Arithmetic problems for elementary teachers and their pupils

A.1. *The magic square.* Put the digits from 1 to 9, inclusive (each appearing once), into a 3×3 array in such a way that the sum of the numbers in each row, in each column, and along each of the two diagonals is the same.

Note: This is a real oldie, going back about 4500 years in ancient China (apparently a sage noticed this array of numbers carved into the back of a turtle as it crawled out of a stream). It is still a nice exercise for children to work on, particularly in groups. There are many different approaches that they may take, some based on some sort of reasoning, and some on intuition. The key is to work out what each row and column sum must be. There is a nice dessert: take the product of the elements of each row and add the three products together; do the same for the products of the columns. The three row products have the same sum as the three column products, and this sum turns out to be the square of the common sums of the rows and columns.

A.2. (a) Are there any whole numbers greater than 10 that are less than the sum of their digits?

(b) Are there any whole numbers greater than 10 that are less the the product of their digits?

(c) Are there any whole numbers greater than 10 that are equal to the sum of the squares of their digits?

A.3. There are two distinct pairs of whole numbers for which the sum of the numbers in one pair is equal to the product of the numbers in the other pair. Find them.

Note: This can be done in an elementary way. Arithmetically, the key to the solution is to realize that for one of the pairs, the product has to be less than the sum. For what sort of pairs of integers is this possible?

However, it is also instructive to take an algebraic approach and look for integer solutions of the pair of equations

$$x + y = uv \qquad \text{and} \qquad xy = u + v.$$

Subtracting one equation from the other leads to

$$(x-1)(y-1) + (u-1)(v-1) = 2.$$

This equation says that 2 is the sum of two nonnegative integers. In how many ways can this happen?

A.4. Here is a "silhouette" of a long multiplication calculation, where each X stands for a digit. Substitute digits for the X's to obtain a valid multiplication and check your answer by interchanging the order of the numbers multiplied.

			X	X	X	X
				X	X	X
			–	–	–	–
			X	X	X	X
	X	X	X	X	X	
	X	X	X	X		
–	–	–	–	–	–	–
X	X	X	X	X	X	X

Note: There are many answers to this problem, so as an exercise, students can check each others' work.

A.5. Place the digits 1 to 9 in the following triangular array in such a way that

(a) the number of the numbers along each side is the same;
(b) the sum of the squares of the numbers along each side is the same.

X
X X
X X
X X X X

Note: This can be done by a group of children using the hunt and peck method as they get a feel for the situation. However, a more systematic approach is available: the key to the problem is to realize that the sum of the three corner elements of the triangle as well as the sum of the squares of these numbers has to be divisible by 3. However, it seems miraculous that there actually *is* a solution to this problem.

A.6. (a) From the digits from 1 to 8 inclusive, using each exactly once, form two numbers and subtract the smaller from the larger. How can you do this so that the difference is as small as possible? What is the second smallest difference that can occur?

(b) Give an example of a four-digit difference that cannot occur. Give an example of a difference that can occur from two different subtractions. What is the maximum number of times a difference can occur?

A.7. Using the digits from 1 to 8 inclusive, as in the previous problem, form two numbers for which the product is as small as possible and is as large as possible.

A.8. Using each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once, make up three numbers A, B, C such that $A + B = C$. There are several possibilities. What are the smallest and largest possible values for C?

A.9. A choirboy, losing the thread of the sermon, allowed his eyes to wander to the hymnboard. There were three hymns that Sunday, and he noticed that each of the nine nonzero digits was used exactly once and that the numbers of the hymns were in the ratio 1:3:5. What were the hymns?

A.10. Pick one odd and one even digit. Using only these two digits, make up a ten-digit number that is evenly divisible by 1024.

A.11. There is a number whose cube (third power) and fourth power together use each of the ten digits exactly once. Find it.

A.12. Find a number such that the number and its square together have nine digits, with each nonzero digit appearing exactly once.

A.13. Find all numbers that are equal to the sum of the squares of its digits. Find all numbers that are equal to the sum of the cubes of its digits. Do the same for higher powers.

A.14. Find all the numbers $ABC\ldots$ whose digits are $A, B, C, \ldots$ written from left to right with the following properties:

(1) All the digits are distinct.

(2) The largest digit is one less than the number of digits in the number, so that they consist of $0, 1, 2, \ldots$ in some order.

(3) The number A is (evenly) divisible by 1; AB is divisible by 2; ABC is divisible by 3; and so on.

For example, 102 is such a number. However, 102453 does not work; 1 is divisible by 1, 10 is divisible by 2; 102 by 3; 1024 by 4; 10245 by 5; but 102453 is not divisible by 6. However, there is an example with 10 digits.

A.15. Assume that we have 12 rods, each 13 units long. They are to be cut into pieces measuring 3, 4 and 5 units, so that the resulting pieces can be assembled into 13 triangles of sides 3, 4 and 5 units. How should the rods be cut?

A.16. (a) When Suhanna celebrated her birthday this year (1997), she noted that her age was the sum of the digits of the year that she was born. How old is Suhanna this year?

(b) Determine other years this century for which it was possible for a person to have an age equal to the sum of the digits of the year of birth.

A.17. Determine the sum

$$\frac{35}{21} + \frac{17}{33} .$$

Give a description of the method that you used to add these fractions and justify it.

A.18. Write down two fractions with positive integers as numerator and denominator. Form a new fraction whose numerator is the sum of the numerators of the given fractions and whose denominator is the sum of the denominators of the given fraction.

(a) Is the new fraction the sum of the given fractions? Explain why.

(b) On a number line, indicate the positions of the three fractions you have written down. Do this for at least two other examples of pairs of fractions. Is there anything systemic about the relative positions of the three fractions?

(c) [Optional] The fraction $(a + c)/(b + d)$ turns out to be a weighted average of the fractions a/b and c/d, where a, b, c, d are positive. What are the weights?

A.19. (a) Using the algorithm of *long division*, divide 7643 into 8864892. Determine the divisor, dividend, quotient and remainder; write an equation to relate these four numbers.

(b) From the concept of what it means to divide one number into another, explain what the strategy of the algorithm is and why it works.

A.20. Suppose that p and $p + 2$ are both prime numbers exceeding 3. Some examples of such pairs are 5 and 7, 11 and 13, 17 and 19. Form the product $p(p + 2)$ and sum the digits of this number, then sum the digits of what you get, and so on until you reach a single digit number. What property do we discover and why is it so?

A.21. (a) Find a set of positive integers whose sum is 20 and whose product is as large as possible. Explain why your answer is correct. (Note that there is no restriction on the number of elements in the set, but be sure that you select only positive integers.)

(b) Write the number 101 as the sum of positive integers whose product is as large as possible.

A.22. Beside each positive integer, we can write all the positive integers that divide into it evenly. Thus we have:

1	1
2	1, 2
3	1, 3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6

(a) Some whole numbers have an even number of divisors; for example, 20 has six divisors (1, 2, 4, 5, 10, 20). Some have an odd number of divisors; for eample, 36 has nine divisors (1, 2, 3, 4, 6, 9, 12, 18, 26). How would you characterize those numbers that have an odd number of divisors?

(b) Is it possible to find 400 numbers between 1,000,000 and 2,000,000 that have an odd number of divisors? List five of them.

(c) Is it possible to find 400 numbers between 1,000,000 and 2,000,000 thaqt have an even number of divisors? List five of them.

A.23. Write out the first few multiples of the following numbers: (a) 142857; (b) 76923; (c) 588235294117647; (d) 52631578947368421; (e) 32258064516129. Do you notice anything? Try to find some more numbers that exhibit the same features. (Hint: Keep finding multiples of the numbers until the pattern is broken; it will be broken in the same way each time.)

A.24. In assigning residence rooms, a college gives preference to pairs of students in this order: AA, AB, AC, BB, BC, AD, CC, BD, CD, DD, in which AA means two fourth-year students, AB means a fourth- and a third-year student, $\cdots$, DD means two first-year students. Determine the smallest positive integer values to assign to A, B, C, D in such a way that the sums A+A, A+B, A+C, B+B, B+C, A+D, C+C, B+D, C+D, D+D are all positive integers with no two equal and they decrease in magnitude.

A.25. Find five rectangles, having sides with lengths chosen from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (each number used only once for a pair of adjacent sides) which can be put together with no overlapping and no empty holes to form a square.

A.26. A 5×9 rectangle is partitioned into a set of 10 rectangles, each having dimensions equal to positive integers. Prove that some two members of the smaller rectangles in the partition must be congruent (*i.e.*, have the same dimensions).

A.27. (a) Verify that

$$3516 \times 8274 = 4728 \times 6153$$

and that

$$992 \times 483 \times 156 = 651 \times 384 \times 299.$$

Is carrying out the multiplication the only way to verify the equations?

(b) The examples in (a) are such that the numbers on the left side are the ones on the right with the digits written in reverse order. Find more examples of products of two numbers, for which each number has two digits and for which one number has two and the other three digits.

A.28. Let n be a positive integer.

(a) Explain why the number $n(n+1)(n+2)$ (*i.e.* the product of three consecutive integers) is divisible by 6.

(b) Write down a table of values of $n(n+1)(n+2)$ for $1 \leq n \leq 16$ and beside each of these the largest proper divisor of $n(n+1)(n+2)$ (a proper divisor is a number, apart from $n(n+1)(n+2)$ itself that divide evenly into $n(n+1)(n+2)$). Discuss your method of approaching this problem.

(c) What is the largest positive integer that divides (evenly) into $n(n+1)(n+2)(n+3)$ for *every* positive integer n?

A.29. We will define a transformation on all of the numbers less than 100 as follows. Suppose the number is presented in the usual base-ten numeration. The transformation takes it to the number obtained by multiplying the units digits by 2 and adding it to the tens digit. For example, the number 37 gets carried to the number $17 = 2 \times 7 + 3$.

(a) What is the largest number that gets carried to a strictly bigger number by this transformation? Be sure to explain your reasoning to arrive at this.

(b) Suppose we begin with a certain number less than 100 and iterate (repeat) the transformation over and over. The set of numbers obtained in order is called an orbit. For example, the orbit that begins with 37 continues with 17 followed by 15 and then followed by 11. Describe what kinds of orbits occur with this transformation.

(c) Since the image of any one- or two-digit number under this transformation is a similar number, explain how the pigeonhole principle allows us to deduce that an orbit cannot consist of numbers that are all different, and deduce that each orbit eventually becomes periodic (repeats the same numbers over and over). What are the possible periodic orbits?

(d) Make a similar study of the transformation that takes a number less than 100 to 3 times the units digit plus the tens digit, and the transformation that takes a number less than 100 to 4 times the units digit plus the tens digit.

A.30. Are there integers M, N, K, such that $M + N = K$ and

(i) each of them consists of the seven digits $1, 2, 3, \cdots, 7$ exactly once?

(ii) each of them contains each of the nine digits $1, 2, 3, \cdots, 9$ exactly once?

(From Round 27 of the International Mathematical Talent Search.)

A.31. Give, in terms of years, days, hours, minutes and seconds, the age of a person who is one billion ($= 10^9$) seconds old.

A.32. Pick any pair of individual, say A and B (this may consist for example of you and a relative or a friend, or you and your dog). Supposing that both individuals live sufficiently long, determine for how many days the age last birthday of the older is exactly twice the age last birthday of the younger. Do this for three distinct pairs. Make a conjecture and explain why it works. What happens in the case of twins? (*For the purposes of this problem, we will neglect leap year and assume that each year is exactly 365 days long. The age last birthday of an individual is an integer which remains constant for a period of 365 days; it represents the number of complete years the person has survived. For example, on September 30, 1999, the age last birthday of an individual born on March 12, 1983 is 16.*)

A.33. In tracing ancestry, each person finds that they have two parents in the previous generation, four grandparents in the second generation back, eight great-grandparents in the third generation back. Of course, these persons might not always be distinct. The child of married cousins will have fewer than eight distinct great-grandparents. For the purposes of this problem, we will neglect foster parents and consider only the natural line.

(a) Consider the oldest direct ancestor with which you have had actual physical contact during your life. When was that person born? Mention a major historical event that occurred about the time of that person's birth.

(b) Consider going back exactly ten generations. What is the maximum number of ancestors that you would have in this generation? Give an estimate of when this generation would have lived and justify your answer. (You will need some idea of your background culture to determine the likely gap between generations; this will depend on the age that people customarily bear children.)

(c) Answer (b) for exactly twenty generations back.

(d) The number of your ancestors doubles with each generation you go back; however, the population of the world (or that part of the world from which your family hailed) was likely smaller in the olden days. How many generations should you go back in order to be sure of finding an ancestor who is related to you through two different lines. (Most of us probably came from a family which had inhabited a fairly small corner of the world until recently, when travelling became easier. Make a reasonable estimate of the population of this small corner and justify your answer.)

A.34. (a) Produce as many *essentially different* 3×3 square arrays of positive integers, in which each of the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ appears exactly once, and the sum of the numbers in each row, each column and down each of the two diagonals is a multiple of 9. [*By essentially different I mean that no one of them can be obtained from another by a rotation or a reflection. This is a little bit like a magic square; the only difference is that the sums are multiples of 9 rather than equal to 15. A good way to begin might be to write down all possible triples of distinct positive digits that sum to a multiple of 9; these will constitute the rows, columns and diagonals.*]

(b) Consider any of the arrays obtained in (a). Replace any number in the array by its difference when subtracted by 9 (thus, you would replace 2 by 7, and so on). This will give a square array that contains the digit 0. How would you modify this resultant array to obtain an array of the desired type? Notice that this gives a pairing of acceptable arrays. Indicate how the arrays that you obtained in (a) are paired off.

(c) Make a general observation about the central number in all the arrays that you have found. How do you account for this observation?

A.35. Canadian Airlines International has a direct flight in each direction between Montreal Mirabel Airport and Paris Charles de Gaulle Airport. Flight CP56 left Montreal at 19:20 and arrived in Paris at 8:15 the following morning, both local times. Flight CP57 leaves Paris at 14:00 and arrives in Montreal at 15:40, on the same day, both local times. Time difference between local times in the two cities is 6 hours.

(a) Explain why local times in the two cities are different and why one city is "behind" the other.

(b) Do both journeys take the same amount of time? If not, what is the time difference? Suggest a reason that the scheduled flight times are different.

A.36. In the days before compact discs and tapes, people used to listen to music recorded on vinyl discs from which the sound was picked up by a tone-arm as the disc rotated on a turntable. The tone-arm followed a continuous groove that tracked towards the centre of the record. However, we will think of the record of having a set of grooves as we move from the periphery to the centre (even through there is actually a single groove that spirals in). The records often had a diameter of 12 inches and rotated at the rate of 33 revolutions per minute. You are being asked to estimate the length of the groove.

To be more specific, let us consider a 33-rpm recording of the Sonata in B minor for flute and harpsichord obbligato by J.S. Bach, which according to the record jacket, requires for its performance a time period of 19 minutes and 30 seconds. The distance from the centre (hole) of the record to the outer groove is $5\frac{3}{4}$ inches (14.5 cm) and from the centre to the inner groove is $3\frac{3}{4}$ inches (7 cm.).

(a) About how many revolutions are made by the record during the performance of the Bach piece? Approximate the number of grooves to the inch and the distance between two adjacent grooves.

(b) Estimate the length of the groove. If the groove could be detached from the record and laid out in a straight line southwards from the corner of St. George and Bloor Street, estimate where the other end of the groove would come to.

(c) What is the speed of the record under the tone arm at the beginning of the performance? at the end of the performance?

A.37. The average age of the children in Miss Ruler's class is 10.3 years, while the average age of the boys is 10.6 years. Is it possible for the average age of the girls to be also greater than 10.3 years?

A.38. A newspaper article described a Grade 4 pupil whose allowance was deemed to be too low. This pupil canvassed seven other pupils whose allowances ranged from \$3 to \$10 inclusive and discovered that the average of the seven allowances was \$3.18. Is this possible? Explain.

A.39. Eugenia wrote two tests, answering 50% of the questions on one of the tests and 80% of the questions on the other. When the two tests were put together, it was found that she had answered 65% of the questions overall. Note that 65 is the average of 50 and 80; does this mean that there was the same number of questions on both papers? Discuss.

A.40. Three numbers have altogether nine digits, with each of the nonzero digits appearing exactly once. The numbers are in the ratio 1:3:5. What are they?

A.41. Any year whose number is not divisible by 4 has 365 days. If the number of a year is divisible by 400, it is a leap year and has 366 days; however, if the number of a year is divisible by 100 but not by 400, then it is not a leap year and has 365 days. (Thus, 1700, 1800 and 1900 were not leap years, but 2000 is a leap year.) All the other years, those divisible by 4 but not by 100, are leap years and have 366 days.

(a) Consider any person of your acquaintance (it could be yourself). Write down the date of that person's birth and the day of the week in 1998 that that person's birthday occurred on. Use this information to determine the day of the week on which that person was born.

(b) You are a manufacturer of "permanent" calendars; these are books of calendars for consecutive years that can be used continuously by the purchaser. If there were no leap years, one could simply use the calendars for seven consecutive years; you would use them consecutively and when you got to the end of the book, you would just start over. Forgetting about the exceptional non-leap-years (1700, 1800, 1900, 2100, etc.), how many consecutive years would you need to put calendars for in your book to get a complete cycle which would repeat over and over?

(c) Taking account of the leap year situation as descibed in the heading of this problem, how long is a complete cycle of years?

(d) Is the chance that the thirteenth of a month falls on a Friday exactly one in seven? Explain your answer.

A.42. Leap years are those years whose numbers are divisible by 4, except for those years whose numbers are divisible by 100 but not by 400. Thus 1700, 1800 and 1900 were not leap years, but 2000 will be.

(a) Determine the number of days that elapse from Jan. 1, 1600 until Dec. 31, 1999, inclusive. How many weeks is this?

(b) Determine the day of the week upon which the first day of January will fall in the year 2000. What day of the week did the first day of January fall on in the year 1600?

(c) Argue that in the long term, the probability that the thirteenth of the month falls on a Friday is not 1/7. (You can do this without doing part (d) if you wish.)

(d) Verify that over *any* period of 400 consecutive years, the thirteenth of the month falls on various days of the week with the following frequencies; Sunday - 687; Monday - 685; Tuesday - 685; Wednesday - 687; Thursday - 684; Friday - 688; Saturday - 684.

(e) Is it true or false that each calendar year must have at least one month on which the thirteenth occurs on Friday?

(f) What is the maximum number of times that the thirteenth of the month can occur on a Friday in a given calendar year?

(g) What is the maximum number of months that can occur between two months for which the thirteenth falls on a Friday?

A.43. A class had to compute $3 \times 5 + 6 \times 2$. Amy, using her *Whiz-bang* $2843.2786GHK$ pocket calculator punched in $3 \times 5 + 6 \times 2 =$ and got 42. Beth, using her *Acme* $767428.312PZZ7999$ pocket calculator punched in the same symbols and got 27. Carolyn has a *Punch-drunk* $TIPSY4962.111398$ pocket calculator. It does not have an "=" button, but does have an **Enter** button. She cleared her calculator, obtaining 0 on the monitor, and then punched in $3 \times 5 + 6 \times 2$ **Enter**. As she did so, she read on the monitor, respectively: 3, 0, 5, 5, 6, 30, 2, 2. Thus, her final result was 2.

(a) Which is the "correct" answer? Explain why.

(b) Account for the different results. What do you think that each calculator was programmed to do?

(c) Suggest to Carolyn what she might try to get a more reasonable answer on her calculator. It might help to know that, had Carolyn next pressed "+", the monitor would have read 32, and that had she next pressed "×", the monitor would have read 60.

A.44. (a) Put the all the numbers from 1 to 16 into a row in such an order that the sum of any two adjacent numbers in the row is a perfect square. (Each number must appear exactly once.)

(b) Is it possible to put all the numbers from 1 to 16 into a ring in such an order that the sum of any two adjacent numbers is a perfect square?

(c) For five numbers n of your choice, explain why it is or is not possible to arrange the numbers from 1 to n inclusive in a line such that any two adjacent numbers add to a perfect square.

A.45. Esther wanted to send a New Year's card to her friend Tamara who lived on Academy Street. Unfortunately, all she could recall was that the house number was 29 times the sum of its digits. Help Esther out.

A.46. Annie, Brian, Cathy, David and Ellen competed in six different sporting events, and in each event, 4 points were awarded for first place, 3 for second, 2 for third, 1 for fourth and 0 for last place. There were no ties in individual competitions.

At the end, Annie had the most points, and had 8 points more than Cathy. In addition, every individual had a total score that was a prime number. If David finishes second overall, how many points did he get?

A.47. (a) Consider the identity $x^2 - y^2 = (x - y)(x + y)$. You want to teach its use to a Grade 6 class (conveying its meaning without necessarily using algebraic notation). Explain how you would approach this topic and how you would convince the pupils that it always works.

(b) The identity in (a) can be used for mental arithmetic. First, one has to learn how to square one- and two-digit numbers in your head. Devise some rules that would help you to do this, and practise to test their effectiveness. Next, having the squares of two-numbers at your fingertips, we can use the identity to find the product of unequal two-digit numbers. For example,

$$38 \times 42 = (40 - 2) \times (40 + 2) = 40^2 - 2^2 = 1600 - 4 = 1596 \ .$$

Investigate which positive integers this will work for, *i.e.*, which positive integers can actually be written as the difference of two squares.

(c) Let $a = 4565486027761$, $b = 1061652293520$ and $c = 4687298610289$. Verify that $a + b$ and c are perfect squares and that $c^2 = a^2 + b^2$. You may use a pocket calculator; explain how you go about your task.

(d) A piece of metal rail 4800 cm long is clamped into position at both ends. On a hot day, the length of the rail expands by 2 cm. About how much will the centre of the rail be buckled up.

A.48. (a) We define a transformation T on the set of positive integers as follows: for each positive integer n, $T(n)$ is the sum of the squares of its digits. Thus $T(372) = 3^2 + 7^2 + 2^2 = 9 + 49 + 4 = 62$. Starting with any numnber that we want, we can form a chain of numbers by applying the transformation T over and over again. For example,

$$397 \longrightarrow 62 \longrightarrow 40 \longrightarrow 16 \longrightarrow 37 \longrightarrow \cdots$$

is the beginning of such a chain. Examine several examples, and describe what the ultimate result is when you apply T over and over again to produce a chain? Do the numbers eventually increase without bound? Do you ever get to a number that T takes to itself? Do you ever get to a "loop" of numbers that repeats over and over?

(b) Let U be the transformation on the set of positive integers that takes each number to the sum of the *cubes* of its digits. For example, $U(326) = 3^3 + 2^3 + 6^3 = 27 + 8 + 216 = 251$. Again, we can form chains of numbers by applying U over and over again, like

$$326 \longrightarrow 251 \longrightarrow 134 \longrightarrow 92 \longrightarrow 737 \longrightarrow 713 \longrightarrow \cdots \ .$$

Study the possible behaviours that such a chain can exhibit.

A.49. (a) The postal rates for properly addressed letters to domestic addresses are 47 cents, 75 cents, 94 cents, \$1.55 and \$2.05. Suppose that the post office wished to issue only two denominations of stamps which can be combined to make up these amounts. There are many ways in which this can be done. For example, if the post office issued only 5-cent and 2-cent stamps, we could make up 47 cents of postage with 9 5-cent and 1 2-cent stamp, and similarly make up the other postage amounts. However, this would require a lot of stamps on the envelopes. Determine the optimum choice for the two denominations which would minimize the number of stamps that would have to be put on the envelope.

(b) Answer (a), except for the case of postage amounts to US addresses: 60 cents, 85 cents, \$1.30, \$2.45, \$4.20.

A.50. Consider the following table of values:

n	x_n	y_n	z_n
0	0	1	0
1	1	1	1
2	2	3	6
3	5	7	35
4	12	17	204
5	29	41	1189
6	70	99	6930

This table gives the values of three sequences, that are *indexed* with the variable n, which indicates how far along in the sequence the term is. For example, the sequence $\{x_n : n = 0, 1, 2, 3, \cdots\}$ is equal to $\{0, 1, 2, 5, 12, 29, 70, \cdots\}$. The zeroth term of the sequence, labelled x_0 is 0, the first term, x_1 is 1, the fourth term, x_4 is 12. Each of the sequences is constructed according to some regular plan and you may wish to figure out what subsequent terms in the sequences will be.

However, when the sequences are continued according to the plan, you will observe that there are many interesting ways in which terms in each sequence are connected to terms in the same sequence and in the other sequences. For example, looking along the terms, we note that

$$5 = 2 + 3 = 6 - 1$$

$$29 = 12 + 17 = 35 - 6$$

$$169 = 70 + 99 = 204 - 35 \ .$$

This is part of a pattern that continues further, and can in fact be described by the equations

$$x_{2n+1} = x_{2n} + y_{2n} = z_{n+1} - z_n \ ,$$

where n is a positive integer. The numerical equations are the particular cases of this that you get when $n = 1, 2, 3$.

Here is another set of numerical equations:

$$1^2 + 1^2 = 2 \times 1$$

$$1^2 + 3^2 = 2 \times 5$$

$$3^2 + 7^2 = 2 \times 29 \ .$$

Give the general equation that relates the sequence $\{x_n\}$ and $\{y_n\}$, as illustrated by these numerical examples.

Find as many other possible general relationships as you can among the three sequences. Note, however, that this is an exercise in mathematical observation and pattern recognition, and that no proofs of any of the properties you find are necessary. (In particular, the pythagoren equations $3^2 + 4^2 = 5^2$ and $20^2 + 21^2 = 29^2$ are related to the sequences, which can be used to find other pythagorean triples for which the smallest two numbers differ by 1.)

A.51. In the British system of measurement, 12 inches = 1 foot, 3 feet = 1 yard, $5\frac{1}{2}$ yards = 1 rod, 4 rods = 1 chain, 10 chains = 1 furlong and 8 furlongs = 1 mile. To convert from British to metric units, note that 1 metre = 39.37 inches.

When the Town of York (now the City of Toronto) was laid out in 1793, a checkerboard arrangement of square blocks of lands, called *concessions* was surveyed. Each of these had a side length of 100 chains, and were separated by a square grid of roads, *concession lines*, so that these lines were thus separated by a distance of 100 chains. The base east-west line was Lot Street (now Queen Street), and the east-west concession lines are Lakeshore Road West (in Etobicoke), Queen St., Bloor St. (Danforth Ave.), St. Clair Ave., Eglinton Ave., Lawrence Ave., Wilson Ave. (York Mills Rd.), Sheppard Ave., Finch Ave., Steeles Ave. The base north-south line was Yonge St., and the concessions lines to the east are Bayview Ave., Leslie St., Woodbine Ave. and Victoria Park Ave. The north-south concession lines to the west are Bathurst St., Dufferin St., Keele St. (Parkside Dr.), Jane St., Royal York Rd., Kipling Ave and Brown's Line. To do the following questions, it might be handy to have a map of Toronto; one that would be suitable to your purpose would be a TTC route map available at any subway station.

(a) Express the distance between concession lines in each of the following units: (i) feet, (ii) miles, (iii) kilometres.

(b) A person in Long Branch wishes to take a streetcar from Brown's Line and Lakeshore Road (which angles into an extension of Queen Street as it passes through Mimico) to Queen and Neville Park (at Victoria Park). In New Toronto and Long Branch, Lakeshore Road is a concession line south of Queensway-Queen Street. Is it reasonable to expect the journey to be completed within one hour? Discuss.

(c) Suppose that one were to travel along the Bloor-Danforth subway line from Kipling to Victoria Park stations. What would be a reasonable amount of time to allot for the journey? Discuss.

(d) A nineteenth century farmer lives in Weston, near the corner of Weston Road and Lawrence Avenue (just west of Jane St.) and wishes to take his produce to market at King St. and Jarvis St. in downtown Toronto. Given the poor condition of the roads, he would be lucky to reach a speed of five miles per hour with his horse and cart. Making a reasonable assumption about a possible route, estimate how long he would need to make the journey one way. Do you think he might have to stay downtown overnight?

A.52. (a) What is the smallest multiple of 97 for which all the digits are even?

(b) What is the smallest multiple of 97, bigger than 97 itself, for which all the digits are odd?

(c) Answer the questions, analogous to (a) and (b), for 997 and 9997 instead of 97.

A.53. (a) Make up a six-digit number ABCDEF as follows. Choose the four leftmost digits, A, B, C, D in any way whatsoever. Select the rightmost digits E and F so that the numbers

$$ABCDEF\ , \qquad BCAEFD\ , \qquad CABFDE$$

are all multiples of 37. Investigate, giving at least five different examples. Do you think that you will succeed, regardless of your choice of A, B, C, D?

(b) Is it always possible to arrange, in addition, that $CBAFED$, $BACEDF$ and $ACBDFE$ are also multiples of 37?

[Here is an example: Suppose that you decide that the four digits are 2569. Then we can take the last two digits to be 65 and note that $256965 = 37 \times 6945$, $562659 = 37 \times 15207$, $625596 = 37 \times 16908$, $652569 = 37 \times 17637$, $526695 = 37 \times 14235$ and $265956 = 37 \times 7188$.]

A.54. In this question, you are going to create two columns of integers, labelled I and II. You will make an arbitrary choice of the first two numbers in Column I, and the first number in Column II will be the second number minus the first. From then on, you extend the columns using this rule: Suppose that, at some level, you have the number x in Column I and the number y in Column II. Then, at the next level, the number $x + y$ will appear in Column I and $y + 2$ will appear in Column II.

Here is an example: Let us start Column I with the numbers 9 and 6, so that the first entry of Column

II is -3. Then, we will get (observing that $6 = 9 - 3$, $5 = 6 - 1$, $6 = 5 + 1$, $9 = 6 + 3$, and so on):

Column I	Column II
9	-3
6	-1
5	1
6	3
9	5
14	7
21	9
30	11
41	13
54	15
69	17
86	19
105	21
...	...

Give at least three other examples of your own. In each case, continue on until you find that the number in Column I at least exceeds the product of the first two numbers in Column I. In the foregoing example, the product 54 of the first two numbers 9 and 6 actually appears in the column. What is the situation for your examples? Is there any kind of regularity? How far down do you have to go to be sure that the product of the first two numbers in Column I also appears in Column I? Make a conjecture. Look also at the product of other adjacent pairs of numbers in Column I.

A.55. Suppose that you were to write in increasing order and in lowest terms all of the fractions whose values lie between 0 and 1 inclusive and whose denominators do not exceed 25. What would be the two fractions closest, one on the left and the other on the right, to the fraction 13/19?

[If we were to ask that the denominators not exceed 6, the the list would be:

$$0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1 .$$

A.56. Paul Erdös once conjectured that, for every positive integer n exceeding 2, the fraction $4/n$ could be written as the sum of three distinct reciprocals of positive integers, *i.e.*,

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

where a, b, c are *distinct* positive whole numbers. For example:

$$\frac{4}{3} = \frac{1}{1} + \frac{1}{4} + \frac{1}{12}$$

$$\frac{4}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$$

$$\frac{4}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$$

$$\frac{4}{9} = \frac{1}{4} + \frac{1}{6} + \frac{1}{36}.$$

Can you find different representation for each of these examples? How small can you make the largest denominator of the three? Continue on, giving a list of presentations as the sum of exactly three integer reciprocals for every denominator up to 25.

Comment on any patterns that you might find that would help you find a suitable representation for higher denominators. For example, suppose that n were even; can you give an algorithm that will produce a result? (What about 4/426?) You should be able to find patterns that cover every velue of n except those that exceed a multiple of 24 by 1. However, you should be able to handle individual cases, such as 4/25, 4/49 and 4/73.

A.57. What is the product of the following fractions?

$$\left(\frac{2^3+1}{2^3-1}\right) \times \left(\frac{3^3+1}{3^3-1}\right) \times \left(\frac{4^3+1}{4^3-1}\right) \times \left(\frac{5^3+1}{5^3-1}\right) \times \cdots \times \left(\frac{99^3+1}{99^3-1}\right).$$

A.58. Consider the following triangular array of numbers:

11		15		1		13		8
	4		14		12		5	
		10		2		7		
			8		5			
				3				

In each row but the top row, each number is the (positive) difference of the two numbers directly above it. You will observe that the numbers have been chosen between 1 and 15 inclusive, but in this example, both the numbers 8 and 5 appear twice and the numbers 6 and 9 do not appear at all.

(a) Make a triangle consisting of the numbers 1, 2, 3, each used exactly once, with two numbers in the top row and one in the second such that the number in the second row is the difference of the numbers in the top row.

(b) Make a triangle consisting of the numbers 1, 2, 3, 4, 5, 6, each used exactly once, with three numbers in the top row, two in the second and one in the third, so that each number in the bottom two rows is the difference of the two directly above it.

(c) Make a triangle consisting of the numbers from 1 to 10 inclusive, with four numbers in the top row and the number of entries in each subsequent row decreasing by 1, so that each number in the bottom three rows is the difference of the two directly above it.

(d) Make a triangle of numbers like the one in the example at the head of the problem, but using each of the numbers from 1 to 15 inclusive exactly one, so that each number in the bottom four rows is the difference of the two directly above it.

A.59. Consider the following table of powers of 2 and powers of 5:

Exponent	Powers of 2	Powers of 5
1	2	5
2	4	25
3	8	125
4	16	625
5	32	3125
6	64	15625
7	128	78125

Observe the number of digits in these powers. As you move from one exponent to the next, if the number of digits of the power of 2 increases by 1 then the number of digits of the power of 5 does not change, while if the number of digits of the power of 5 increases by 1, then the number of digits of the power of 2 does not change. Why is this?

A.60. It is an interesting fact that $2+2 = 2 \times 2$. Find other pairs of numbers (including fractions) whose sum is equal to their product.

Note: With algebra, this is an easy problem to solve. However, for pupils without this tool, quite a bit of insight is needed to find some examples. However, once one has been found, pupils may be able to guess at a pattern to find others.

A.61. The Prague Astronomical Clock. In the Old Town City Hall in the Czech capital, Prague, there is a remarkable clock constructed by the clockmaker, Mikulus of Kadan, and mathematician, Jan Sindel, in 1410. The clock strikes each hour over a 24-hour cycle; the timing is control by the joint operation of two gears to control the timing of the chimes. A detailed description of the operation appears in an article *The mathematics behind Prague's horloge* by M. Krizek, A. Selcova and L. Somer in *Mathematical Culture*; this is available on the website *www.global-sci.org*. A Wikipedia entry provides the history of this device.

The smaller of the two gears has indentations separated by the distances 1, 2, 3, 4, 3, 2. The "Prague Clock Sequence" formed by repeating this cycle over and over has the interesting property that it can be partitioned into consecutive subsets whose sums give all of the positive integers in turn:

$$(1)(2)(3)(4)(3+2)(1+2+3)(4+3)(2+1+2+3)(4+3+2)(1+2+3+4)(3+2+1+2+3)(4+3+2+1+2)\ldots$$

Why is it possible to continue in this way, getting sets whose sums will yield all the positive integers in turn indefinitely?

Find other "Prague clock sequences" that have the same property; they must be periodic (the same set of numbers repeating over and over) and be partitioned to give each positive integer. A simple example is the sequence all of whose entries is 1. The cycle for the original Prague clock sequence has sum 15. You can begin by finding a sequences whose cycle has sum 3, then one whose cycle has sum 5. This may suggest a way to proceed.

A.62. A customer goes into a 7-11 corner store and makes four purchases. The clerk works out the total cost on a pocket calculator, and says that the bill is \$7.11. However, the customer noticed that the clerk had multiplied rather than added the four costs, and pointed this out. The clerk apologized, and then added the four numbers getting the same total, \$7.11! What did each of the four items cost?

Note: The costs are given in the usual way as a decimal fraction. The presence of the fractions is a confounding feature, and it is advisable to covert the situation to where the costs are given in cents. This makes it into a problem about whole numbers. However, caution is needed: the sum of the costs in cents is not equal to the product of the costs in cents; an adjustment involving a power of 10 is necessary. This is a hard problem, and is beswt handled by a group of students who can contribute ideas.

A.63. Let $A = 4444^{4444}$, the product of 4444 numbers, each equal to 4444. Let B be equal to the sum of the digits of A; C be equal to the sum of the digits of B and D be the sum of the digits of C. What is D?

Note: This problem actually appeared on an international olympiad, so it is unreasonable to expect ordinary students to solve it. However, its solution involves elementary arithmetical ideas, and I walk students and teachers through it to indicate how a talented student would approach the situation. Once the strategy is laid out, it is possible for students in an ordinary class to fill in the steps.

We can begin the discussion by asking why we cannot approach the problem in the usual way, *i.e.* working out A and then performing the necessary digital summing. First, it would require an inordinate amount of time. Secondly, if we make a slip along the way, then we are done. (Of course, we could have a high speed computer do this, but that would destroy the fun.) So we need to use guile.

A student with a little bit of background will recognize that this is a situation for "casting out nines". People who were educated in the first half of the twentieth century will know about this: if we take any positive integer, add its digits, and then repeat the process over and over until we come down to a single digit, then that digit will be the remainder when we divide the original number by 9. (This used to be used to check whether arithmetic computations might have an error.) When we cast out nines in this situation, we find that A, B, C and D each have a remainder of 7 when divided by 9.

Act II is to get an estimate of how big A, B, C, D are. The key idea is that the sum of the digits of any number cannot exceed 9 times the number of digits. Thus, we need to begin by getting an upper estimate for the number of digits of each of the number. We know how many digits there are in a power of 10, and we also know that $4444^{4444} < 10000^{4444} = 10^{17776}$. Thus A has fewer than $17777 < 20000$ digits and so $B < 180000$. We find that $C < 54$ and $D < 14$. So we are looking for a number that is less than 14 that leaves a remainder of 7 when you divide by 9.

A.64. The list of odd primes in order is $3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots$. Suppose we take the sum of two consecutive primes in this list, say $11 + 13 = 24 = 2 \times 3 \times 4$. No matter what consecutive pair we pick, the sum always can be written as the product of at least three whole numbers bigger than 1. Why is this?

Note: This is a problem where you have to pay attention to each of the main words in the problem, and understand their significance. A *prime* is a number that has only two divisors, itself and 1; any number that is not prime can be written as the product of three numbers bigger than 1. Two primes are *consecutive* when there are no primes in between them; every number in between is composite, when means that it can be written as a product of two numbers, each bigger than 1. The fact that the two primes to be added are odd means that their sum is *even*, and so divisible by 2. If we take the sum of two numbers, we get the average and the average of any two distinct numbers always lies between them.

The good news about a mathematical problem is that you are transported into a world that is hermetically sealed from the outside. The whole environment is contained within the statement of the problem, and you do not need to rely on extraneous experiences to solve it (apart from background mathematical knowledge, which can be considered as part of the hypotheses of the problem).