The Alabama Paradox

One proposed electoral reform is to adopt proportional representation. Each political party makes a list of candidates and each elector votes for a party. The number of seats in a legislature assigned to a party is proportional to the ballots cast in its favour. Thus, a party with a third of the votes should get a third of the seats.

A similar system can be used in a federal country, such as the United States or Germany, where the number of seats representing a state in a governing chamber is proportional to the population of the state. Sounds simple and fair. The principle seems exemplary, but the execution is complicated.

Consider, for example, the town of Backenac with 10000 voters. Unusually, the Backenacians vote for parties rather than individual candidates. The Aardvark Party, the Bactrian Party, the Cheetah Party and the Dromedary Party have to compete for nine seats in the town council. In a recent election, the number of votes cast for the respective parties is given by $(A, B, C, D) = (590, 810, 3780, 4820)$. Working out the seat share of each party in the council proportional to the number of votes received gives respectively $(a, b, c, d) = (0.531, 0.729, 3.402, 4.338)$. (You can check that these numbers add to 9.)

We have a problem. How do we use these figures to assign a whole number of seats to each party? This is how the Backenacians did it. They first gave 3 seats to the Cheetahs and 4 to the Dromedaries, these being the largest whole numbers less than their share. They then listed the fractional amounts in decreasing order: $0.729, 0.531, 0.402, 0.338$. There are two seats left to be assigned, and these went to Bactrians and the Aardvarks who had the largest two fractional amounts. The final assignment of seats to the four parties were: Aardvark – 1; Bactrian –1; Cheetah – 3; Dromedary – 4.

However, at the same election, there was a referendum that permitted the council to have ten members instead of nine. So the apportionment of seats had to be recalculated. Now the share of seats became $(a, b, c, d) = (0.59, 0.81, 3.78, 4.82)$. Following the same method as before, the Cheetahs were given 3 and the Dromedaries 4 seats. The largest three fractional amounts belonged to the Bactrian, Cheetah and Dromedary parties, so these parties wound up with the three remaining seats. The assignment of seats became: Aardvark – 0; Bactrian – 1; Cheetah – 4; Dromedary – 5.

Of course, the Aardvarks screamed bloody murder. How could the number of seats be increased while they got one fewer seat than before?

The Aardvarks were the victims of what is known as the Alabama Paradox. The apportionment method used by the Backenacians was used by the United States in the latter half of the nineteenth century for the House of Representatives. In 1880, a proposed increase in the number of representatives put the state of Alabama in precisely this position. When a similar anomaly occurred later in the century, this system was exchanged for another. There are many different systems of apportionment, and these do not always give the same results. How can we judge among them?

There are three things that we want in any system of apportionment. (1) There is no possibility of the Alabama paradox. (2) The quota rule must be obeyed; this means that the number of seats assigned must be the integer immediately below or immediately above the theoretical share of the seats. (3) If an election results in one party P getting more votes than before and a second party Q getting fewer, then no seat will be transferred from P to Q. In 1982, it was shown that these three conditions are incompatible. Whatever system you come up with cannot satisfy all three. So there really is no perfect system and we just have to do what seems to be most reasonable.

There is a branch of mathematics called *decision theory* that deals with a whole variety of social issues. Its role is to show what is possible, and to analyze the implications of different policy choices. The use of mathematics requires us to clarify our goals and assumptions. Often, we learn that tradeoffs are absolutely necessary. At the end of the day, mathematics may not prescribe a clear choice, but we may have a number of options to consider and be better able to make the necessary moral, ethical and political judgments.

There are a lot of dimensions to apportionment that cannot be discussed here. Google "Apportionment problems", "Alabama paradox" and "Apportionment House of Representatives" for further information on this fascinating topic.