Confidence in Mathematics

Suppose we write down any number with more than one digit, like 295. Now take the product of its digits: $2 \times 9 \times 5 = 90$. The result in this case is smaller than the number that we started with. If we try this a few more times, we keep getting a smaller result. For example, 2899 gives us the result $2 \times 8 \times 9 \times 9 = 1296$. Will this always happen?

In other words: given any number with at least two digits, is always true that it is larger than the product of its digits?

The first reaction of many readers might be that there is no way they can get anywhere with this question. Which is a pity, since there is no more to solving this problem than solving any number of the difficulties that one encounters in everyday life. So the first thing to realize is that you are perfectly capable of understanding what is going on. If you have confidence that the solution is within reach, then you are well on your way. The second thing is to accept that the answer might not come immediately. You might have to think about it awhile, and the solution might depend very much on how you look at the situation. It might be necessary to look at a few more examples to give you the proper insight. With these comments, I will leave you to it.

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In many of the examples you look at, you might have noticed that the resulting number has fewer digits, which will certainly make it smaller. However, this does not always happen. Look at four-digit numbers, for instance. The product of the digits is largest when the number is 9999, and this gives us the product $9 \times 9 \times 9 \times 9 = 6561$, which also has four digits. While this initial idea does not always work, maybe it can be adapted.

For sake of argument, take a four-digit number that begins with a 5. Then this number is surely at least $5000 = 5 \times 10^3$, while the product of its digits cannot exceed 5×9^3. Since $9^3 < 10^3$, we are in business. Can you apply the same idea to any number whose first digit is 6, say? Or any other digit?

At this point, we can understand why our little result is true. From the viewpoint of informal mathematics, we are finished. However, to make the argument clearly general and applicable to any number bigger than 9, we use the special language of mathematics — algebra.

Suppose that n is a number with $k \geq 2$ digits for which the first digit is a. Then n is at least as large as $a \times 10^{k-1}$. However, as none of its digits can exceed 9, the product of the digits of n is no larger than $a \times 9^{k-1}$. Since $9 < 10$, $9^{k-1} < 10^{k-1}$, so that the product of the digits of n is less than n.

If you are having trouble negotiating the symbolism, go through the argument with an example, such as 5867. In this case, $n = 5867$, $k = 4$ and $a = 5$, and you can replace the symbols with the numbers.