The last ones standing.

Ms. Applephile lined up her class of thirty pupils and numbered them off – "one, two, three, $\cdots$, thirty" – so that each pupil was assigned a different number from 1 to 30, inclusive. She told all those whose number was a multiple of 1 to stand up. Everyone stood up. Then she told all those whose number was a multiple of 2 to sit down. So all the evenly numbered pupils sat down.

Then she said, "I want all those multiples of 3 who are standing to sit down, and all those multiples of 3 who are seated to stand up." So pupil 3 sat down, 6 stood up, 9 sat down, and so on. She then gave similar instructions in turn to pupils who were multiples of 4, 5, 6, and so on up to 30. Those that were standing had to sit down, and those that were sitting down had to stand.

At the end of this, who was left standing and why? Suppose instead of thirty, we have a million people numbered from 1 to 1,000,000, and we went through the multiples of all whole numbers from one to a million. What can we say about those who were left standing?

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To get the picture, let us look at the pupils numbered 12 and 16. The pupil with the number 12 will stand as a multiple of 1, sit as a multiple of 2, then stand (3), sit (4), stand (6) and finally sit (12). The pupil with the number 16 will stand (1), sit (2), stand (4), sit (8) and finally stand (16).

Those left standing will have a number that is a perfect square: 1, 4, 9, 16, 25. A pupil will have to move when any of the divisors of her number is mentioned by the teacher, so the number of movements depends on the number of divisors. If the number has evenly many divisors, like 12, the pupil will end up seated; if the number has oddly many divisors, like 16, the pupil will be finally standing. It turns out that the squares are characterized as those numbers with an odd number of divisors.

Why is this so? For every divisor of a number, we can identify another divisor (not necessarily different), such that the number is the product of the two divisors. This gives us a way of pairing them off. Take, for example, 12: the pairs of divisors are (1, 12), (2, 6), (3, 4), When we do this for a square, like 16, one divisor gets paired off with itself (because the number **is** a square). For 16, the pairing is (1, 16), (2, 8), (4, 4). For a nonsquare numbers, each pair has distinct numbers in it, so the number of divisors, being twice the number of pairs, is even. For a square number, the square root is a wallflower and gets paired with itself. So the divisors can be paired off with one left over, and the number of divisors is odd.