GIMPS and the prime form Hell

One of the most fascinating areas, to amateur and professionals alike, of mathematics is the theory of numbers, in particular of prime numbers. You will recall that these are positive whole numbers with exactly two divisors, itself and 1. The smallest prime numbers are 2, 3, 5, 7, 11, 13, 17, 19. Other numbers, like 4, 6, 8, 9, 10, 12, are said to be composite. (The number 1 belongs to neither category.)

One of the oldest results in mathematics (given by Euclid 2300 years ago) is that there are infinitely many primes. To put it another way, no matter how many primes you can find there is always one more. Here is how the argument goes. We know that 2 and 3 are prime. The number 7, which is one more that the product of 2 and 3, cannot be divisible by either 2 or 3. It must either be prime, or divisible by some other prime. We know that 7 is prime, so we next look at the number $2 \times 3 \times 7 + 1 = 43$. This is not divisible by any of the primes we have already, so it has to be divisible by some other prime. Actually, 43 turns out to be prime. So now we check out $2 \times 3 \times 7 \times 43 + 1 = 1807$. Any prime divisor of 1807 has to be different from 2, 3, 7 and 43. Now 1807 is not prime; it is the product 13×139, two new primes. We can turn the crank again and look at $2 \times 3 \times 7 \times 43 \times 13 \times 139 + 1$ and get a new prime. This can go on as long as we want.

It is not easy however to identify large numbers as prime or composite, even with modern high-powered computers. One way to find primes was given by a French monk with a passion for numbers, Marin Mersenne (1588-1648), who noted that if you raise 2 to a prime power and subtract 1, you often got a prime: $2^2 - 1 = 3$; $2^3 - 1 = 7$; $2^5 - 1 = 31$; $2^7 - 1 = 127$. This does not always work; $2^{11} - 1 = 2047 = 23 \times 89$. However, the largest primes that have been identified so far are these socalled Mersenne primes of the form $2^p - 1$ where p is a prime.

There is a continuing search for Mersenne primes; the largest, discovered in January, 2013, is $2^{57885161} - 1$, a number with over seventeen million digits. A complete set of Mersenne primes is known up to the 44th smallest one: $2^{32582657} - 1$. Between this and the 2013 record are three more Mersenne primes as well as numbers of the Mersenne form whose status is unknown.

All this computation may seem to be playfulness, but it has a serious purpose. We are dealing with extremely large numbers and sophisticated problems in computing, so the quest for large primes offers a chance to develop and test algorithms that may have a more practical purpose. It is possible for anyone to join in. The *Great International Mersenne Prime Search* (GIMPS) is a network of people among whom the task of finding primes is parcelled out. You can find out more from the website *www.mersenne.org*. For a complete listing of Mersenne primes along with the date of discovery, go to *www.mersenne.org/primes*.

Of course, there are loads of other primes as well, one of which bears the name of one of the seven lords of Hell. This is *Belphagor's Prime*:

$$1000000000000066600000000000001.$$

This palindrome has 31 digits, including two clusters of thirteen zeros with the Number of the Beast, 666, in the middle.

A sidelight to Mersenne primes is that they can be used to find perfect numbers. These are numbers like $6 = 1+2+3$ and $28 = 1+2+4+7+14$ that are the sums of

all of their smaller divisors. Each even perfect number has the form $2^{p-1} \times (2^p - 1)$ where p and $2^p - 1$ are both primes. It is not known whether there are any odd perfect numbers.

Here are a couple of facts that high school mathematics students may wish to prove:

1. If n is a composite number, then $2^n - 1$ is also composite.

2. The sum of two consecutive odd primes (all numbers in between are composite) can be written as the product of three numbers greater than 1. For example, $31 + 37 = 68 = 2 \times 2 \times 17$.