-y)^2 - 2a(x+y) + a^2 - (b^2 - 4ac). \end{aligned}$$

Since for the function $h(x,y)$ to generate an allemand, we need it to be symmetric and of degree 2 in *each* variable, we can allow $h(x,y)$ to be cubic or even quartic in the variables together. We look at the cubic situation next.

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We will now make a more systematic study of the allemand generated by the function $h(x, y) = x^2 + y^2 + \beta xy + \delta$, where $|\beta| \neq 2$. When $|\beta| < 2$, we can write

$$\beta = -2\cos\theta = -(e^{i\theta} + e^{-i\theta}).$$

Then recursion $x_{n+1} = 2(\cos\theta)x_n - x_{n-1}$ is satisfied by $x_n = \cos nx$, which corresponds to $\delta = \cos^2\theta - 1 = -\sin^2\theta$, and by $x_n = \sin n\theta$, which corresponds to $\delta = -\sin^2\theta$. Alternatively, we have the allemand $x_n = e^{in\theta}$ which corresponds to $\delta = 0$. We get a periodic allemand when $\theta = 2\pi/m$ for some integer m. The general solution of the linear recursion is

$$x_n = a\cos n\theta + b\sin n\theta$$

where the corresponding $\delta = \frac{1}{2}(a^2 + b^2)(\cos 2\theta - 1) = (a^2 + b^2)\sin^2\theta$.

When $|\beta| > 2$, then we can set $\beta = -2\cosh s = -(e^s + e^{-s}$. In this case, the recursion $x_{n+1} = (2\cosh s)x_n - x_{n-1}$ is satisfied by $x_n = e^{ns}$ with corresponding $\delta = 0$, and by $x_n = \cosh s$ with $\delta = (e - e^{-1})^2 = 4\sinh^2 s$.

Related to this, we can find a polynomial allemands by considering the function

$$h(x, y) = x^2 + y^2 - 2txy + \delta(t).$$

The linear recursion satisfied by the allemand is

$$x_{n+1} = 2tx_n - x_{n-1},$$

which is satisfied by the sequences of Chebyshev functions $T_n(t)$ and $U_n(t)$ of the first and second kind. For $T_n(t)$, the corresponding value of δ is $t^2 - 1$ and for $U_n(t)$, it is -1.

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Another generalization is to increase the humber of variables in the function h. Let

$$h(x, y, z) = x^2 + y^2 + z^2 + \beta(xy + yz + zx) + \gamma(x + y + z) + \delta$$
$$= x^2 + (\beta(y + z) + \gamma)x + (y^2 + z^2 + \beta yz + \gamma(y + z) + \delta).$$

We create an allemand with a double seed x_0 and x_1, and for each value of n, x_{n+1} and x_{n-2} are the roots of the quadratic

$$h(x, x_n, x_{n-1}) = 0.$$

This allemand satisfies the recursions

$$x_{n+1} = -(\beta(x_n + x_{n-1}) + \gamma + x_{n-2})$$

and

$$x_{n+1} = \frac{x_n^2 + x_{n-1}^2 + \beta(x_n x_{n-1}) + \gamma(x_n + x_{n-1}) + \delta}{x_{n-2}}.$$

In fact, our choice of which root is x_n and which is x_{n-2} determines two different allemands (*i.e.* one is not the reverse of the other). As for the two variable case, by a translation of the variables, we see that it suffices to consider $h(x, y)$ of the form

$$h(x, y, z) = x^2 + y^2 + z^2 + \beta(xy + yz + zx) + \delta$$

when $\beta \neq -1$.

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Example 6. Let

$$h(x^2+y^2+z^2) = x^2+y^2+z^2+3(xy+yz+zx)+1.$$

This satisfies the recursions $x_{n+1} = -3(x_n x_{n-1}) - x_{n-2}$ and

$$x_{n+1} = \frac{x_n^2 + x_{n-1}^2 + 3x_n x_{n-1} + 1}{x_{n-2}}.$$

Some allemands generated by this function are

$$\{\dots, -3, 2, -1, 0, 1, -2, 3, -4, 5, \dots\}$$
$$\{\dots, -2, 1, -1, 0, 2, -5, 9, -14, \dots\}$$
$$\{\dots, -9, 5, -2, 0, 1, -1, 0, 2, \dots\}$$
$$\{\dots, -5, 0, 2, -1, -3, 10, \cdots\}$$

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CUBIC ALLEMANDS

In 1942, the British mathematician, R.C. Lyness, introduced a sequence that exhibits an unexpected periodicity. This bilateral sequence of nonzero numbers $\{x_n\}$ satisfies $x_{n-1}x_{n+1} - x_n = 1$ for every integer n.

This relation can be rewritten

$$x_{n+1} = \frac{x_n + 1}{x_{n-1}}.$$

If we start with a pair of numbers x, y, the sequence turns out to be 5-periodic with period

$$x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}.$$

If we add these five quantities together, we get the function

$$f_1(x,y) = \frac{x^2y + xy^2 + x^2 + y^2 + 2(x+y) + 1}{xy}$$

which satisfies $f_1(x,y) = f_1(y,(y+1)/x)$.

(An interesting extension of this is that fact that the sequence defined by

$$x_{n+1} = \frac{x_n + x_{n-1} + 1}{x_{n-2}}$$

has period 8. This unfortunately does not seem to generalize.)

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Next, consider a sequence satisfying the recursion

$$x_{n+1} = \frac{x_n}{x_{n-1}}.$$

This has period 6 with the period $(x, y, \frac{y}{x}, \frac{1}{x}, \frac{1}{y}, \frac{x}{y})$.

Adding the six terms gives the function

$$f_0(x, y) = \frac{x^2y + xy^2 + x^2 + y^2 + x + y}{xy}.$$

which satisfies $f_0(x, y) = f_0(y, y/x)$.

These two example motivate considering the recursion

$$x_{n+1} = \frac{x_n + c}{x_{n-1}}$$

and checking out the function

$$f_c(x, y) = \frac{x^2y + xy^2 + x^2 + y^2 + (c + 1)(x + y) + c}{xy}.$$

We find that, although in general, the sequence is not periodic, nevertheless

$$f_c(x, y) = f_c(y, (y + c)/x).$$

Thus $f_c(x_n, x_{n+1}) = k$ for some constant k along the sequence $\{x_n\}$.

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Put another way, $\{x_n\}$ is an allemand generated by the cubic polynomial

$$h_{c,k}(x,y) = x^2y + xy^2 + x^2 + y^2 - kxy + (c+1)(x+y) + c$$
$$= (y+1)x^2 + (y^2 - ky + (c+1))x + (y+1)(y+c).$$

We suppose that the allemand never contains -1, which leads to a degeneracy.

We have a number of results involving periodicity:

(1) If $c \neq 0, 1$, then the allemand is not generally periodic, although periodicity may occur for specific pairs (c, k).

(2) The allemand S_c satisfied by $x_{n+1} = (x_n + c)/x_{n-1}$ is the constant sequence $\{r\}$ if and only if $r^2 = r + c$, and the corresponding value of k is $2(r+1) + 2(c+1)r^{-1} + cr^{-2}$.

(3) The allemand S_c has period 2 whose entries are the roots of the quadratic equation $t^2 + t + 1 = c$. The corresponding value of k is 3. (When $c = 1$, the situation degenerates.)

(4) The allemand S_c has the 3-period $(-1, -1, 1-c)$ and the corresponding value of k is $-(c+1)$. The case $c = 1$ is degenerate.

(5) Period 5 allemands occur if and only if $c = 1$ for any value of k.

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(6) When $c \neq 0$, there are no allemands S_c of smallest period 6.

(7) There is a allemand S_c of smallest period 7 if and only if $k = -(c^{-1} + 1 + c)$.

(8) There is an allemand S_c of smallest period 8 if and only if $k = -(2c + c^{-1})$.

(9) There is an allemand S_c of smallest period 9 if and only if $k = c^2 - 3c - c^{-1}$.

(10) There is an allemand S_c of smallest period 10 if and only if

$$k = -\left(\frac{1 + c + 3c^2 + c^3}{c(c+1)}\right).$$

If we fix a positive value of $c \neq 1$, and allow k to vary, we get loci of $h_{c,k}(x, y) = 0$ part of which are nested loops in the positive quadrant. The mapping induced on these loops is sometimes periodic, the smallest possible period being 9. The rotation number (the number of circuits needed to come back to the original point divided by the period) varies between 1/4 and 1/5. For example, when $c = 6$, there is an allemand with period 9 and rotation number 2/9 when $k = 18 - 1/6$.

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Generalizing, we define an **allemand** as a bilateral sequence $\{x_n : n \in \mathbb{Z}\}$ formed from a *seed* x_0 and a symmetric polynomial $h(x, y)$ which is quadratic in each for which, given any integer n, x_{n-1} and x_{n+1} are the two roots of the equation $h(x, x_n) = 0$ (or, equivalently $h(x_n, x) = 0$).

Observe that, once x_0 and x_1 have been fixed, then the remainder of the sequence is uniquely determined.

We have already considered quadratic allemands. Now we can look at the situation where $h(x, y)$ is cubic in x and y together, but quadratic in each variable.

Let

$$\begin{aligned} h(x,y) &= x^2y + xy^2 + \alpha(x^2 + y^2) + \beta xy + \gamma(x + y) + \delta \\ &= (y + \alpha)x^2 + (y^2 + \beta y + \gamma)x + (\alpha y^2 + \gamma y + \delta). \end{aligned}$$

As described above, we can determine a bilateral sequence by specifying the function h and a seed x_0. We will suppose that α is not a term in the sequence.

We now consider allemands that avoid the term $-\alpha$. Any such allemand corresponding to function $h(x, y)$ satisfies both of the recursions

$$x_{n+1} + x_{n-1} = -\left[\frac{x_n^2 + \beta x_n + \gamma}{x_n + \alpha}\right] \qquad (3)$$

$$x_{n+1}x_{n-1} = \frac{\alpha x_n^2 + \gamma x_n + \delta}{x_n + \alpha} \qquad (4)$$

As in the quadratic case, a cubic allemand satisfies two related recursions.

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Proposition. We have

$$h\left(\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)}, y\right) = \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)} h(x,y)$$

and

$$h\left(-\left[\frac{\alpha y^2+\beta y+\gamma}{y+\alpha}\right] - x, y\right) = h(x,y)$$

with the result that $\frac{h(x,y)}{xy}$ is invariant along any recursion satisfying (4) and $h(x,y)$ is invariant along any recursion satisfying (3).

It follows that, given a sequence $\{x_n\}$ satisfying (4), we can select β so that $h(x_n, x_{n+1}) = 0$ and so it is an allemand.

Similarly, given a sequence $\{x_n\}$ satisfying (3), we can select δ so that $h(x_n, x_{n+1}) = 0$ and so it is an allemand.

Any recursion satisfying either (5) or (6) will satisfy the other with a suitable choice of parameters.

Proof.

$$\begin{aligned}
h\left(\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)}, y\right) &= (y+\alpha)\frac{(\alpha y^2+\gamma y+\delta)^2}{x^2(y+\alpha)^2} \\
&\quad + (y^2+\beta y+\gamma)\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)} \\
&\quad + (\alpha y^2+\gamma y+\delta) \\
&= \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)} \\
&\quad \left[(\alpha y^2+\gamma y+\delta) + x(y^2+\beta y+\gamma) + x^2(y+\alpha)\right] \\
&= \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)} h(x,y).
\end{aligned}$$

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$$
\begin{aligned}
h&\left(-\left[\frac{y^2+\beta y+\gamma}{y+\alpha}\right]-x,y\right)\\
&=(y+\alpha)\left[x^2+\frac{2x(y^2+\beta y+\gamma)}{y+\alpha}+\frac{(y^2+\beta y+\gamma)^2}{(y+\alpha)^2}\right]\\
&\quad-(y^2+\beta y+\gamma)\left[\frac{y^2+\beta y+\gamma}{y+\alpha}+x\right]\\
&\quad+(\alpha y^2+\gamma y+\delta)=h(x,y).
\end{aligned}
$$

The remaining statements of the proposition follow from these relations. If (4) holds, we select β so that $h(x_k,x_{k+1})=0$ when $x_kx_{k+1}\neq 0$; it then follows by induction that $h(x_n,x_{n+1})=0$ and so the recursion satisfies (3). A similar argument applies if we assume that (3) holds.

If $H(x,y)=h(x+\kappa,y+\kappa)$, and if $\{x_n\}$ is an allemand for h, then $\{x_n-\kappa\}$ is an allemand for H; thus there is a one-one correspondence between the allemands for the two functions. Since

$$
\begin{aligned}
H(x,y)&=x^2y+xy^2+(\alpha+\kappa)(x^2+y^2)+(\beta+4\kappa)xy\\
&\quad+(3\kappa^2+2\alpha\kappa+\beta\kappa+\gamma)(x+y)+h(\kappa,\kappa)
\end{aligned}
$$

we can, without loss of generality, select κ to make a desired coefficient vanish.

We started this note with an example of an allemand S_1 of period 5. Before we examine this particular period in more detail, let us look at situations where other periodicities can occur.

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Constant allemands. Suppose $h(x,y) = x^2y + xy^2 + \alpha(x^2 + y^2) + \beta xy + \gamma(x+y) + \delta$ has a constant allemand for which $x_n = \kappa$ for all n.

A necessary condition for this is $\kappa^3 = \gamma\kappa + \delta$.

Conversely, for any γ and δ with $\delta \neq 0$, we choose real κ for which $\kappa^3 = \gamma\kappa + \delta$ and then select α and β so that $2\alpha + \beta = -3\kappa - \gamma/\kappa$ to obtain a constant allemand.

To obtain the zero sequence as a constant allemand, it is necessary and sufficient to take $\gamma = \delta = 0$.

This allemand satisfies the pair of recursion relations

$$x_{n+1} + x_{n-1} = -\frac{x_n(x_n + \beta)}{x_n + \alpha}$$

$$x_{n+1}x_{n-1} = \frac{\alpha x_n^2}{x_n + \alpha} \ .$$

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Here are a few simple cases where periodicity occurs:

Example 7. Let $h(x, y) = x^2y + xy^2 + \alpha(x^2 + y^2 + xy)$ Any allemand satisfies $x_{n+1} + x_n + x_{n-1} = 0$ and so is periodic with period 3, the periodic segment being of the form $\{u, v, -u-v\}$ with u, v, α related by $(u^2+uv+v^2)\alpha+(u+v)uv = 0$.

Example 8. Let $h(x, y) = x^2y+xy^2+\beta xy+\delta$ In this case, we find that $x_{n+1}+x_n+x_{n-1} = -\beta$ and $x_{n+1}x_nx_{n-1} = \delta$ so that the allemand must have period 3. An example is given by $h(x, y) = x^2 + y^2x + 4xy + 6$ which yields the allemand with period $(1, -2, -3)$.

Example 9. $h(x, y) = x^2y + xy^2 + \beta xy + \gamma(x + y)$

This yields an allemand of period 4, regardless of the starting value. If u, v are consecutive terms, then the next two terms are $\gamma/u, \gamma/v$.

Now we move to the period 5 situation.

Proposition 3. *An allemand generated by the function $h(x,y) = x^2y + xy^2 + \beta xy + \gamma(x+y) + \delta$ is periodic of period dividing 5 regardless of seed if and only if $\gamma^3 + \delta^2 = \beta\gamma\delta$.*

Proof. We can work from either recurrence satisfied by the allemand. From the first, we have

$$x_{n+2} + x_{n+1} + x_n = -\beta - \frac{\gamma}{x_{n+1}}$$
$$x_n + x_{n-1} + x_{n-2} = -\beta - \frac{\gamma}{x_{n-1}}$$

whence

$$\begin{aligned}
x_{n+2} + x_{n+1} + x_n + x_{n-1} + x_{n-2} &= -2\beta - \gamma\left(\frac{x_{n+1} + x_{n-1}}{x_{n+1}x_{n-1}}\right) - x_n \\
&= -2\beta + \gamma\left(\frac{x_n^2 + \beta x_n + \gamma}{\gamma x_n + \delta}\right) - x_n \\
&= \frac{-(\beta\gamma + \delta)x_n + (\gamma^2 - 2\beta\delta)}{\gamma x_n + \delta} \\
&= -\left(\beta + \frac{\delta}{\gamma}\right) + \left(\frac{\gamma^3 + \delta^2 - \beta\gamma\delta}{\gamma^2 x_n + \gamma\delta}\right).
\end{aligned}$$

The allemand has period 5 if and only if the sum of any five consecutive terms is constant. This will occur if and only if the term involving x_n vanishes identically, *i.e.*, if and only if the required condition holds.

An alternative argument uses the product of five consecutive terms. We have

$$\begin{aligned}
x_{n+2}x_{n+1}x_n^2x_{n-1}x_{n-2} &= (\gamma x_{n+1} + \delta)(\gamma x_{n-1} + \delta) \\
&= \gamma^2\left(\frac{\gamma x_n + \delta}{x_n}\right) - \gamma\delta\left(x_n + \beta + \frac{\gamma}{x_n}\right) + \delta^2 \\
&= (\gamma^3 + \delta^2 - \beta\gamma\delta) - \gamma\delta x_n
\end{aligned}$$

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whence

$$x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2} = -\gamma\delta + \left(\frac{\gamma^3+\delta^2-\beta\gamma\delta}{x_n}\right) .$$

The result again follows. **QED**

Proposition 3. *An allemand generated by the function* $h(x,y) = x^2y+xy^2+\beta xy+\gamma(x+y)+\delta$ *is periodic of period dividing 5 regardless of seed if and only if* $\gamma^3+\delta^2 = \beta\gamma\delta$.

We check this result against the example that introduced this paper. If $h_{c,k}(x,y) = x^2y+xy^2+x^2+y^2-kxy+(c+1)(x+y)+c$, we compute

$$\begin{aligned} H_{c,k}(x,y) &= h_{c,k}(x-1,y-1) \\ &= x^2y+xy^2-(k+4)xy+(k+c+2)(x+y)-(k+c+2) \\ &= yx^2+[y^2-(k+4)y+(k+c+2)]x+(k+c+2)(y-1). \end{aligned}$$

This has the form under discussion, where $\beta = -(k+4)$, $\gamma = -\delta = k+c+2$. It is readily checked that

$$\gamma^3+\delta^2-\beta\gamma\delta = (k+c+2)^2(c-1) .$$

If $k+c+2 = 0$, then the allemand degenerates. Regardless of the seed, 0 is a root of the quadratic equation, and if we then plug in 0 to get the neighbouring entries, the quadratic degenerates. Therefore, we should suppose that $k+c+2 \neq 0$. Thus, we see that every allemand from $h_{c,k}(x,y)$ is of period 5 if and only if $c = 1$.

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Here are examples for which the allemand can be periodic or nonperiodic depending on the seed.

Example 10. $h(x,y) = x^2y + xy^2 - 3xy - 2(x+y) + 6 = yx^2 + (y^2 - 3y - 2)x + (-2y + 6)$. We have that $h(x,1) = (x-2)^2$; $h(x,2) = 2(x-1)^2$; $h(x,1/2) = (1/4)(x-4)(2x-5)$, $h(x,4) = 2(2x-1)(x+1)$; $h(x,-1) = -(x-4)(x+2)$, $h(x,-2) = -2(x+1)(x-5)$, $h(x,5) = (x+2)(5x-2)$, $h(x,2/5) = (1/10)(x-5)(5x-13)$. This function generates the period 2 allemand $\{\cdots, 1, 2, 1, 2, \cdots\}$ as well as the (apparently) nonperiodic allemand $\{\cdots, 5/2, 1/2, 4, -1, -2, 5, 2/5, \cdots\}$.

Example 11. $h(x,y) = x^2y + xy^2 - \frac{26}{3}xy + 3(x+y) - 26 = yx^2 + (y^2 - \frac{26}{3}y + 3)x + (3y - 26)$. There is a period 2 allemand with period $(-\frac{1}{3}, 9)$.

$$h(x, -\tfrac{1}{3}) = -\tfrac{1}{3}x^2 + (\tfrac{1}{9} + \tfrac{26}{9} + 3)x - 27 = -\tfrac{1}{3}(x-9)^2.$$

$$h(x,9) = 9x^2 + (81 - 78 + 3)x + 1 = (3x+1)^2.$$

If we take 12 as the seed, we find that

$$h(x,12) = 12x^2 + 43x + 10 = (4x+1)(3x+10),$$

so that the allemand contains the chunk $\{\ldots, -1/4, 12, -10/3, \ldots$.

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There is a lot of untilled soil here. Let me list some question that can be investigated.

(1) What familiar sequences are in fact allemands? We have seen for example that the sequence of integers and of triangular numbers are allemands.

(2) What are the conditions for an allemand to be periodic regardless of the seed?

(3) When can an allemand be periodic for some seeds, and what can be said about the periods?

(4) Investigating the dynamic of the planar transformation $(x, y) - -- > (y, z)$ where $h(x, y) = h(y, z) = 0$. For example, when is it conjugate to the mapping $\theta \to c\theta$ on the unit circle in the complex plane parametrized by the argument θ?

(5) Is there anything useful or interesting that can be said about quartic allemands implemented by
$h(x, y) = x^2y^2+\epsilon(x^2y+xy^2)+\alpha(x^2+y^2)+\beta xy+\gamma(x+y)+\delta$?

References

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