

July 23, 2007

Polynomials: Problems

This is a broad collection of problems having to do with polynomials. You are not expected to solve them all; the collection is chosen so that it will cater to a wide variety of interests and sophistication. You are to make a selection of problems and submit about 20-25 pages of solutions during the time of the course. You will be graded on the variation of the types of problems (not too many of the same kind), the quality of the solution (whether mundane or showing some ingenuity), the clarity of the presentation. If by any chance you come across the solution to any of these problems, you should give the reference and try to find a better solution. You may work together on the problems, provided all the participants are acknowledged in the solution along with the contribution each made towards the solution. You should plan to submit the problems in three or four instalments about a week apart. Solutions can be sent ahead of time to E.J. Barbeau, Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4 or by pdf file to barbeau@math.utoronto.ca, no later than June 15, 2007. The final due date is Tuesday, August 8.

As an alternative to problems on this list, you may solve problems in books, where the solution is not provided in the book. For example, the books by Marden and Prasolov have problem sections.

1. Prove that, for any natural number n , the equation

$$x(x+1)(x+2)\cdots(x+2n-1) + (x+2n+1)(x+2n+2)\cdots(x+4n) = 0$$

does not have real solutions.

2. Prove that, if the quadratic equation $x^2 + ax + b + 1 = 0$ has nonzero integer solutions, then $a^2 + b^2$ is a composite integer.
3. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.
4. Find all ordered pairs (x, y) that are solutions of the following system of two equations (where a is a parameter):

$$x - y = 2$$

$$\left(x - \frac{2}{a}\right)\left(y - \frac{2}{a}\right) = a^2 - 1.$$

Find all values of the parameter a for which the system has exactly two solution pairs consisting of nonnegative numbers. Find the minimum value of these solutions $x + y$ over all these values of a .

5. Prove that, if in the equation

$$2x^3 + 2y^3 - 3x^2 - 3y^2 + 1 = 0$$

any rational number is assigned to x , then there is a rational value of y for which the equation is satisfied.

6. One solution of the equation $2x^3 + ax^2 + bx + 8 = 0$ is $1 + \sqrt{3}$. Given that a and b are rational numbers, determine its other two solutions.
7. Find integers a, b, c with $a \neq 0$ for which the quadratic function $f(x) = ax^2 + bx + c$ satisfies

$$f(f(1)) = f(f(2)) = f(f(3)).$$

8. Prove that the equation $x^4 + 5x^3 + 6x^2 - 4x - 16 = 0$ has exactly two real solutions.

9. Let $p(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0$ be a polynomial with integer coefficients. Suppose that r is rational with $p(r) = 0$. Show that

$$c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \dots, c_n r^n + c_{n-1} r^{n-1} + \dots + c_1 r$$

are all integers.

10. Determine all functions f from the real numbers to the real numbers that satisfy

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y .$$

11. Let $x, y > 0$ be such that $x^3 + y^3 \leq x - y$. Prove that $x^2 + y^2 \leq 1$.

12. Suppose that p is a real parameter and that

$$f(x) = x^3 - (p + 5)x^2 - 2(p - 3)(p - 1)x + 4p^2 - 24p + 36 .$$

(a) Check that $f(3 - p) = 0$.

(b) Find all values of p for which two of the roots of the equation $f(x) = 0$ (expressed in terms of p) can be the lengths of the two legs of a right triangle with a hypotenuse of $4\sqrt{2}$.

13. Determine five integer values of p for which the polynomial $x^2 + 2002x - 1002p$ has integer roots.

14. Determine all integer pairs (x, y) for which $x^3 + 9xy + 127 = y^3$.

15. Find all values of the parameter a for which the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four real solutions which are in geometric progression.

16. For which real value of x is the function

$$(1 - x)^5(1 + x)(1 + 2x)^2$$

maximum? Determine its maximum value.

17. Let $p(z)$ be a polynomial of degree n all of whose zeros have absolute value 1. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z)$ have absolute value 1.

18. Suppose that a, b, c, A, B, C are all real with $Aa \neq 0$, and that $|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$ for all real x . Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC| .$$

19. Are there polynomials $a(x), b(x), c(y), d(y)$ for which

$$1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y) ?$$

20. Prove that for all integers numbers a, b, c , there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not a square.

21. (a) Let $f(z) = (2 - z^2)^{-1}$. Prove that all periodic points of f are real.

(b) Let $f_\lambda(z) = (\lambda - z^2)^{-1}$. For which positive real values of λ does f_λ have only real periodic points?

22. Let n be a positive integer and D denote the operation of taking the derivative. The *Legendre polynomial* $P_n(x)$ is defined by $P_0(x) = 1$ and

$$P_n(x) = \frac{1}{2^n n!} D^n [(x^2 - 1)^n] .$$

(a) Determine $P_1(x)$, $P_2(x)$, $P_3(x)$.

(b) What is the degree of $P_n(x)$?

(c) Show that

$$P_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{(2n-2r)!}{2^{nr}(n-r)!(n-2r)!} x^{n-2r}.$$

(d) Show that $y = P_n(x)$ is a solution of the differential equation

$$(1-x^2)D^2y - 2xDy + n(n+1)y = 0.$$

(e) Show that

$$(x-y) \sum_{m=0}^n (2m+1)P_m(y)P_m(x) = (n+1)[P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)].$$

(f) Prove that all the zeros of $P_n(x)$ are simple, real and lie in $(-1, 1)$.

23. Let c be a complex number with complex conjugate \bar{c} . Show that the quartic equation

$$z^4 - 2cz^3 + 2\bar{c}z - 1 = 0$$

has a root not on the unit circle ($|z| = 1$) if and only if $(\operatorname{Re}c)^{1/3} + (\operatorname{Im}c)^{1/3}i$ lies outside the circle.

24. Let $p(z)$ be a polynomial of positive degree. Prove that

$$\lim_{z \rightarrow \infty} \frac{zp'(z)}{p(z)} = n$$

and that

$$\frac{1}{2\pi i} \int_{C_R} \frac{p'(z)}{p(z)} dz = n$$

where C_R is the circle centred at the origin with radius R and R is sufficiently large. Deduce from this that the Fundamental Theorem of Algebra holds.

25. For which complex numbers c does the polynomial $p(z) = z^3 - cz^2 + \bar{c}z - 1$ possess exactly three distinct zeros with absolute value equal to 1?

26. Let k be the smallest positive integer with the property: *There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial*

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

27. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$ and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 1$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

28. If $p(x) = a_0 + a_1x + \cdots + a_mx^m$ is a polynomial with real coefficients, set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $f(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that (i) $g(0) = 1$ and $\Gamma(f(x)^n) = \Gamma(g(x)^n)$ for every integer $n \geq 1$.

29. Let $f(t) = t^2 - t - 1$. Determine $g(t)$ such that the degree of g does not exceed 4 and $f(t)g(t) \equiv 1 \pmod{5}$ for every integer t .
30. Let k be a positive integer exceeding 1, and let $f(x)$ be a polynomial of positive degree with nonnegative integer coefficients. Suppose that $S(n) = \{f(1), f(2), \dots, f(n)\}$ for each positive integer n .
- (a) Prove that there are infinitely many positive integers n for which $S(n)$ can be partitioned into k subsets in such a way that the sum of the numbers in each of the subsets is the same.
- (b) Prove or disprove that there is a positive integer N , such that, for every integer $n \geq N$, $S(n)$ can be partitioned as in (a) whenever k divides the sum $f(1) + f(2) + \dots + f(n)$.
31. Suppose that n is an odd positive integer, and that E_n is the smallest field of real numbers which is closed under the operation $x \rightarrow x^{1/n}$.
- (a) Suppose that $f(x)$ is an irreducible polynomial over the integers. Prove that $f(x)$ has at most one root in E_3 .
- (b) Are there fields E_n and polynomials irreducible over the integers which have more than one root in E_n ?
32. Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that $f(0) = 0$ and $f(1) = 1$ and that, for every positive integer n , $f(n) \equiv 0$ or $f(n) \equiv 1$ modulo p . Prove that $d \geq p - 1$. Give an example of such a polynomial.
33. (a) Find all natural numbers n for which the Diophantine equation $(x + y + z)^2 = nxyz$ has a positive integer solution (x, y, z) .
- (b) Determine the set of all triples of integers (x, y, z) with $1 \leq x \leq y \leq z \leq 1000$ for which $x^2 + y^2 + z^2$ is a multiple of xyz .
34. Suppose that a and b are the roots of the quadratic $x^2 + px + 1$ and that c and d are roots of the quadratic $x^2 + qx + 1$. Determine $(a - c)(b - c)(a + d)(b + d)$ as a function of p and q .
35. Find a nonzero polynomial $P(x, y)$ for which $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real a .
36. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, 3, \dots\}$ of polynomials by

$$f_0(x) \equiv 1$$

$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \geq 0$.

- (a) Prove that, for all n, x ,

$$f_n(x) = x^n f_n(1/x).$$

- (b) Determine a formula for the coefficient of x^k ($0 \leq k \leq n$) in $f_n(x)$.

37. (a) Suppose that n is a positive integer. Prove that

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x(x+k)^{k-1} (y-k)^{n-k}.$$

- (b) Prove that

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x(x - kz)^{k-1} (y + kz)^{n-k}.$$

38. Solve for integers x, y, z , the three systems

$$1 = x + y + z = x^3 + y^3 + z^k ,$$

where $k = 1, 2, 3$ separately.

39. The real numbers u and v satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 .$$

Determine $u + v$.

40. Solve for positive real values of x, y, t :

$$(x^2 + y^2)^2 + 2tx(x^2 + y^2) = t^2y^2 .$$

Determine the smallest value of t for which there is a solution in positive integers x, y, t . Are there infinitely many solutions for which the values of x, y, t are integers?

41. Let a, b, c be three real numbers for which $0 \leq c \leq b \leq a \leq 1$ and let w be a complex root of the polynomial $z^3 + az^2 + bz + c$. Must $|w| \leq 1$?

42. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3} .$$

Give an example to show that equality is possible.

43. Let x, y be positive real numbers for which $x + y = 1$. Prove that, for positive integers m and n exceeding 1,

$$(1 - x^m)^n + (1 - y^n)^m > 1 .$$

44. Is there a quadratic polynomial g with rational coefficients for which $g^4 + 1$ is reducible over the rationals?

45. Determine all the roots of the equation

$$\frac{(x^2 - x + 1)^3}{x^2(x - 1)^2} = \frac{(a^2 - a + 1)^3}{a^2(a - 1)^2} ,$$

where $a \neq 0, 1$ is a given parameter.

46. Let $p(x)$ be a monic polynomial of positive degree n with constant term equal to 1 and all coefficients nonnegative. Suppose further that all its n zeros are real. Prove that, for each positive integer m , $p(m) \geq (m + 1)^n$.

47. Determine all polynomials of the form

$$n!x^n + a_{n-1}x^{n-1} + \cdots + a_1x + (-1)^n(n + 1)n$$

with integer coefficients that have n real zeros x_1, x_2, \dots, x_n for which $k \leq x_k \leq k + 1$ for $1 \leq k \leq n$.

48. Solve the equation

$$x^2 + \left(\frac{x}{x + 1} \right)^2 = 1 .$$

49. Determine a real number k for which the quintic polynomial

$$x^5 + y^5 + z^5 + k(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)$$

is divisible by $x + y + z$.

50. Let x_1, x_2, \dots, x_n satisfy the equations

$$\sum_{j=1}^n \frac{x_j}{i+j} = \frac{4}{2i+1}$$

for $1 \leq i \leq n$. Determine

$$\sum_{j=1}^n \frac{x_j}{2j+1}.$$

51. Let $p(x)$ be a real polynomial with $p(x) \geq 0$ for $0 \leq x \leq 1$. Prove that there exist real polynomials $u(x)$ and $v(x)$ that are nonnegative for all real x and satisfy

$$p(x) = xu(x) + (1-x)v(x).$$

52. Let

$$f(x) = x^5 + 5x^4 + 5x^3 + 5x^2 + 1$$

and

$$g(x) = x^5 + 5x^4 + 3x^3 - 5x^2 - 1.$$

Determine all primes p for which there is an integer x with $0 \leq x < p$ for which both $f(x)$ and $g(x)$ are divisible by p . For each such p , determine all such x .

53. Suppose that $p(x)$ is a polynomial with real coefficients and that N is the number of distinct integers k for which $p(k)^2 = 1$. Prove that $N \leq (\deg p) + 2$.

54. Determine all solutions in integers of the equation

$$x^3 + (x+1)^3 + \dots + (x+7)^3 = y^3.$$

55. Solve over the reals the system

$$(x^2 - 6x + 13)y = 20$$

$$(y^2 - 6y + 13)z = 20$$

$$(z^2 - 6z + 13)x = 20$$

56. Find all real polynomials $p(x)$ for which there exists a unique polynomial $q(x)$ for which $q(0) = 0$ and

$$x + q(y + p(x)) = y + q(x + p(y)).$$

57. The monic polynomial $p(x) = x^n + \dots + a_1x + a_0$ of degree exceeding 1 has n distinct real negative zeros. Prove that $a_1p(1) > 2n^2a_0$.

58. Let $f(x) = x^3 - 3x + 1$. How many distinct real roots does the equation $f(f(x)) = 0$ have?

59. Solve over the complexes the system

$$x^4 + y^2 - xy^3 - (9/8)x = 0$$

$$y^4 + x^2 - x^3y - (9/8)y = 0$$

60. Solve for real x, y, u, v the system

$$x^2 + y^2 + u^2 + v^2 = 4$$

$$xu + yv + xv + yu = 0$$

$$xyu + yuv + uvx + vxy = -2$$

$$xyuv = -1 .$$

61. Find all positive real solutions of the system

$$xy + yz + zx = 12$$

$$xyz = 2 + x + y + z .$$

62. The sequence of polynomials $\{P_n\}$ is defined by $P_0(x, y) = 1$ and

$$P_{n+1}(x, y) = (x + y - 1)(y + 1)P_n(x, y + 2) + (y - y^2)P_n(x, y)$$

for $n \geq 0$, Prove that, for all n and for all x, y , $P_n(x, y) = P_n(y, x)$.

63. Let $f_0(x) = x^2 - 2$ and $f_n(x) = f(f_{n-1})(x)$ for $n \geq 1$. Prove that all of the zeros of $f_n(x)$ are real and distinct.

64. Let $p(z)$ be a polynomial of degree n at least 2 with distinct zeros z_1, z_2, \dots, z_n . Prove that

$$\sum_{k=1}^n \frac{1}{p'(z_k)} = 0 .$$

65. Determine the values of the parameter c for which the mapping $Q_c(z) = z^2 + c$ under iteration has (a) an attractive fixed point; (b) an attractive orbit of period 2. Sketch the locus of these values in the complex c -plane.

66. (a) Suppose that n is a positive integer greater than 1 and that $bc \neq 0$. Prove that $a + bz + cz^n$ has at least one zero in $|z| \leq 2|a/b|$.

b) Suppose that $2 \leq m < n$ and $bcd \neq 0$. Prove that $a + bz + cz^m + dz^n$ has at least one zero in $|z| \leq (17/3)|a/b|$.

67. (a) Let $p(z)$ be a polynomial of degree n . Prove that the multiple points of each curve $\operatorname{Re} p(z) = a$, $\Im p(z) = b$, $|p(z)| = c$ and $\arg p(z) = d$, where a, b, c, d are constants, lie at the zeros of $p'(z)$.

(b) Let K be the arc of $\operatorname{Im} p(z) = 0$ joining a pair of zeros of $p(z)$. Prove that at least one zero of $p'(z)$ lies on K .

68. Let r be the minimum with respect to all the zeros of the derivative $p'(z)$ of a polynomial $p(z)$ lying within $|z| \leq r$. Prove that at least one zero of $p(z)$ lies in $|z| \geq r$.

69. (a) Prove that, if all of the zeros of $p(z) = a_0 + a_1z + \dots + a_nz^n$ lie on the positive or negative real axis, then so do all the zeros of $q(z) = a_0 + a_1z + (a_2/2!)z^2 + \dots + (a_n/n!)z^n$.

(b) Same as (a) with the set replaced by a convex sector with vertex at the origin.

70. Determine all pairs of positive integers (a, b) for which $2ab^2 - b^3 + 1$ is positive and divides a^2 .

71. Suppose that $f(x)$ is a polynomial over the reals for which $f(0) \neq 0$ and $f(x^n)$ can be written nontrivially as the product of two polynomials with nonnegative coefficients for some natural number n . Prove that the same is true of $f(x)$.
72. Let I be an ideal of the ring of all polynomials over \mathbf{Z} for which the elements of I do not have a common divisor of degree greater than 0 and I contains a polynomial with constant term 1. Prove that, for some natural number m , I contains the polynomial $1 + x + x^2 + \cdots + x^{m-1}$.

73. Let

$$f(x) = a_0 + a_1x + a_2x^2 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13}$$

and

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_{11}x^{11} + b_{12}x^{12} + b_{13}x^{13}$$

be polynomials over the same field for which $a_{13}b_3 \neq 0$. Prove that the degree of their greatest common divisor is at most 6.

74. Let $p(z)$ be a polynomial of degree less than k over \mathbf{C} and let ζ_i ($1 \leq i \leq k$) be the k -th roots of unity. Prove that

$$\frac{1}{k} \sum_{i=1}^k P(\zeta_i) = P(0).$$

75. Let $f(x)$ be a polynomial with rational coefficients. Show that there is a nontrivial polynomial $g(x)$ over \mathbf{Q} such that $f(x)g(x) = a_2x^2 + a_3x^3 + \cdots + a_px^p$ is a polynomial in which all the nonzero terms have prime exponents.
76. Show that the polynomials $x^4 + x + 1$ and $x^4 + x^3 + x^2 + 6x + 1$ are irreducible over the rationals.
77. Let p be a prime congruent to 3 modulo 4, and suppose that $x^4 + 1$ factors into a product of two quadratics over \mathbf{Z}_p . Prove that each of these two quadratics is irreducible over \mathbf{Z}_p .
78. Let n be a positive integer and let f be a polynomial over \mathbf{R} of degree n . Prove that there are real numbers a_0, a_1, \dots, a_n for which the polynomial

$$\sum_{i=0}^n a_i x^{2^i}$$

is divisible by f .

79. Suppose that $p(z) = a_0 + a_1z + \cdots + a_nz^n$ is a polynomial for which $|p(z)| \leq 1$ when $|z| \leq 1$. Prove that $|a_0| + |a_n| \leq 1$.
80. Prove that, if the quadratic polynomial $az^2 + bz + c$ has nonzero coefficients, then its zeros must satisfy $|z| \leq |b/a| + |c/b|$.
81. Let S be the set of all nonnegative real quadratic functions $p(x)$ defined on $[-1, 1]$ for which $\int_{-1}^1 p(t) dt = 1$. Prove that

$$\min (\max p(t)) = \frac{2}{3}$$

where the maximum is taken over all polynomials in S and the minimum is taken over all $t \in [0, 1]$.

82. Suppose that $p(x)$ is a polynomial of degree exceeding 1 with integer coefficients. Suppose k is a positive integer and that $q(x)$ is the k -fold composite of $p(x)$. Prove that there are at most k integers m for which $q(m) = m$.
83. (a) Let n be a positive integer and let $p(x)$ be that polynomial of degree not exceeding $n - 1$ for which $p(k) = 1/k$ for $1 \leq k \leq n$. Determine $p(n + 1)$.

- (b) Let n be a positive integer and let $q(x)$ be that polynomial of degree not exceeding $n - 1$ for which $q(k) = 1/k^2$ for $1 \leq k \leq n$. Determine $q(n + 1)$.
84. Let n be a positive integer and let $p(x)$ be that polynomial of degree not exceeding $n - 1$ for which $p(k) = 2^k$ for $1 \leq k \leq n$. Determine $p(0)$ and $p(n + 1)$.
85. A fourth degree polynomial p satisfies $p(1) = 1$, $p(2) = 3$, $p(3) = 5$, $p(4) = 4$ and $p(5) = 2$. Determine $p(3/4)$.
86. (a) Determine the best uniform linear polynomial approximation to 2^x on $[-1, 1]$, *i.e.*, find that polynomial $p(x)$ of degree 1 that minimizes

$$\|p(x) - 2^x\|_\infty = \max \{|p(x) - 2^x| : -1 \leq x \leq 1\} .$$

- (b) Determine the best uniform polynomial approximation of degree not exceeding 2 to 2^x on $[-1, 1]$.
87. Determine points α, β, γ in $(0, 1)$ and constants a, b, c, d, e such that the expression

$$af(0) + bf(\alpha) + cf(\beta) + df(\gamma) + ef(1)$$

is equal to $\int_0^1 f(x)dx$, whenever $f(x)$ is a polynomial of degree not exceeding 7.

88. Sketch the Bézier curve for the ordered set

$$\{(1, 0), (0, 1), (-1, 0), (0, -1), (0, 1)\}$$

in the plane, where $(1, 0)$ is the first and last point in the set. If you have a computer, sketch some Bézier approximations for the unit circle in the plane. If you have not a computer, describe how you would set up the computation.