

COUNTING THE ZEROS

If we write down the numbers from 1 to 10, we need to use 11 digits. If we write down the numbers from 1 to 100, we need to use 192 digits: one each for the number from 1 to 9, two each for each of the ninety numbers from 10 to 99, and three for the number 100. How many of these digits are zeros? The digit zero appears only in the numbers 10, 20, 30, 40, 50, 60, 70, 80, 90, 100. Thus, there are 11 zeros among the digits. This is the same as the number of digits in writing the numbers from 1 to 10.

Now let us look at the numbers from 1 to 1000. To write all these numbers down, we need to use $9 + 180 + 2700 + 4 = 2893$ digits to accommodate the 9 single-digit numbers, the 90 double-digit numbers, the 900 triple-digit numbers and the number 1000. How many zeros did we need? There are 9 numbers of the form $d0$, 9 numbers of the form $d00$ (each with two zeros), 81 numbers of the form $d0k$, 81 numbers of the form $dk0$, where d and k are one of the nine nonzero digits, and the number 1000. The number of zeros we need is $9 + 18 + 81 + 81 + 3 = 192$. This is the same as the number digits needed to write the numbers from 1 to 100.

These are two instances of a general fact: *the number of digits to write down all the numbers from 1 to any power of 10 is equal to the number of zeros among the digits used to write down all the numbers from 1 to the next higher power of 10.*

You might think about this for a while, before I explain why it works at the next step.

We know that there are 2889 digits needed to write down all the numbers from 1 to 999. If we continue from here writing down the numbers up to 9999, we have 9000 numbers, each requiring 4 digits, so the number of digits needed to write all the numbers from 1 up to 999 is $2889 + 4 \times 9 \times 1000$.

How many zeros do we need in writing down the same set of numbers? Let us change the problem slightly and write down all the numbers from 0 to 9999 as three digit numbers by putting zeros in the left position: 0000, 0001, 0002, 0003, \dots , 9998, 9999. We now require 4×10000 digits; each digit appears equally often, 4×1000 times. Thus the nine nonzero digits appear $4 \times 9 \times 1000$ times; this is the same number of times if we wrote down the numbers without the extra zero on the left.

So we have used, in writing down the numbers from 1 to 9999, $2889 + 4 \times 9000$ digits, of which 4×9000 of them are known to be nonzero. Therefore the number of zeros used must be 2889. If we add in four for the number of zeros in 10000, we see that we need 2993 zeros altogether, the same as the number of digits for the numbers up to 1000.

I will leave it to you to take the argument up to 100,000.