

## OLYMON

### COMPLETE PROBLEM SET

No solutions. See yearly files.

April, 2000 - December, 2007

#### Problems 1-527

*Notes:* The *inradius* of a triangle is the radius of the *incircle*, the circle that touches each side of the polygon. The *circumradius* of a triangle is the radius of the *circumcircle*, the circle that passes through its three vertices.

A set of lines of *concurrent* if and only if they have a common point of intersection.

The word *unique* means *exactly one*. A *regular octahedron* is a solid figure with eight faces, each of which is an equilateral triangle. You can think of gluing two square pyramids together along the square bases. The symbol  $[u]$  denotes the greatest integer that does not exceed  $u$ .

An *acute triangle* has all of its angles less than  $90^\circ$ . The *orthocentre* of a triangle is the intersection point of its altitudes. Points are *collinear* iff they lie on a straight line.

For any real number  $x$ ,  $[x]$  (the *floor* of  $x$ ) is equal to the greatest integer that is less than or equal to  $x$ .

A real-valued function  $f$  defined on an interval is *concave* iff  $f((1-t)u+tv) \geq (1-t)f(u)+tf(v)$  whenever  $0 < t < 1$  and  $u$  and  $v$  are in the domain of definition of  $f(x)$ . If  $f(x)$  is a one-one function defined on a domain into a range, then the *inverse* function  $g(x)$  defined on the set of values assumed by  $f$  is determined by  $g(f(x)) = x$  and  $f(g(y)) = y$ ; in other words,  $f(x) = y$  if and only if  $g(y) = x$ .

A sequence  $\{x_n\}$  *converges* if and only if there is a number  $c$ , called its *limit*, such that, as  $n$  increases, the number  $x_n$  gets closer and closer to  $c$ . If the sequence is *increasing* (i.e.,  $x_{n+1} \geq x_n$  for each index  $n$ ) and *bounded above* (i.e., there is a number  $M$  for which  $x_n \leq M$  for each  $n$ ), then it must converge. [Do you see why this is so?] Similarly, a decreasing sequence that is bounded below converges. [Supply the definitions and justify the statement.] An infinite *series* is an expression of the form  $\sum_{k=a}^{\infty} x_k = x_a + x_{a+1} + x_{a+2} + \dots + x_k + \dots$ , where  $a$  is an integer, usually 0 or 1. The  $n$ th *partial sum* of the series is  $s_n \equiv \sum_{k=a}^n x_k$ . The series has sum  $s$  if and only if its sequence  $\{s_n\}$  of partial sums converges and has limit  $s$ ; when this happens, the series *converges*. If the sequence of partial sums fails to converge, the series *diverges*. If every term in the series is nonnegative and the sequence of partial sums is bounded above, then the series converges. If a series of nonnegative terms converges, then it is possible to rearrange the order of the terms without changing the value of the sum.

A *rectangular hyperbola* is a hyperbola whose asymptotes are at right angles.

A function  $f : A \rightarrow B$  is a *bijection* iff it is one-one and onto; this means that, if  $f(u) = f(v)$ , then  $u = v$ , and, if  $w$  is some element of  $B$ , then  $A$  contains an element  $t$  for which  $f(t) = w$ . Such a function has an *inverse*  $f^{-1}$  which is determined by the condition

$$f^{-1}(b) = a \Leftrightarrow b = f(a) .$$

The sides of a right-angled triangle that are adjacent to the right angle are called *legs*. The *centre of gravity* or *centroid* of a collection of  $n$  mass particles is the point where the cumulative mass can be regarded as concentrated so that the motion of this point, when exposed to outside forces such as gravity, is identical to that of the whole collection. To illustrate this point, imagine that the mass particles are connected to a point by rigid non-material sticks (with mass 0) to form a structure. The point where the tip of a needle

could be put so that this structure is in a state of balance is its centroid. In addition, there is an intuitive definition of a centroid of a lamina, and of a solid: The centroid of a lamina is the point, which would cause equilibrium (balance) when the tip of a needle is placed underneath to support it. Likewise, the centroid of a solid is the point, at which the solid “balances”, *i.e.*, it will not revolve if force is applied. The centroid,  $G$  of a set of points is defined vectorially by

$$\overrightarrow{OG} = \frac{\sum_{i=1}^n m_i \cdot \overrightarrow{OM}_i}{\sum_{i=1}^n m_i}$$

where  $m_i$  is the mass of the particle at a position  $M_i$  (the summation extending over the whole collection). Problem 181 is related to the centroid of an assembly of three particles placed at the vertices of a given triangle. The *circumcentre* of a triangle is the centre of its circumscribed circle. The *orthocentre* of a triangle is the intersection point of its altitudes. An *unbounded* region in the plane is one not contained in the interior of any circle.

An *isosceles* tetrahedron is one for which the three pairs of opposite edges are equal. For integers  $a$ ,  $b$  and  $n$ ,  $a \equiv b$ , modulo  $n$ , iff  $a - b$  is a multiple of  $n$ .

A real-valued function on the reals is *increasing* if and only if  $f(u) \leq f(v)$  whenever  $u < v$ . It is *strictly increasing* if and only if  $f(u) < f(v)$  whenever  $u < v$ .

The inverse tangent function is denoted by  $\tan^{-1} x$  or  $\arctan x$ . It is defined by the relation  $y = \tan^{-1} x$  if and only if  $\pi/2 < y < \pi/2$  and  $x = \tan y$ .

The absolute value  $|x|$  is equal to  $x$  when  $x$  is nonnegative and  $-x$  when  $x$  is negative; always  $|x| \geq 0$ . The *floor of  $x$* , denoted by  $\lfloor x \rfloor$  is equal to the greatest integer that does not exceed  $x$ . For example,  $\lfloor 5.34 \rfloor = 5$ ,  $\lfloor -2.3 \rfloor = -3$  and  $\lfloor 5 \rfloor = 5$ . A geometric figure is said to be *convex* if the segment joining any two points inside the figure also lies inside the figure.

Given a triangle, extend two nonadjacent sides. The circle tangent to these two sides and to the third side of the triangle is called an *excircle*, or sometimes an *escribed circle*. The centre of the circle is called the *excentre* and lies on the angle bisector of the opposite angle and the bisectors of the external angles formed by the extended sides with the third side. Every triangle has three excircles along with their excentres.

The *incircle* of a polygon is a circle inscribed inside of the polygon that is tangent to all of the sides of a polygon. While every triangle has an incircle, this is not true of all polygons.

The *greatest common divisor* of two integers  $m$ ,  $n$ , denoted by  $\gcd(m, n)$  is the largest positive integer which divides (evenly) both  $m$  and  $n$ . The *least common multiple* of two integers  $m$ ,  $n$ , denoted by  $\text{lcm}(m, n)$  is the smallest positive integer which is divisible by both  $m$  and  $n$ .

Let  $n$  be a positive integer. It can be written uniquely as a sum of powers of 2, *i.e.* in the form

$$n = \epsilon_k \cdot 2^k + \epsilon_{k-1} \cdot 2^{k-1} + \cdots + \epsilon_1 \cdot 2 + \epsilon_0$$

where each  $\epsilon_i$  takes one of the values 0 and 1. This is known as the *binary representation* of  $n$  and is denoted  $(\epsilon_k, \epsilon_{k-1}, \dots, \epsilon_0)_2$ . The numbers  $\epsilon_i$  are known as the *(binary) digits* of  $n$ .

The *circumcircle* of a triangle is the centre of the circle that passes through the three vertices of the triangle; the *incentre* of a triangle is centre of the circle within the triangle that is tangent to the three sides; the *orthocentre* of a triangle is the intersection point of its three altitudes.

The function  $f$  defined on the real numbers and taking real values is *increasing* if and only if, for  $x < y$ ,  $f(x) \leq f(y)$ .

1. Let  $M$  be a set of eleven points consisting of the four vertices along with seven interior points of a square of unit area.

(a) Prove that there are three of these points that are vertices of a triangle whose area is at most  $1/16$ .

(b) Give an example of a set  $M$  for which no four of the interior points are collinear and each nondegenerate triangle formed by three of them has area at least  $1/16$ .

2. Let  $a, b, c$  be the lengths of the sides of a triangle. Suppose that  $u = a^2 + b^2 + c^2$  and  $v = (a + b + c)^2$ . Prove that

$$\frac{1}{3} \leq \frac{u}{v} < \frac{1}{2}$$

and that the fraction  $1/2$  on the right cannot be replaced by a smaller number.

3. Suppose that  $f(x)$  is a function satisfying

$$|f(m+n) - f(m)| \leq \frac{n}{m}$$

for all rational numbers  $n$  and  $m$ . Show that, for all natural numbers  $k$ ,

$$\sum_{i=1}^k |f(2^k) - f(2^i)| \leq \frac{k(k-1)}{2}.$$

4. Is it true that any pair of triangles sharing a common angle, inradius and circumradius must be congruent?
5. Each point of the plane is coloured with one of 2000 different colours. Prove that there exists a rectangle all of whose vertices have the same colour.
6. Let  $n$  be a positive integer,  $P$  be a set of  $n$  primes and  $M$  a set of at least  $n+1$  natural numbers, each of which is divisible by no primes other than those belonging to  $P$ . Prove that there is a nonvoid subset of  $M$ , the product of whose elements is a square integer.

7. Let

$$S = \frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \frac{3^2}{5 \cdot 7} + \cdots + \frac{500^2}{999 \cdot 1001}.$$

Find the value of  $S$ .

8. The sequences  $\{a_n\}$  and  $\{b_n\}$  are such that, for every positive integer  $n$ ,

$$a_n > 0, \quad b_n > 0, \quad a_{n+1} = a_n + \frac{1}{b_n}, \quad b_{n+1} = b_n + \frac{1}{a_n}.$$

Prove that  $a_{50} + b_{50} > 20$ .

9. There are six points in the plane. Any three of them are vertices of a triangle whose sides are of different length. Prove that there exists a triangle whose smallest side is the largest side of another triangle.
10. In a rectangle, whose sides are 20 and 25 units of length, are placed 120 squares of side 1 unit of length. Prove that a circle of diameter 1 unit can be placed in the rectangle, so that it has no common points with the squares.
11. Each of nine lines divides a square into two quadrilaterals, such that the ratio of their area is 2:3. Prove that at least three of these lines are concurrent.
12. Each vertex of a regular 100-sided polygon is marked with a number chosen from among the natural numbers  $1, 2, 3, \dots, 49$ . Prove that there are four vertices (which we can denote as  $A, B, C, D$  with

respective numbers  $a, b, c, d$ ) such that  $ABCD$  is a rectangle, the points  $A$  and  $B$  are two adjacent vertices of the rectangle and  $a + b = c + d$ .

13. Suppose that  $x_1, x_2, \dots, x_n$  are nonnegative real numbers for which  $x_1 + x_2 + \dots + x_n < \frac{1}{2}$ . Prove that

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) > \frac{1}{2},$$

14. Given a convex quadrilateral, is it always possible to determine a point in its interior such that the four line segments joining the point to the midpoints of the sides divide the quadrilateral into four regions of equal area? If such a point exists, is it unique?
15. Determine all triples  $(x, y, z)$  of real numbers for which

$$x(y + 1) = y(z + 1) = z(x + 1).$$

16. Suppose that  $ABCDEFZ$  is a regular octahedron whose pairs of opposite vertices are  $(A, Z)$ ,  $(B, D)$  and  $(C, E)$ . The points  $F, G, H$  are chosen on the segments  $AB, AC, AD$  respectively such that  $AF = AG = AH$ .

(a) Show that  $EF$  and  $DG$  must intersect in a point  $K$ , and that  $BG$  and  $EH$  must intersect in a point  $L$ .

(b) Let  $EG$  meet the plane of  $AKL$  in  $M$ . Show that  $AKML$  is a square.

17. Suppose that  $r$  is a real number. Define the sequence  $x_n$  recursively by  $x_0 = 0, x_1 = 1, x_{n+2} = rx_{n+1} - x_n$  for  $n \geq 0$ . For which values of  $r$  is it true that

$$x_1 + x_3 + x_5 + \cdots + x_{2m-1} = x_m^2$$

for  $m = 1, 2, 3, 4, \dots$ .

18. Let  $a$  and  $b$  be integers. How many solutions in real pairs  $(x, y)$  does the system

$$\lfloor x \rfloor + 2y = a$$

$$\lfloor y \rfloor + 2x = b$$

have?

19. Is it possible to divide the natural numbers  $1, 2, \dots, n$  into two groups, such that the squares of the members in each group have the same sum, if (a)  $n = 40000$ ; (b)  $n = 40002$ ? Explain your answer.

20. Given any six irrational numbers, prove that there are always three of them, say  $a, b, c$ , for which  $a + b, b + c$  and  $c + a$  are irrational.

21. The natural numbers  $x_1, x_2, \dots, x_{100}$  are such that

$$\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20.$$

Prove that at least two of the numbers are equal.

22. Let  $\mathbf{R}$  be a rectangle with dimensions  $11 \times 12$ . Find the least natural number  $n$  for which it is possible to cover  $\mathbf{R}$  with  $n$  rectangles, each of size  $1 \times 6$  or  $1 \times 7$ , with no two of these having a common interior point.

23. Given 21 points on the circumference of a circle, prove that at least 100 of the arcs determined by pairs of these points subtend an angle not exceeding  $120^\circ$  at the centre.
24.  $ABC$  is an acute triangle with orthocentre  $H$ . Denote by  $M$  and  $N$  the midpoints of the respective segments  $AB$  and  $CH$ , and by  $P$  the intersection point of the bisectors of angles  $CAH$  and  $CBH$ . Prove that the points  $M$ ,  $N$  and  $P$  are collinear.
25. Let  $a, b, c$  be non-negative numbers such that  $a + b + c = 1$ . Prove that

$$\frac{ab}{c+1} + \frac{bc}{a+1} + \frac{ca}{b+1} \leq \frac{1}{4} .$$

When does equality hold?

26. Each of  $m$  cards is labelled by one of the numbers  $1, 2, \dots, m$ . Prove that, if the sum of labels of any subset of cards is not a multiple of  $m + 1$ , then each card is labelled by the same number.
27. Find the least number of the form  $|36^m - 5^n|$  where  $m$  and  $n$  are positive integers.
28. Let  $A$  be a finite set of real numbers which contains at least two elements and let  $f : A \rightarrow A$  be a function such that  $|f(x) - f(y)| < |x - y|$  for every  $x, y \in A, x \neq y$ . Prove that there is  $a \in A$  for which  $f(a) = a$ . Does the result remain valid if  $A$  is not a finite set?
29. Let  $A$  be a nonempty set of positive integers such that if  $a \in A$ , then  $4a$  and  $\lfloor \sqrt{a} \rfloor$  both belong to  $A$ . Prove that  $A$  is the set of all positive integers.
30. Find a point  $M$  within a regular pentagon for which the sum of its distances to the vertices is minimum.
31. Let  $x, y, z$  be positive real numbers for which  $x^2 + y^2 + z^2 = 1$ . Find the minimum value of

$$S = \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} .$$

32. The segments  $BE$  and  $CF$  are altitudes of the acute triangle  $ABC$ , where  $E$  and  $F$  are points on the segments  $AC$  and  $AB$ , respectively.  $ABC$  is inscribed in the circle  $\mathbf{Q}$  with centre  $O$ . Denote the orthocentre of  $ABC$  by  $H$ , and the midpoints of  $BC$  and  $AH$  be  $M$  and  $K$ , respectively. Let  $\angle CAB = 45^\circ$ .
- (a) Prove, that the quadrilateral  $MEKF$  is a square.
- (b) Prove that the midpoint of both diagonals of  $MEKF$  is also the midpoint of the segment  $OH$ .
- (c) Find the length of  $EF$ , if the radius of  $\mathbf{Q}$  has length 1 unit.
33. Prove the inequality  $a^2 + b^2 + c^2 + 2abc < 2$ , if the numbers  $a, b, c$  are the lengths of the sides of a triangle with perimeter 2.
34. Each of the edges of a cube is 1 unit in length, and is divided by two points into three equal parts. Denote by  $\mathbf{K}$  the solid with vertices at these points.
- (a) Find the volume of  $\mathbf{K}$ .
- (b) Every pair of vertices of  $\mathbf{K}$  is connected by a segment. Some of the segments are coloured. Prove that it is always possible to find two vertices which are endpoints of the same number of coloured segments.
35. There are  $n$  points on a circle whose radius is 1 unit. What is the greatest number of segments between two of them, whose length exceeds  $\sqrt{3}$ ?
36. Prove that there are not three rational numbers  $x, y, z$  such that

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0 .$$

37. Let  $ABC$  be a triangle with sides  $a, b, c$ , inradius  $r$  and circumradius  $R$  (using the conventional notation). Prove that

$$\frac{r}{2R} \leq \frac{abc}{\sqrt{2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}}.$$

When does equality hold?

38. Let us say that a set  $S$  of nonnegative real numbers is *hunky-dory* if and only if, for all  $x$  and  $y$  in  $S$ , either  $x + y$  or  $|x - y|$  is in  $S$ . For instance, if  $r$  is positive and  $n$  is a natural number, then  $S(n, r) = \{0, r, 2r, \dots, nr\}$  is hunky-dory. Show that every hunky-dory set with finitely many elements is  $\{0\}$ , is of the form  $S(n, r)$  or has exactly four elements.
39. (a)  $ABCDEF$  is a convex hexagon, each of whose diagonals  $AD, BE$  and  $CF$  pass through a common point. Must each of these diagonals bisect the area?
- (b)  $ABCDEF$  is a convex hexagon, each of whose diagonals  $AD, BE$  and  $CF$  bisects the area (so that half the area of the hexagon lies on either side of the diagonal). Must the three diagonals pass through a common point?
40. Determine all solutions in integer pairs  $(x, y)$  to the diophantine equation  $x^2 = 1 + 4y^3(y + 2)$ .
41. Determine the least positive number  $p$  for which there exists a positive number  $q$  such that

$$\sqrt{1+x} + \sqrt{1-x} \leq 2 - \frac{x^p}{q}$$

for  $0 \leq x \leq 1$ . For this least value of  $p$ , what is the smallest value of  $q$  for which the inequality is satisfied for  $0 \leq x \leq 1$ ?

42.  $G$  is a connected graph; that is, it consists of a number of vertices, some pairs of which are joined by edges, and, for any two vertices, one can travel from one to another along a chain of edges. We call two vertices *adjacent* if and only if they are endpoints of the same edge. Suppose there is associated with each vertex  $v$  a nonnegative integer  $f(v)$  such that all of the following hold:
- (1) If  $v$  and  $w$  are adjacent, then  $|f(v) - f(w)| \leq 1$ .
  - (2) If  $f(v) > 0$ , then  $v$  is adjacent to at least one vertex  $w$  such that  $f(w) < f(v)$ .
  - (3) There is exactly one vertex  $u$  such that  $f(u) = 0$ .
- Prove that  $f(v)$  is the number of edges in the chain with the fewest edges connecting  $u$  and  $v$ .
43. Two players play a game: the first player thinks of  $n$  integers  $x_1, x_2, \dots, x_n$ , each with one digit, and the second player selects some numbers  $a_1, a_2, \dots, a_n$  and asks what is the value of the sum  $a_1x_1 + a_2x_2 + \dots + a_nx_n$ . What is the minimum number of questions used by the second player to find the integers  $x_1, x_2, \dots, x_n$ ?
44. Find the permutation  $\{a_1, a_2, \dots, a_n\}$  of the set  $\{1, 2, \dots, n\}$  for which the sum

$$S = |a_2 - a_1| + |a_3 - a_2| + \dots + |a_n - a_{n-1}|$$

has maximum value.

45. Prove that there is no polynomial  $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_0$  with integer coefficients  $a_i$  for which  $p(m)$  is a prime number for every integer  $m$ .
46. Let  $a_1 = 2, a_{n+1} = \frac{a_n+2}{1-2a_n}$  for  $n = 1, 2, \dots$ . Prove that
- (a)  $a_n \neq 0$  for each positive integer  $n$ ;
  - (b) there is no integer  $p \geq 1$  for which  $a_{n+p} = a_n$  for every integer  $n \geq 1$  (i.e., the sequence is not periodic).

47. Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $a_1 a_2 \cdots a_n = 1$ . Prove that

$$\sum_{k=1}^n \frac{1}{s - a_k} \leq 1$$

where  $s = 1 + a_1 + a_2 + \cdots + a_n$ .

48. Let  $A_1 A_2 \cdots A_n$  be a regular  $n$ -gon and  $d$  an arbitrary line. The parallels through  $A_i$  to  $d$  intersect its circumcircle respectively at  $B_i$  ( $i = 1, 2, \dots, n$ ). Prove that the sum

$$S = |A_1 B_1|^2 + \cdots + |A_n B_n|^2$$

is independent of  $d$ .

49. Find all ordered pairs  $(x, y)$  that are solutions of the following system of two equations (where  $a$  is a parameter):

$$\begin{aligned} x - y &= 2 \\ \left(x - \frac{2}{a}\right) \left(y - \frac{2}{a}\right) &= a^2 - 1. \end{aligned}$$

Find all values of the parameter  $a$  for which the solutions of the system are two pairs of nonnegative numbers. Find the minimum value of  $x + y$  for these values of  $a$ .

50. Let  $n$  be a natural number exceeding 1, and let  $A_n$  be the set of *all* natural numbers that are not relatively prime with  $n$  (i.e.,  $A_n = \{x \in \mathbf{N} : \gcd(x, n) \neq 1\}$ ). Let us call the number  $n$  *magic* if for each two numbers  $x, y \in A_n$ , their sum  $x + y$  is also an element of  $A_n$  (i.e.,  $x + y \in A_n$  for  $x, y \in A_n$ ).

(a) Prove that 67 is a magic number.

(b) Prove that 2001 is **not** a magic number.

(c) Find all magic numbers.

51. In the triangle  $ABC$ ,  $AB = 15$ ,  $BC = 13$  and  $AC = 12$ . Prove that, for this triangle, the angle bisector from  $A$ , the median from  $B$  and the altitude from  $C$  are concurrent (i.e., meet in a common point).

52. One solution of the equation  $2x^3 + ax^2 + bx + 8 = 0$  is  $1 + \sqrt{3}$ . Given that  $a$  and  $b$  are rational numbers, determine its other two solutions.

53. Prove that among any 17 natural numbers chosen from the sets  $\{1, 2, 3, \dots, 24, 25\}$ , it is always possible to find two whose product is a perfect square.

54. A circle has exactly one common point with each of the sides of a  $(2n + 1)$ -sided polygon. None of the vertices of the polygon is a point of the circle. Prove that at least one of the sides is a tangent of the circle.

55. A textbook problem has the following form: *A man is standing in a line in front of a movie theatre. The fraction  $x$  of the line is in front of him, and the fraction  $y$  of the line is behind him, where  $x$  and  $y$  are rational numbers written in lowest terms. How many people are there in the line?* Prove that, if the problem has an answer, then that answer must be the least common multiple of the denominators of  $x$  and  $y$ .

56. Let  $n$  be a positive integer and let  $x_1, x_2, \dots, x_n$  be integers for which

$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \leq (2n - 1)(x_1 + x_2 + \cdots + x_n) + n^2.$$

Show that

- (a)  $x_1, x_2, \dots, x_n$  are all nonnegative;
- (b)  $x_1 + x_2 + \dots + x_n + n + 1$  is not a perfect square.
57. Let  $ABCD$  be a rectangle and let  $E$  be a point in the diagonal  $BD$  with  $\angle DAE = 15^\circ$ . Let  $F$  be a point in  $AB$  with  $EF \perp AB$ . It is known that  $EF = \frac{1}{2}AB$  and  $AD = a$ . Find the measure of the angle  $\angle EAC$  and the length of the segment  $EC$ .
58. Find integers  $a, b, c$  such that  $a \neq 0$  and the quadratic function  $f(x) = ax^2 + bx + c$  satisfies
- $$f(f(1)) = f(f(2)) = f(f(3)) .$$
59. Let  $ABCD$  be a concyclic quadrilateral. Prove that
- $$|AC - BD| \leq |AB - CD| .$$
60. Let  $n \geq 2$  be an integer and  $M = \{1, 2, \dots, n\}$ . For every integer  $k$  with  $1 \leq k \leq n - 1$ , let
- $$x_k = \sum \{\min A + \max A : A \subseteq M, A \text{ has } k \text{ elements}\}$$
- where  $\min A$  is the smallest and  $\max A$  is the largest number in  $A$ . Determine  $\sum_{k=1}^n (-1)^{k-1} x_k$ .
61. Let  $S = 1!2!3! \cdots 99!100!$  (the product of the first 100 factorials). Prove that there exists an integer  $k$  for which  $1 \leq k \leq 100$  and  $S/k!$  is a perfect square. Is  $k$  unique? (*Optional*: Is it possible to find such a number  $k$  that exceeds 100?)
62. Let  $n$  be a positive integer. Show that, with three exceptions,  $n! + 1$  has at least one prime divisor that exceeds  $n + 1$ .
63. Let  $n$  be a positive integer and  $k$  a nonnegative integer. Prove that
- $$n! = (n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \dots \pm \binom{n}{n}k^n .$$
64. Let  $M$  be a point in the interior of triangle  $ABC$ , and suppose that  $D, E, F$  are points on the respective side  $BC, CA, AB$ . Suppose  $AD, BE$  and  $CF$  all pass through  $M$ . (In technical terms, they are *cevians*.) Suppose that the areas and the perimeters of the triangles  $BMD, CME, AMF$  are equal. Prove that triangle  $ABC$  must be equilateral.
65. Suppose that  $XTY$  is a straight line and that  $TU$  and  $TV$  are two rays emanating from  $T$  for which  $\angleXTU = \angleUTV = \angleVTY = 60^\circ$ . Suppose that  $P, Q$  and  $R$  are respective points on the rays  $TY, TU$  and  $TV$  for which  $PQ = PR$ . Prove that  $\angleQPR = 60^\circ$ .
66. (a) Let  $ABCD$  be a square and let  $E$  be an arbitrary point on the side  $CD$ . Suppose that  $P$  is a point on the diagonal  $AC$  for which  $EP \perp AC$  and that  $Q$  is a point on  $AE$  produced for which  $CQ \perp AE$ . Prove that  $B, P, Q$  are collinear.
- (b) Does the result hold if the hypothesis is weakened to require only that  $ABCD$  is a rectangle?
67. (a) Consider the infinite integer lattice in the plane (*i.e.*, the set of points with integer coordinates) as a graph, with the edges being the lines of unit length connecting nearby points. What is the minimum number of colours that can be used to colour all the vertices and edges of this graph, so that

- (i) each pair of adjacent vertices gets two distinct colours; AND
- (ii) each pair of edges that meet at a vertex get two distinct colours; AND
- (iii) an edge is coloured differently than either of the two vertices at the ends?
- (b) Extend this result to lattices in real  $n$ -dimensional space.
68. Let  $a, b, c > 0$ ,  $a < bc$  and  $1 + a^3 = b^3 + c^3$ . Prove that  $1 + a < b + c$ .
69. Let  $n, a_1, a_2, \dots, a_k$  be positive integers for which  $n \geq a_1 > a_2 > a_3 > \dots > a_k$  and the least common multiple of  $a_i$  and  $a_j$  does not exceed  $n$  for all  $i$  and  $j$ . Prove that  $ia_i \leq n$  for  $i = 1, 2, \dots, k$ .
70. Let  $f(x)$  be a concave strictly increasing function defined for  $0 \leq x \leq 1$  such that  $f(0) = 0$  and  $f(1) = 1$ . Suppose that  $g(x)$  is its inverse. Prove that  $f(x)g(x) \leq x^2$  for  $0 \leq x \leq 1$ .
71. Suppose that lengths  $a, b$  and  $i$  are given. Construct a triangle  $ABC$  for which  $|AC| = b$ ,  $|AB| = c$  and the length of the bisector  $AD$  of angle  $A$  is  $i$  ( $D$  being the point where the bisector meets the side  $BC$ ).
72. The centres of the circumscribed and the inscribed spheres of a given tetrahedron coincide. Prove that the four triangular faces of the tetrahedron are congruent.
73. Solve the equation:

$$\left(\sqrt{2 + \sqrt{2}}\right)^x + \left(\sqrt{2 - \sqrt{2}}\right)^x = 2^x .$$

74. Prove that among any group of  $n + 2$  natural numbers, there can be found two numbers so that their sum or their difference is divisible by  $2n$ .
75. Three consecutive natural numbers, larger than 3, represent the lengths of the sides of a triangle. The area of the triangle is also a natural number.
- (a) Prove that one of the altitudes “cuts” the triangle into two triangles, whose side lengths are natural numbers.
- (b) The altitude identified in (a) divides the side which is perpendicular to it into two segments. Find the difference between the lengths of these segments.
76. Solve the system of equations:

$$\log x + \frac{\log(xy^8)}{\log^2 x + \log^2 y} = 2 ,$$

$$\log y + \frac{\log(x^8/y)}{\log^2 x + \log^2 y} = 0 .$$

(The logarithms are taken to base 10.)

77.  $n$  points are chosen from the circumference or the interior of a regular hexagon with sides of unit length, so that the distance between any two of them is **not** less than  $\sqrt{2}$ . What is the largest natural number  $n$  for which this is possible?
78. A truck travelled from town  $A$  to town  $B$  over several days. During the first day, it covered  $1/n$  of the total distance, where  $n$  is a natural number. During the second day, it travelled  $1/m$  of the remaining distance, where  $m$  is a natural number. During the third day, it travelled  $1/n$  of the distance remaining after the second day, and during the fourth day,  $1/m$  of the distance remaining after the third day. Find the values of  $m$  and  $n$  if it is known that, by the end of the fourth day, the truck had travelled  $3/4$  of the distance between  $A$  and  $B$ . (Without loss of generality, assume that  $m < n$ .)

79. Let  $x_0, x_1, x_2$  be three positive real numbers. A sequence  $\{x_n\}$  is defined, for  $n \geq 0$  by

$$x_{n+3} = \frac{x_{n+2} + x_{n+1} + 1}{x_n} .$$

Determine all such sequences whose entries consist solely of positive integers.

80. Prove that, for each positive integer  $n$ , the series

$$\sum_{k=1}^{\infty} \frac{k^n}{2^k}$$

converges to twice an odd integer not less than  $(n+1)!$ .

81. Suppose that  $x \geq 1$  and that  $x = [x] + \{x\}$ , where  $[x]$  is the greatest integer not exceeding  $x$  and the fractional part  $\{x\}$  satisfies  $0 \leq \{x\} < 1$ . Define

$$f(x) = \frac{\sqrt{[x]} + \sqrt{\{x\}}}{\sqrt{x}} .$$

- (a) Determine the small number  $z$  such that  $f(x) \leq z$  for each  $x \geq 1$ .  
 (b) Let  $x_0 \geq 1$  be given, and for  $n \geq 1$ , define  $x_n = f(x_{n-1})$ . Prove that  $\lim_{n \rightarrow \infty} x_n$  exists.
82. (a) A regular pentagon has side length  $a$  and diagonal length  $b$ . Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3 .$$

(b) A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let  $a$  be the length of a side,  $b$  be the length of a shorter diagonal and  $c$  be the length of a longer diagonal of a regular heptagon (so that  $a < b < c$ ). Prove that:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 .$$

83. Let  $\mathcal{C}$  be a circle with centre  $O$  and radius 1, and let  $\mathfrak{F}$  be a closed convex region inside  $\mathcal{C}$ . Suppose from each point  $\mathcal{C}$ , we can draw two rays tangent to  $\mathfrak{F}$  meeting at an angle of  $60^\circ$ . Describe  $\mathfrak{F}$ .
84. Let  $ABC$  be an acute-angled triangle, with a point  $H$  inside. Let  $U, V, W$  be respectively the reflected image of  $H$  with respect to axes  $BC, AC, AB$ . Prove that  $H$  is the orthocentre of  $\Delta ABC$  if and only if  $U, V, W$  lie on the circumcircle of  $\Delta ABC$ ,
85. Find all pairs  $(a, b)$  of positive integers with  $a \neq b$  for which the system

$$\cos ax + \cos bx = 0$$

$$a \sin ax + b \sin bx = 0$$

has a solution. If so, determine its solutions.

86. Let  $ABCD$  be a convex quadrilateral with  $AB = AD$  and  $CB = CD$ . Prove that

- (a) it is possible to inscribe a circle in it;  
 (b) it is possible to circumscribe a circle about it if and only if  $AB \perp BC$ ;  
 (c) if  $AB \perp AC$  and  $R$  and  $r$  are the respective radii of the circumscribed and inscribed circles, then the distance between the centres of the two circles is equal to the square root of  $R^2 + r^2 - r\sqrt{r^2 + 4R^2}$ .

87. Prove that, if the real numbers  $a, b, c$ , satisfy the equation

$$\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor$$

for each positive integer  $n$ , then at least one of  $a$  and  $b$  is an integer.

88. Let  $I$  be a real interval of length  $1/n$ . Prove that  $I$  contains no more than  $\frac{1}{2}(n+1)$  irreducible fractions of the form  $p/q$  with  $p$  and  $q$  positive integers,  $1 \leq q \leq n$  and the greatest common divisor of  $p$  and  $q$  equal to 1.
89. Prove that there is only one triple of positive integers, each exceeding 1, for which the product of any two of the numbers plus one is divisible by the third.
90. Let  $m$  be a positive integer, and let  $f(m)$  be the smallest value of  $n$  for which the following statement is true:  
*given any set of  $n$  integers, it is always possible to find a subset of  $m$  integers whose sum is divisible by  $m$*   
 Determine  $f(m)$ .
91. A square and a regular pentagon are inscribed in a circle. The nine vertices are all distinct and divide the circumference into nine arcs. Prove that at least one of them does not exceed  $1/40$  of the circumference of the circle.
92. Consider the sequence 200125, 2000125, 20000125,  $\dots$ ,  $200 \dots 00125$ ,  $\dots$  (in which the  $n$ th number has  $n+1$  digits equal to zero). Can any of these numbers be the square or the cube of an integer?
93. For any natural number  $n$ , prove the following inequalities:

$$2^{(n-1)/(2^{n-2})} \leq \sqrt{2} \sqrt[4]{4} \sqrt[8]{8} \dots \sqrt[2^n]{2^n} < 4.$$

94.  $ABC$  is a right triangle with arms  $a$  and  $b$  and hypotenuse  $c = |AB|$ ; the area of the triangle is  $s$  square units and its perimeter is  $2p$  units. The numbers  $a, b$  and  $c$  are positive integers. Prove that  $s$  and  $p$  are also positive integers and that  $s$  is a multiple of  $p$ .
95. The triangle  $ABC$  is isosceles with equal sides  $AC$  and  $BC$ . Two of its angles measure  $40^\circ$ . The interior point  $M$  is such that  $\angle MAB = 10^\circ$  and  $\angle MBA = 20^\circ$ . Determine the measure of  $\angle CMB$ .
96. Find all prime numbers  $p$  for which all three of the numbers  $p^2 - 2$ ,  $2p^2 - 1$  and  $3p^2 + 4$  are also prime.
97. A triangle has its three vertices on a rectangular hyperbola. Prove that its orthocentre also lies on the hyperbola.
98. Let  $a_1, a_2, \dots, a_{n+1}, b_1, b_2, \dots, b_n$  be nonnegative real numbers for which  
 (i)  $a_1 \geq a_2 \geq \dots \geq a_{n+1} = 0$ ,  
 (ii)  $0 \leq b_k \leq 1$  for  $k = 1, 2, \dots, n$ .  
 Suppose that  $m = \lfloor b_1 + b_2 + \dots + b_n \rfloor + 1$ . Prove that

$$\sum_{k=1}^n a_k b_k \leq \sum_{k=1}^m a_k.$$

99. Let  $E$  and  $F$  be respective points on sides  $AB$  and  $BC$  of a triangle  $ABC$  for which  $AE = CF$ . The circle passing through the points  $B, C, E$  and the circle passing through the points  $A, B, F$  intersect at  $B$  and  $D$ . Prove that  $BD$  is the bisector of angle  $ABC$ .
100. If 10 equally spaced points around a circle are joined consecutively, a convex regular inscribed decagon  $P$  is obtained; if every third point is joined, a self-intersecting regular decagon  $Q$  is formed. Prove that the difference between the length of a side of  $Q$  and the length of a side of  $P$  is equal to the radius of the circle. [With thanks to Ross Honsberger.]
101. Let  $a, b, u, v$  be nonnegative. Suppose that  $a^5 + b^5 \leq 1$  and  $u^5 + v^5 \leq 1$ . Prove that

$$a^2u^3 + b^2v^3 \leq 1 .$$

[With thanks to Ross Honsberger.]

102. Prove that there exists a tetrahedron  $ABCD$ , all of whose faces are similar right triangles, each face having acute angles at  $A$  and  $B$ . Determine which of the edges of the tetrahedron is largest and which is smallest, and find the ratio of their lengths.
103. Determine a value of the parameter  $\theta$  so that

$$f(x) \equiv \cos^2 x + \cos^2(x + \theta) - \cos x \cos(x + \theta)$$

is a constant function of  $x$ .

104. Prove that there exists exactly one sequence  $\{x_n\}$  of positive integers for which

$$x_1 = 1 , \quad x_2 > 1 , \quad x_{n+1}^3 + 1 = x_n x_{n+2}$$

for  $n \geq 1$ .

105. Prove that within a unit cube, one can place two regular unit tetrahedra that have no common point.
106. Find all pairs  $(x, y)$  of positive real numbers for which the least value of the function

$$f(x, y) = \frac{x^4}{y^4} + \frac{y^4}{x^4} - \frac{x^2}{y^2} - \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x}$$

is attained. Determine that minimum value.

107. Given positive numbers  $a_i$  with  $a_1 < a_2 < \dots < a_n$ , for which permutation  $(b_1, b_2, \dots, b_n)$  of these numbers is the product

$$\prod_{i=1}^n \left( a_i + \frac{1}{b_i} \right)$$

maximized?

108. Determine all real-valued functions  $f(x)$  of a real variable  $x$  for which

$$f(xy) = \frac{f(x) + f(y)}{x + y}$$

for all real  $x$  and  $y$  for which  $x + y \neq 0$ .

109. Suppose that

$$\frac{x^2 + y^2}{x^2 - y^2} + \frac{x^2 - y^2}{x^2 + y^2} = k .$$

Find, in terms of  $k$ , the value of the expression

$$\frac{x^8 + y^8}{x^8 - y^8} + \frac{x^8 - y^8}{x^8 + y^8} .$$

110. Given a triangle  $ABC$  with an area of 1. Let  $n > 1$  be a natural number. Suppose that  $M$  is a point on the side  $AB$  with  $AB = nAM$ ,  $N$  is a point on the side  $BC$  with  $BC = nBN$ , and  $Q$  is a point on the side  $CA$  with  $CA = nCQ$ . Suppose also that  $\{T\} = AN \cap CM$ ,  $\{R\} = BQ \cap AN$  and  $\{S\} = CM \cap BQ$ , where  $\cap$  signifies that the singleton is the intersection of the indicated segments. Find the area of the triangle  $TRS$  in terms of  $n$ .
111. (a) Are there four different numbers, not exceeding 10, for which the sum of any three is a prime number?  
 (b) Are there five different natural numbers such that the sum of every three of them is a prime number?
112. Suppose that the measure of angle  $BAC$  in the triangle  $ABC$  is equal to  $\alpha$ . A line passing through the vertex  $A$  is perpendicular to the angle bisector of  $\angle BAC$  and intersects the line  $BC$  at the point  $M$ . Find the other two angles of the triangle  $ABC$  in terms of  $\alpha$ , if it is known that  $BM = BA + AC$ .
113. Find a function that satisfies all of the following conditions:  
 (a)  $f$  is defined for every positive integer  $n$ ;  
 (b)  $f$  takes only positive values;  
 (c)  $f(4) = 4$ ;  
 (d)

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)} .$$

114. A natural number is a multiple of 17. Its binary representation (*i.e.*, when written to base 2) contains exactly three digits equal to 1 and some zeros.  
 (a) Prove that there are at least six digits equal to 0 in its binary representation.  
 (b) Prove that, if there are exactly seven digits equal to 0 and three digits equal to 1, then the number must be even.
115. Let  $U$  be a set of  $n$  distinct real numbers and let  $V$  be the set of all sums of distinct pairs of them, *i.e.*,

$$V = \{x + y : x, y \in U, x \neq y\} .$$

What is the smallest possible number of distinct elements that  $V$  can contain?

116. Prove that the equation

$$x^4 + 5x^3 + 6x^2 - 4x - 16 = 0$$

has exactly two real solutions.

117. Let  $a$  be a real number. Solve the equation

$$(a - 1) \left( \frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x \cos x} \right) = 2 .$$

118. Let  $a, b, c$  be nonnegative real numbers. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc .$$

When does equality hold?

119. The medians of a triangle  $ABC$  intersect in  $G$ . Prove that

$$|AB|^2 + |BC|^2 + |CA|^2 = 3(|GA|^2 + |GB|^2 + |GC|^2) .$$

120. Determine all pairs of nonnull vectors  $\mathbf{x}$ ,  $\mathbf{y}$  for which the following sequence  $\{a_n : n = 1, 2, \dots\}$  is (a) increasing, (b) decreasing, where

$$a_n = |\mathbf{x} - n\mathbf{y}| .$$

121. Let  $n$  be an integer exceeding 1. Let  $a_1, a_2, \dots, a_n$  be positive real numbers and  $b_1, b_2, \dots, b_n$  be arbitrary real numbers for which

$$\sum_{i \neq j} a_i b_j = 0 .$$

Prove that  $\sum_{i \neq j} b_i b_j < 0$  .

122. Determine all functions  $f$  from the real numbers to the real numbers that satisfy

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for any real numbers  $x$ ,  $y$ .

123. Let  $a$  and  $b$  be the lengths of two opposite edges of a tetrahedron which are mutually perpendicular and distant  $d$  apart. Determine the volume of the tetrahedron.

124. Prove that

$$\frac{(1^4 + \frac{1}{4})(3^4 + \frac{1}{4})(5^4 + \frac{1}{4}) \cdots (11^4 + \frac{1}{4})}{(2^4 + \frac{1}{4})(4^4 + \frac{1}{4})(6^4 + \frac{1}{4}) \cdots (12^4 + \frac{1}{4})} = \frac{1}{313} .$$

125. Determine the set of complex numbers  $z$  which satisfy

$$\text{Im}(z^4) = (\text{Re}(z^2))^2 ,$$

and sketch this set in the complex plane. (Note: Im and Re refer respectively to the imaginary and real parts.)

126. Let  $n$  be a positive integer exceeding 1, and let  $n$  circles (*i.e.*, circumferences) of radius 1 be given in the plane such that no two of them are tangent and the subset of the plane formed by the union of them is connected. Prove that the number of points that belong to at least two of these circles is at least  $n$ .

127. Let

$$A = 2^n + 3^n + 216(2^{n-6} + 3^{n-6})$$

and

$$B = 4^n + 5^n + 8000(4^{n-6} + 5^{n-6})$$

where  $n > 6$  is a natural number. Prove that the fraction  $A/B$  is reducible.

128. Let  $n$  be a positive integer. On a circle,  $n$  points are marked. The number 1 is assigned to one of them and 0 is assigned to the others. The following operation is allowed: Choose a point to which 1 is assigned and then assign  $(1 - a)$  and  $(1 - b)$  to the two adjacent points, where  $a$  and  $b$  are, respectively, the numbers assigned to these points before. Is it possible to assign 1 to all points by applying this operation several times if (a)  $n = 2001$  and (b)  $n = 2002$ ?

129. For every integer  $n$ , a nonnegative integer  $f(n)$  is assigned such that

- (a)  $f(mn) = f(m) + f(n)$  for each pair  $m, n$  of natural numbers;  
 (b)  $f(n) = 0$  when the rightmost digit in the decimal representation of the number  $n$  is 3; and  
 (c)  $f(10) = 0$ .

Prove that  $f(n) = 0$  for any natural number  $n$ .

130. Let  $ABCD$  be a rectangle for which the respective lengths of  $AB$  and  $BC$  are  $a$  and  $b$ . Another rectangle is circumscribed around  $ABCD$  so that each of its sides passes through one of the vertices of  $ABCD$ . Consider all such rectangles and, among them, find the one with a maximum area. Express this area in terms of  $a$  and  $b$ .
131. At a recent winter meeting of the Canadian Mathematical Society, some of the attending mathematicians were friends. It appeared that every two mathematicians, that had the same number of friends among the participants, did not have a common friend. Prove that there was a mathematician who had only one friend.
132. Simplify the expression

$$\sqrt[5]{3\sqrt{2} - 2\sqrt{5}} \cdot \sqrt[10]{\frac{6\sqrt{10} + 19}{2}}.$$

133. Prove that, if  $a, b, c, d$  are real numbers,  $b \neq c$ , both sides of the equation are defined, and

$$\frac{ac - b^2}{a - 2b + c} = \frac{bd - c^2}{b - 2c + d},$$

then each side of the equation is equal to

$$\frac{ad - bc}{a - b - c + d}.$$

Give two essentially different examples of quadruples  $(a, b, c, d)$ , not in geometric progression, for which the conditions are satisfied. What happens when  $b = c$ ?

134. Suppose that

$$a = zb + yc$$

$$b = xc + za$$

$$c = ya + xb.$$

Prove that

$$\frac{a^2}{1 - x^2} = \frac{b^2}{1 - y^2} = \frac{c^2}{1 - z^2}.$$

Of course, if any of  $x^2, y^2, z^2$  is equal to 1, then the conclusion involves undefined quantities. Give the proper conclusion in this situation. Provide two essentially different numerical examples.

135. For the positive integer  $n$ , let  $p(n) = k$  if  $n$  is divisible by  $2^k$  but not by  $2^{k+1}$ . Let  $x_0 = 0$  and define  $x_n$  for  $n \geq 1$  recursively by

$$\frac{1}{x_n} = 1 + 2p(n) - x_{n-1}.$$

Prove that every nonnegative rational number occurs exactly once in the sequence  $\{x_0, x_1, x_2, \dots, x_n, \dots\}$ . ■

136. Prove that, if in a semicircle of radius 1, five points  $A, B, C, D, E$  are taken in consecutive order, then

$$|AB|^2 + |BC|^2 + |CD|^2 + |DE|^2 + |AB||BC||CD| + |BC||CD||DE| < 4.$$

137. Can an arbitrary convex quadrilateral be decomposed by a polygonal line into two parts, each of whose diameters is less than the diameter of the given quadrilateral?
138. (a) A room contains ten people. Among any three, there are two (mutual) acquaintances. Prove that there are four people, any two of whom are acquainted.
- (b) Does the assertion hold if “ten” is replaced by “nine”?
139. Let  $A, B, C$  be three pairwise orthogonal faces of a tetrahedron meeting at one of its vertices and having respective areas  $a, b, c$ . Let the face  $D$  opposite this vertex have area  $d$ . Prove that

$$d^2 = a^2 + b^2 + c^2 .$$

140. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction  $x$  of the line was in front of him, while  $1/n$  of the line was behind him. On Tuesday, the same fraction  $x$  of the line was in front of him, while  $1/(n+1)$  of the line was behind him. On Wednesday, the same fraction  $x$  of the line was in front of him, while  $1/(n+2)$  of the line was behind him. Determine a value of  $n$  for which this is possible.
141. In how many ways can the rational  $2002/2001$  be written as the product of two rationals of the form  $(n+1)/n$ , where  $n$  is a positive integer?
142. Let  $x, y > 0$  be such that  $x^3 + y^3 \leq x - y$ . Prove that  $x^2 + y^2 \leq 1$ .
143. A sequence whose entries are 0 and 1 has the property that, if each 0 is replaced by 01 and each 1 by 001, then the sequence remains unchanged. Thus, it starts out as 010010101001 $\dots$ . What is the 2002th term of the sequence?
144. Let  $a, b, c, d$  be rational numbers for which  $bc \neq ad$ . Prove that there are infinitely many rational values of  $x$  for which  $\sqrt{(a+bx)(c+dx)}$  is rational. Explain the situation when  $bc = ad$ .
145. Let  $ABC$  be a right triangle with  $\angle A = 90^\circ$ . Let  $P$  be a point on the hypotenuse  $BC$ , and let  $Q$  and  $R$  be the respective feet of the perpendiculars from  $P$  to  $AC$  and  $AB$ . For what position of  $P$  is the length of  $QR$  minimum?
146. Suppose that  $ABC$  is an equilateral triangle. Let  $P$  and  $Q$  be the respective midpoint of  $AB$  and  $AC$ , and let  $U$  and  $V$  be points on the side  $BC$  with  $4BU = 4VC = BC$  and  $2UV = BC$ . Suppose that  $PV$  are joined and that  $W$  is the foot of the perpendicular from  $U$  to  $PV$  and that  $Z$  is the foot of the perpendicular from  $Q$  to  $PV$ .
- Explain how that four polygons  $APZQ, BUWP, CQZV$  and  $UVW$  can be rearranged to form a rectangle. Is this rectangle a square?

147. Let  $a > 0$  and let  $n$  be a positive integer. Determine the maximum value of

$$\frac{x_1 x_2 \cdots x_n}{(1+x_1)(x_1+x_2) \cdots (x_{n-1}+x_n)(x_n+a^{n+1})}$$

subject to the constraint that  $x_1, x_2, \dots, x_n > 0$ .

148. For a given prime number  $p$ , find the number of distinct sequences of natural numbers (positive integers)  $\{a_0, a_1, \dots, a_n \dots\}$  satisfying, for each positive integer  $n$ , the equation

$$\frac{a_0}{a_1} + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_n} + \frac{p}{a_{n+1}} = 1 .$$

149. Consider a cube concentric with a parallelepiped (rectangular box) with sides  $a < b < c$  and faces parallel to that of the cube. Find the side length of the cube for which the difference between the volume of the union and the volume of the intersection of the cube and parallelepiped is minimum.

150. The area of the bases of a truncated pyramid are equal to  $S_1$  and  $S_2$  and the total area of the lateral surface is  $S$ . Prove that, if there is a plane parallel to each of the bases that partitions the truncated pyramid into two truncated pyramids within each of which a sphere can be inscribed, then

$$S = (\sqrt{S_1} + \sqrt{S_2})(\sqrt[4]{S_1} + \sqrt[4]{S_2})^2 .$$

151. Prove that, for any natural number  $n$ , the equation

$$x(x+1)(x+2)\cdots(x+2n-1) + (x+2n+1)(x+2n+2)\cdots(x+4n) = 0$$

does not have real solutions.

152. Andrew and Brenda are playing the following game. Taking turns, they write in a sequence, from left to right, the numbers 0 or 1 until each of them has written 2002 numbers (to produce a 4004-digit number). Brenda is the winner if the sequence of zeros and ones, considered as a binary number (*i.e.*, written to base 2), can be written as the sum of two integer squares. Otherwise, the winner is Andrew. Prove that the second player, Brenda, can always win the game, and explain her winning strategy (*i.e.*, how she must play to ensure winning every game).

153. (a) Prove that, among any 39 consecutive natural numbers, there is one the sum of whose digits (in base 10) is divisible by 11.

(b) Present some generalizations of this problem.

154. (a) Give as neat a proof as you can that, for any natural number  $n$ , the sum of the squares of the numbers  $1, 2, \dots, n$  is equal to  $n(n+1)(2n+1)/6$ .

(b) Find the least natural number  $n$  exceeding 1 for which  $(1^2 + 2^2 + \dots + n^2)/n$  is the square of a natural number.

155. Find all real numbers  $x$  that satisfy the equation

$$3^{[(1/2)+\log_3(\cos x+\sin x)]} - 2^{\log_2(\cos x-\sin x)} = \sqrt{2} .$$

[The logarithms are taken to bases 3 and 2 respectively.]

156. In the triangle  $ABC$ , the point  $M$  is from the inside of the angle  $BAC$  such that  $\angle MAB = \angle MCA$  and  $\angle MAC = \angle MBA$ . Similarly, the point  $N$  is from the inside of the angle  $ABC$  such that  $\angle NBA = \angle NCB$  and  $\angle NBC = \angle NAB$ . Also, the point  $P$  is from the inside of the angle  $ACB$  such that  $\angle PCA = \angle PBC$  and  $\angle PCB = \angle PAC$ . (The points  $M$ ,  $N$  and  $P$  each could be inside or outside of the triangle.) Prove that the lines  $AM$ ,  $BN$  and  $CP$  are concurrent and that their intersection point belongs to the circumcircle of the triangle  $MNP$ .

157. Prove that if the quadratic equation  $x^2 + ax + b + 1 = 0$  has nonzero integer solutions, then  $a^2 + b^2$  is a composite integer.

158. Let  $f(x)$  be a polynomial with real coefficients for which the equation  $f(x) = x$  has no real solution. Prove that the equation  $f(f(x)) = x$  has no real solution either.

159. Let  $0 \leq a \leq 4$ . Prove that the area of the bounded region enclosed by the curves with equations

$$y = 1 - |x - 1|$$

and

$$y = |2x - a|$$

cannot exceed  $\frac{1}{3}$ .

160. Let  $I$  be the incentre of the triangle  $ABC$  and  $D$  be the point of contact of the inscribed circle with the side  $AB$ . Suppose that  $ID$  is produced outside of the triangle  $ABC$  to  $H$  so that the length  $DH$  is equal to the semi-perimeter of  $\triangle ABC$ . Prove that the quadrilateral  $AHBI$  is concyclic if and only if angle  $C$  is equal to  $90^\circ$ .

161. Let  $a, b, c$  be positive real numbers for which  $a + b + c = 1$ . Prove that

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{1}{2}.$$

162. Let  $A$  and  $B$  be fixed points in the plane. Find all positive integers  $k$  for which the following assertion holds:

*among all triangles  $ABC$  with  $|AC| = k|BC|$ , the one with the largest area is isosceles.*

163. Let  $R_i$  and  $r_i$  be the respective circumradius and inradius of triangle  $A_iB_iC_i$  ( $i = 1, 2$ ). Prove that, if  $\angle C_1 = \angle C_2$  and  $R_1r_2 = r_1R_2$ , then the two triangles are similar.

164. Let  $n$  be a positive integer and  $X$  a set with  $n$  distinct elements. Suppose that there are  $k$  distinct subsets of  $X$  for which the union of any four contains no more than  $n - 2$  elements. Prove that  $k \leq 2^{n-2}$ .

165. Let  $n$  be a positive integer. Determine all  $n$ -tuples  $\{a_1, a_2, \dots, a_n\}$  of positive integers for which  $a_1 + a_2 + \dots + a_n = 2n$  and there is no subset of them whose sum is equal to  $n$ .

166. Suppose that  $f$  is a real-valued function defined on the reals for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real  $x$  and  $y$ . Prove that  $f(x) \geq 0$  for all real  $x$ .

167. Let  $u = (\sqrt{5} - 2)^{1/3} - (\sqrt{5} + 2)^{1/3}$  and  $v = (\sqrt{189} - 8)^{1/3} - (\sqrt{189} + 8)^{1/3}$ . Prove that, for each positive integer  $n$ ,  $u^n + v^{n+1} = 0$ .

168. Determine the value of

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ.$$

169. Prove that, for each positive integer  $n$  exceeding 1,

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1.$$

170. Solve, for real  $x$ ,

$$x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x = 4.$$

171. Let  $n$  be a positive integer. In a round-robin match,  $n$  teams compete and each pair of teams plays exactly one game. At the end of the match, the  $i$ th team has  $x_i$  wins and  $y_i$  losses. There are no ties. Prove that

$$x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2.$$

172. Let  $a, b, c, d, e, f$  be different integers. Prove that

$$(a - b)^2 + (b - c)^2 + (c - d)^2 + (d - e)^2 + (e - f)^2 + (f - a)^2 \geq 18.$$

173. Suppose that  $a$  and  $b$  are positive real numbers for which  $a + b = 1$ . Prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

Determine when equality holds.

174. For which real value of  $x$  is the function

$$(1 - x)^5(1 + x)(1 + 2x)^2$$

maximum? Determine its maximum value.

175.  $ABC$  is a triangle such that  $AB < AC$ . The point  $D$  is the midpoint of the arc with endpoints  $B$  and  $C$  of that arc of the circumcircle of  $\triangle ABC$  that contains  $A$ . The foot of the perpendicular from  $D$  to  $AC$  is  $E$ . Prove that  $AB + AE = EC$ .

176. Three noncollinear points  $A$ ,  $M$  and  $N$  are given in the plane. Construct the square such that one of its vertices is the point  $A$ , and the two sides which do not contain this vertex are on the lines through  $M$  and  $N$  respectively. [Note: In such a problem, your solution should consist of a description of the construction (with straightedge and compasses) and a proof in correct logical order proceeding from what is given to what is desired that the construction is valid. You should deal with the feasibility of the construction.]

177. Let  $a_1, a_2, \dots, a_n$  be nonnegative integers such that, whenever  $1 \leq i, 1 \leq j, i + j \leq n$ , then

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1.$$

(a) Give an example of such a sequence which is not an arithmetic progression.

(b) Prove that there exists a real number  $x$  such that  $a_k = \lfloor kx \rfloor$  for  $1 \leq k \leq n$ .

178. Suppose that  $n$  is a positive integer and that  $x_1, x_2, \dots, x_n$  are positive real numbers such that  $x_1 + x_2 + \dots + x_n = n$ . Prove that

$$\sum_{i=1}^n \sqrt[n]{ax_i + b} \leq a + b + n - 1$$

for every pair  $a, b$  of real numbers with each  $ax_i + b$  nonnegative. Describe the situation when equality occurs.

179. Determine the units digit of the numbers  $a^2$ ,  $b^2$  and  $ab$  (in base 10 numeration), where

$$a = 2^{2002} + 3^{2002} + 4^{2002} + 5^{2002}$$

and

$$b = 3^1 + 3^2 + 3^3 + \dots + 3^{2002}.$$

180. Consider the function  $f$  that takes the set of complex numbers into itself defined by  $f(z) = 3z + |z|$ . Prove that  $f$  is a bijection and find its inverse.

181. Consider a regular polygon with  $n$  sides, each of length  $a$ , and an interior point located at distances  $a_1, a_2, \dots, a_n$  from the sides. Prove that

$$a \sum_{i=1}^n \frac{1}{a_i} > 2\pi.$$

182. Let  $M$  be an interior point of the equilateral triangle  $ABC$  with each side of unit length. Prove that

$$MA.MB + MB.MC + MC.MA \geq 1 .$$

183. Simplify the expression

$$\frac{\sqrt{1 + \sqrt{1 - x^2}}((1 + x)\sqrt{1 + x} - (1 - x)\sqrt{1 - x})}{x(2 + \sqrt{1 - x^2})} ,$$

where  $0 < |x| < 1$ .

184. Using complex numbers, or otherwise, evaluate

$$\sin 10^\circ \sin 50^\circ \sin 70^\circ .$$

185. Find all triples of natural numbers  $a, b, c$ , such that all of the following conditions hold: (1)  $a < 1974$ ; (2)  $b$  is less than  $c$  by 1575; (3)  $a^2 + b^2 = c^2$ .

186. Find all natural numbers  $n$  such that there exists a convex  $n$ -sided polygon whose diagonals are all of the same length.

187. Suppose that  $p$  is a real parameter and that

$$f(x) = x^3 - (p + 5)x^2 - 2(p - 3)(p - 1)x + 4p^2 - 24p + 36 .$$

(a) Check that  $f(3 - p) = 0$ .

(b) Find all values of  $p$  for which two of the roots of the equation  $f(x) = 0$  (expressed in terms of  $p$ ) can be the lengths of the two legs in a right-angled triangle with a hypotenuse of  $4\sqrt{2}$ .

188. (a) The measure of the angles of an acute triangle are  $\alpha, \beta$  and  $\gamma$  degrees. Determine (as an expression of  $\alpha, \beta, \gamma$ ) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the orthocentre of the triangle; (ii) the circumcentre of the triangle.

(b) The sides of an arbitrary triangle are  $a, b, c$  units in length. Determine (as an expression of  $a, b, c$ ) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the centre of the inscribed circle of the triangle; (ii) the intersection point of the three segments joining the vertices of the triangle to the points on the opposite sides where the inscribed circle is tangent (be sure to prove that, indeed, the three segments intersect in a common point).

189. There are  $n$  lines in the plane, where  $n$  is an integer exceeding 2. No three of them are concurrent and no two of them are parallel. The lines divide the plane into regions; some of them are closed (they have the form of a convex polygon); others are unbounded (their borders are broken lines consisting of segments and rays).

(a) Determine as a function of  $n$  the number of unbounded regions.

(b) Suppose that some of the regions are coloured, so that no two coloured regions have a common side (a segment or ray). Prove that the number of regions coloured in this way does not exceed  $\frac{1}{3}(n^2 + n)$ .

190. Find all integer values of the parameter  $a$  for which the equation

$$|2x + 1| + |x - 2| = a$$

has exactly one integer among its solutions.

191. In **Olymonland** the distances between every two cities is different. When the transportation program of the country was being developed, for each city, the closest of the other cities was chosen and a highway was built to connect them. All highways are line segments. Prove that
- no two highways intersect;
  - every city is connected by a highway to no more than 5 other cities;
  - there is no closed broken line composed of highways only.
192. Let  $ABC$  be a triangle,  $D$  be the midpoint of  $AB$  and  $E$  a point on the side  $AC$  for which  $AE = 2EC$ . Prove that  $BE$  bisects the segment  $CD$ .
193. Determine the volume of an isosceles tetrahedron for which the pairs of opposite edges have lengths  $a, b, c$ . Check your answer independently for a regular tetrahedron.
194. Let  $ABC$  be a triangle with incentre  $I$ . Let  $M$  be the midpoint of  $BC$ ,  $U$  be the intersection of  $AI$  produced with  $BC$ ,  $D$  be the foot of the perpendicular from  $I$  to  $BC$  and  $P$  be the foot of the perpendicular from  $A$  to  $BC$ . Prove that

$$|PD||DM| = |DU||PM| .$$

195. Let  $ABCD$  be a convex quadrilateral and let the midpoints of  $AC$  and  $BD$  be  $P$  and  $Q$  respectively, Prove that

$$|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 = |AC|^2 + |BD|^2 + 4|PQ|^2 .$$

196. Determine five values of  $p$  for which the polynomial  $x^2 + 2002x - 1002p$  has integer roots.
197. Determine all integers  $x$  and  $y$  that satisfy the equation  $x^3 + 9xy + 127 = y^3$ .
198. Let  $p$  be a prime number and let  $f(x)$  be a polynomial of degree  $d$  with integer coefficients such that  $f(0) = 0$  and  $f(1) = 1$  and that, for every positive integer  $n$ ,  $f(n) \equiv 0$  or  $f(n) \equiv 1$ , modulo  $p$ . Prove that  $d \geq p - 1$ . Give an example of such a polynomial.
199. Let  $A$  and  $B$  be two points on a parabola with vertex  $V$  such that  $VA$  is perpendicular to  $VB$  and  $\theta$  is the angle between the chord  $VA$  and the axis of the parabola. Prove that

$$\frac{|VA|}{|VB|} = \cot^3 \theta .$$

200. Let  $n$  be a positive integer exceeding 1. Determine the number of permutations  $(a_1, a_2, \dots, a_n)$  of  $(1, 2, \dots, n)$  for which there exists exactly one index  $i$  with  $1 \leq i \leq n$  and  $a_i > a_{i+1}$ .
201. Let  $(a_1, a_2, \dots, a_n)$  be an arithmetic progression and  $(b_1, b_2, \dots, b_n)$  be a geometric progression, each of  $n$  positive real numbers, for which  $a_1 = b_1$  and  $a_n = b_n$ . Prove that

$$a_1 + a_2 + \dots + a_n \geq b_1 + b_2 + \dots + b_n .$$

202. For each positive integer  $k$ , let  $a_k = 1 + (1/2) + (1/3) + \dots + (1/k)$ . Prove that, for each positive integer  $n$ ,

$$3a_1 + 5a_2 + 7a_3 + \dots + (2n + 1)a_n = (n + 1)^2 a_n - \frac{1}{2}n(n + 1) .$$

203. Every midpoint of an edge of a tetrahedron is contained in a plane that is perpendicular to the opposite edge. Prove that these six planes intersect in a point that is symmetric to the centre of the circumsphere of the tetrahedron with respect to its centroid.

204. Each of  $n \geq 2$  people in a certain village has at least one of eight different names. No two people have exactly the same set of names. For an arbitrary set of  $k$  names (with  $1 \leq k \leq 7$ ), the number of people containing at least one of the  $k$  ( $\geq 1$ ) names among his/her set of names is even. Determine the value of  $n$ .

205. Let  $f(x)$  be a convex realvalued function defined on the reals,  $n \geq 2$  and  $x_1 < x_2 < \dots < x_n$ . Prove that

$$x_1 f(x_2) + x_2 f(x_3) + \dots + x_n f(x_1) \geq x_2 f(x_1) + x_3 f(x_2) + \dots + x_1 f(x_n) .$$

206. In a group consisting of five people, among any three people, there are two who know each other and two neither of whom knows the other. Prove that it is possible to seat the group around a circular table so that each adjacent pair knows each other.

207. Let  $n$  be a positive integer exceeding 1. Suppose that  $A = (a_1, a_2, \dots, a_m)$  is an ordered set of  $m = 2^n$  numbers, each of which is equal to either 1 or  $-1$ . Let

$$S(A) = (a_1 a_2, a_2 a_3, \dots, a_{m-1} a_m, a_m a_1) .$$

Define,  $S^0(A) = A$ ,  $S^1(A) = S(A)$ , and for  $k \geq 1$ ,  $S^{k+1} = S(S^k(A))$ . Is it always possible to find a positive integer  $r$  for which  $S^r(A)$  consists entirely of 1s?

208. Determine all positive integers  $n$  for which  $n = a^2 + b^2 + c^2 + d^2$ , where  $a < b < c < d$  and  $a, b, c, d$  are the four smallest positive divisors of  $n$ .

209. Determine all positive integers  $n$  for which  $2^n - 1$  is a multiple of 3 and  $(2^n - 1)/3$  has a multiple of the form  $4m^2 + 1$  for some integer  $m$ .

210.  $ABC$  and  $DAC$  are two isosceles triangles for which  $B$  and  $D$  are on opposite sides of  $AC$ ,  $AB = AC$ ,  $DA = DC$ ,  $\angle BAC = 20^\circ$  and  $\angle ADC = 100^\circ$ . Prove that  $AB = BC + CD$ .

211. Let  $ABC$  be a triangle and let  $M$  be an interior point. Prove that

$$\min \{MA, MB, MC\} + MA + MB + MC < AB + BC + CA .$$

212. A set  $S$  of points in space has at least three elements and satisfies the condition that, for any two distinct points  $A$  and  $B$  in  $S$ , the right bisecting plane of the segment  $AB$  is a plane of symmetry for  $S$ . Determine all possible finite sets  $S$  that satisfy the condition.

213. Suppose that each side and each diagonal of a regular hexagon  $A_1 A_2 A_3 A_4 A_5 A_6$  is coloured either red or blue, and that no triangle  $A_i A_j A_k$  has all of its sides coloured blue. For each  $k = 1, 2, \dots, 6$ , let  $r_k$  be the number of segments  $A_k A_j$  ( $j \neq k$ ) coloured red. Prove that

$$\sum_{k=1}^6 (2r_k - 7)^2 \leq 54 .$$

214. Let  $S$  be a circle with centre  $O$  and radius 1, and let  $P_i$  ( $1 \leq i \leq n$ ) be points chosen on the (circumference of the) circle for which  $\sum_{i=1}^n \overrightarrow{OP_i} = \mathbf{0}$ . Prove that, for each point  $X$  in the plane,  $\sum |XP_i| \geq n$ .

215. Find all values of the parameter  $a$  for which the equation  $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$  has exactly four real solutions which are in geometric progression.

216. Let  $x$  be positive and let  $0 < a \leq 1$ . Prove that

$$(1 - x^a)(1 - x)^{-1} \leq (1 + x)^{a-1} .$$

217. Let the three side lengths of a scalene triangle be given. There are two possible ways of orienting the triangle with these side lengths, one obtainable from the other by turning the triangle over, or by reflecting in a mirror. Prove that it is possible to slice the triangle in one of its orientations into finitely many pieces that can be rearranged using rotations and translations in the plane (but not reflections and rotations out of the plane) to form the other.
218. Let  $ABC$  be a triangle. Suppose that  $D$  is a point on  $BA$  produced and  $E$  a point on the side  $BC$ , and that  $DE$  intersects the side  $AC$  at  $F$ . Let  $BE + EF = BA + AF$ . Prove that  $BC + CF = BD + DF$ .
219. There are two definitions of an ellipse.

(1) An ellipse is the locus of points  $P$  such that the sum of its distances from two fixed points  $F_1$  and  $F_2$  (called *foci*) is constant.

(2) An ellipse is the locus of points  $P$  such that, for some real number  $e$  (called the *eccentricity*) with  $0 < e < 1$ , the distance from  $P$  to a fixed point  $F$  (called a *focus*) is equal to  $e$  times its perpendicular distance to a fixed straight line (called the *directrix*).

Prove that the two definitions are compatible.

220. Prove or disprove: A quadrilateral with one pair of opposite sides and one pair of opposite angles equal is a parallelogram.
221. A *cycloid* is the locus of a point  $P$  fixed on a circle that rolls without slipping upon a line  $u$ . It consists of a sequence of arches, each arch extending from that position on the locus at which the point  $P$  rests on the line  $u$ , through a curve that rises to a position whose distance from  $u$  is equal to the diameter of the generating circle and then falls to a subsequent position at which  $P$  rests on the line  $u$ . Let  $v$  be the straight line parallel to  $u$  that is tangent to the cycloid at the point furthest from the line  $u$ .
- (a) Consider a position of the generating circle, and let  $P$  be on this circle and on the cycloid. Let  $PQ$  be the chord on this circle that is parallel to  $u$  (and to  $v$ ). Show that the locus of  $Q$  is a similar cycloid formed by a circle of the same radius rolling (upside down) along the line  $v$ .
- (b) The region between the two cycloids consists of a number of “beads”. Argue that the area of one of these beads is equal to the area of the generating circle.
- (c) Use the considerations of (a) and (b) to find the area between  $u$  and one arch of the cycloid using a method that does not make use of calculus.

222. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1} \left( \frac{2}{n^2} \right).$$

223. Let  $a, b, c$  be positive real numbers for which  $a + b + c = abc$ . Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2}.$$

224. For  $x > 0, y > 0$ , let  $g(x, y)$  denote the minimum of the three quantities,  $x, y + 1/x$  and  $1/y$ . Determine the maximum value of  $g(x, y)$  and where this maximum is assumed.
225. A set of  $n$  lightbulbs, each with an *on-off* switch, numbered  $1, 2, \dots, n$  are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on or off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For  $k \geq 3$ , switch  $k$  can turn bulb  $k$  on or off if and only if bulb  $k - 1$  is off and bulbs  $1, 2, \dots, k - 2$  are all on; otherwise it does not function.
- (a) Prove that there is an algorithm that will turn all of the bulbs on.

(b) If  $x_n$  is the length of the shortest algorithm that will turn on all  $n$  bulbs when they are initially off, determine the largest prime divisor of  $3x_n + 1$  when  $n$  is odd.

226. Suppose that the polynomial  $f(x)$  of degree  $n \geq 1$  has all real roots and that  $\lambda > 0$ . Prove that the set  $\{x \in \mathbf{R} : |f(x)| \leq \lambda|f'(x)|\}$  is a finite union of closed intervals whose total length is equal to  $2n\lambda$ .

227. Let  $n$  be an integer exceeding 2 and let  $a_0, a_1, a_2, \dots, a_n, a_{n+1}$  be positive real numbers for which  $a_0 = a_n$ ,  $a_1 = a_{n+1}$  and

$$a_{i-1} + a_{i+1} = k_i a_i$$

for some positive integers  $k_i$ , where  $1 \leq i \leq n$ .

Prove that

$$2n \leq k_1 + k_2 + \dots + k_n \leq 3n .$$

228. Prove that, if  $1 < a < b < c$ , then

$$\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) > 0 .$$

229. Suppose that  $n$  is a positive integer and that  $0 < i < j < n$ . Prove that the greatest common divisor of  $\binom{n}{i}$  and  $\binom{n}{j}$  exceeds 1.

230. Let  $f$  be a strictly increasing function on the closed interval  $[0, 1]$  for which  $f(0) = 0$  and  $f(1) = 1$ . Let  $g$  be its inverse. Prove that

$$\sum_{k=1}^9 \left( f\left(\frac{k}{10}\right) + g\left(\frac{k}{10}\right) \right) \leq 9.9 .$$

231. For  $n \geq 10$ , let  $g(n)$  be defined as follows:  $n$  is mapped by  $g$  to the sum of the number formed by taking all but the last three digits of its square and adding it to the number formed by the last three digits of its square. For example,  $g(54) = 918$  since  $54^2 = 2916$  and  $2 + 916 = 918$ . Is it possible to start with 527 and, through repeated applications of  $g$ , arrive at 605?

232. (a) Prove that, for positive integers  $n$  and positive values of  $x$ ,

$$(1 + x^{n+1})^n \leq (1 + x^n)^{n+1} \leq 2(1 + x^{n+1})^n .$$

(b) Let  $h(x)$  be the function defined by

$$h(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ x, & \text{if } x > 1. \end{cases}$$

Determine a value  $N$  for which

$$|h(x) - (1 + x^n)^{\frac{1}{n}}| < 10^{-6}$$

whenever  $0 \leq x \leq 10$  and  $n \geq N$ .

233. Let  $p(x)$  be a polynomial of degree 4 with rational coefficients for which the equation  $p(x) = 0$  has *exactly one* real solution. Prove that this solution is rational.

234. A square of side length 100 is divided into 10000 smaller unit squares. Two squares sharing a common side are called *neighbours*.

(a) Is it possible to colour an even number of squares so that each coloured square has an even number of coloured neighbours?

(b) Is it possible to colour an odd number of squares so that each coloured square has an odd number of coloured neighbours?

235. Find all positive integers,  $N$ , for which:

(i)  $N$  has exactly sixteen positive divisors:  $1 = d_1 < d_2 < \dots < d_{16} = N$ ;

(ii) the divisor with the index  $d_5$  (namely,  $d_{d_5}$ ) is equal to  $(d_2 + d_4) \times d_6$  (the product of the two).

236. For any positive real numbers  $a, b, c$ , prove that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$

237. The sequence  $\{a_n : n = 1, 2, \dots\}$  is defined by the recursion

$$a_1 = 20 \qquad a_2 = 30$$

$$a_{n+2} = 3a_{n+1} - a_n \qquad \text{for } n \geq 1.$$

Find all natural numbers  $n$  for which  $1 + 5a_n a_{n+1}$  is a perfect square.

238. Let  $ABC$  be an acute-angled triangle, and let  $M$  be a point on the side  $AC$  and  $N$  a point on the side  $BC$ . The circumcircles of triangles  $CAN$  and  $BCM$  intersect at the two points  $C$  and  $D$ . Prove that the line  $CD$  passes through the circumcentre of triangle  $ABC$  if and only if the right bisector of  $AB$  passes through the midpoint of  $MN$ .

239. Find all natural numbers  $n$  for which the diophantine equation

$$(x + y + z)^2 = nxyz$$

has positive integer solutions  $x, y, z$ .

240. In a competition, 8 judges rate each contestant “yes” or “no”. After the competition, it turned out, that for any two contestants, two judges marked the first one by “yes” and the second one also by “yes”; two judges have marked the first one by “yes” and the second one by “no”; two judges have marked the first one by “no” and the second one by “yes”; and, finally, two judges have marked the first one by “no” and the second one by “no”. What is the greatest number of contestants?

241. Determine  $\sec 40^\circ + \sec 80^\circ + \sec 160^\circ$ .

242. Let  $ABC$  be a triangle with sides of length  $a, b, c$  opposite respective angles  $A, B, C$ . What is the radius of the circle that passes through the points  $A, B$  and the incentre of triangle  $ABC$  when angle  $C$  is equal to (a)  $90^\circ$ ; (b)  $120^\circ$ ; (c)  $60^\circ$ . (With thanks to Jean Turgeon, Université de Montréal.)

243. The inscribed circle, with centre  $I$ , of the triangle  $ABC$  touches the sides  $BC, CA$  and  $AB$  at the respective points  $D, E$  and  $F$ . The line through  $A$  parallel to  $BC$  meets  $DE$  and  $DF$  produced at the respective points  $M$  and  $N$ . The midpoints of  $DM$  and  $DN$  are  $P$  and  $Q$  respectively. Prove that  $A, E, F, I, P, Q$  lie on a common circle.

244. Let  $x_0 = 4, x_1 = x_2 = 0, x_3 = 3$ , and, for  $n \geq 4, x_{n+4} = x_{n+1} + x_n$ . Prove that, for each prime  $p, x_p$  is a multiple of  $p$ .

245. Determine all pairs  $(m, n)$  of positive integers with  $m \leq n$  for which an  $m \times n$  rectangle can be tiled with congruent pieces formed by removing a  $1 \times 1$  square from a  $2 \times 2$  square.

246. Let  $p(n)$  be the number of partitions of the positive integer  $n$ , and let  $q(n)$  denote the number of finite sets  $\{u_1, u_2, u_3, \dots, u_k\}$  of positive integers that satisfy  $u_1 > u_2 > u_3 > \dots > u_k$  such that

$n = u_1 + u_3 + u_5 + \dots$  (the sum of the ones with odd indices). Prove that  $p(n) = q(n)$  for each positive integer  $n$ .

For example,  $q(6)$  counts the sets  $\{6\}$ ,  $\{6, 5\}$ ,  $\{6, 4\}$ ,  $\{6, 3\}$ ,  $\{6, 2\}$ ,  $\{6, 1\}$ ,  $\{5, 4, 1\}$ ,  $\{5, 3, 1\}$ ,  $\{5, 2, 1\}$ ,  $\{4, 3, 2\}$ ,  $\{4, 3, 2, 1\}$ .

247. Let  $ABCD$  be a convex quadrilateral with no pairs of parallel sides. Associate to side  $AB$  a point  $T$  as follows. Draw lines through  $A$  and  $B$  parallel to the opposite side  $CD$ . Let these lines meet  $CB$  produced at  $B'$  and  $DA$  produced at  $A'$ , and let  $T$  be the intersection of  $AB$  and  $B'A'$ . Let  $U, V, W$  be points similarly constructed with respect to sides  $BC, CD, DA$ , respectively. Prove that  $TUVW$  is a parallelogram.
248. Find all real solutions to the equation

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1.$$

249. The non-isosceles right triangle  $ABC$  has  $\angle CAB = 90^\circ$ . Its inscribed circle with centre  $T$  touches the sides  $AB$  and  $AC$  at  $U$  and  $V$  respectively. The tangent through  $A$  of the circumscribed circle of triangle  $ABC$  meets  $UV$  in  $S$ . Prove that:

(a)  $ST \parallel BC$ ;

(b)  $|d_1 - d_2| = r$ , where  $r$  is the radius of the inscribed circle, and  $d_1$  and  $d_2$  are the respective distances from  $S$  to  $AC$  and  $AB$ .

250. In a convex polygon  $\mathfrak{P}$ , some diagonals have been drawn so that no two have an intersection in the interior of  $\mathfrak{P}$ . Show that there exists at least two vertices of  $\mathfrak{P}$ , neither of which is an endpoint of any of these diagonals.
251. Prove that there are infinitely many positive integers  $n$  for which the numbers  $\{1, 2, 3, \dots, 3n\}$  can be arranged in a rectangular array with three rows and  $n$  columns for which (a) each row has the same sum, a multiple of 6, and (b) each column has the same sum, a multiple of 6.

252. Suppose that  $a$  and  $b$  are the roots of the quadratic  $x^2 + px + 1$  and that  $c$  and  $d$  are the roots of the quadratic  $x^2 + qx + 1$ . Determine  $(a - c)(b - c)(a + d)(b + d)$  as a function of  $p$  and  $q$ .

253. Let  $n$  be a positive integer and let  $\theta = \pi/(2n + 1)$ . Prove that  $\cot^2 \theta, \cot^2 2\theta, \dots, \cot^2 n\theta$  are the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \binom{2n+1}{5}x^{n-2} - \dots = 0.$$

254. Determine the set of all triples  $(x, y, z)$  of integers with  $1 \leq x, y, z \leq 1000$  for which  $x^2 + y^2 + z^2$  is a multiple of  $xyz$ .

255. Prove that there is no positive integer that, when written to base 10, is equal to its  $k$ th multiple when its initial digit (on the left) is transferred to the right (units end), where  $2 \leq k \leq 9$  and  $k \neq 3$ .

256. Find the condition that must be satisfied by  $y_1, y_2, y_3, y_4$  in order that the following set of six simultaneous equations in  $x_1, x_2, x_3, x_4$  is solvable. Where possible, find the solution.

$$\begin{array}{lll} x_1 + x_2 = y_1 y_2 & x_1 + x_3 = y_1 y_3 & x_1 + x_4 = y_1 y_4 \\ x_2 + x_3 = y_2 y_3 & x_2 + x_4 = y_2 y_4 & x_3 + x_4 = y_3 y_4. \end{array}$$

257. Let  $n$  be a positive integer exceeding 1. Discuss the solution of the system of equations:

$$\begin{aligned}
ax_1 + x_2 + \cdots + x_n &= 1 \\
x_1 + ax_2 + \cdots + x_n &= a \\
&\dots \\
x_1 + x_2 + \cdots + ax_i + \cdots + x_n &= a^{i-1} \\
&\dots \\
x_1 + x_2 + \cdots + x_i + \cdots + ax_n &= a^{n-1} .
\end{aligned}$$

258. The infinite sequence  $\{a_n; n = 0, 1, 2, \dots\}$  satisfies the recursion

$$a_{n+1} = a_n^2 + (a_n - 1)^2$$

for  $n \geq 0$ . Find all rational numbers  $a_0$  such that there are four distinct indices  $p, q, r, s$  for which  $a_p - a_q = a_r - a_s$ .

259. Let  $ABC$  be a given triangle and let  $A'BC$ ,  $AB'C$ ,  $ABC'$  be equilateral triangles erected outwards on the sides of triangle  $ABC$ . Let  $\Omega$  be the circumcircle of  $A'B'C'$  and let  $A''$ ,  $B''$ ,  $C''$  be the respective intersections of  $\Omega$  with the lines  $AA'$ ,  $BB'$ ,  $CC'$ .

Prove that  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent and that

$$AA'' + BB'' + CC'' = AA' = BB' = CC' .$$

260.  $TABC$  is a tetrahedron with volume 1,  $G$  is the centroid of triangle  $ABC$  and  $O$  is the midpoint of  $TG$ . Reflect  $TABC$  in  $O$  to get  $T'A'B'C'$ . Find the volume of the intersection of  $TABC$  and  $T'A'B'C'$ .

261. Let  $x, y, z > 0$ . Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(x+y)(y+z)}} + \frac{z}{z + \sqrt{(x+z)(y+z)}} \leq 1 .$$

as above to get a linear polynomial with root  $r$ .

262. Let  $ABC$  be an acute triangle. Suppose that  $P$  and  $U$  are points on the side  $BC$  so that  $P$  lies between  $B$  and  $U$ , that  $Q$  and  $V$  are points on the side  $CA$  so that  $Q$  lies between  $C$  and  $V$ , and that  $R$  and  $W$  are points on the side  $AB$  so that  $R$  lies between  $A$  and  $W$ . Suppose also that

$$\angle APU = \angle AUP = \angle BQV = \angle BVQ = \angle CRW = \angle CWR .$$

The lines  $AP$ ,  $BQ$  and  $CR$  bound a triangle  $T_1$  and the lines  $AU$ ,  $BV$  and  $CW$  bound a triangle  $T_2$ . Prove that all six vertices of the triangles  $T_1$  and  $T_2$  lie on a common circle.

263. The ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are each used exactly once altogether to form three positive integers for which the largest is the sum of the other two. What are the largest and the smallest possible values of the sum?

264. For the real parameter  $a$ , solve for real  $x$  the equation

$$x = \sqrt{a + \sqrt{a + x}} .$$

A complete answer will discuss the circumstances under which a solution is feasible.

265. Note that  $959^2 = 919681$ ,  $919 + 681 = 40^2$ ;  $960^2 = 921600$ ,  $921 + 600 = 39^2$ ; and  $961^2 = 923521$ ,  $923 + 521 = 38^2$ . Establish a general result of which these are special instances.
266. Prove that, for any positive integer  $n$ ,  $\binom{2n}{n}$  divides the least common multiple of the numbers  $1, 2, 3, \dots, 2n - 1, 2n$ .
267. A non-orthogonal reflection in an axis  $a$  takes each point on  $a$  to itself, and each point  $P$  not on  $a$  to a point  $P'$  on the other side of  $a$  in such a way that  $a$  intersects  $PP'$  at its midpoint and  $PP'$  always makes a fixed angle  $\theta$  with  $a$ . Does this transformation preserve lines? preserve angles? Discuss the image of a circle under such a transformation.
268. Determine all continuous real functions  $f$  of a real variable for which

$$f(x + 2f(y)) = f(x) + y + f(y)$$

for all real  $x$  and  $y$ .

269. Prove that the number

$$N = 2 \times 4 \times 6 \times \dots \times 2000 \times 2002 + 1 \times 3 \times 5 \times \dots \times 1999 \times 2001$$

is divisible by 2003.

270. A straight line cuts an acute triangle into two parts (not necessarily triangles). In the same way, two other lines cut each of these two parts into two parts. These steps repeat until all the parts are triangles. Is it possible for all the resulting triangle to be obtuse? (Provide reasoning to support your answer.)
271. Let  $x, y, z$  be natural numbers, such that the number

$$\frac{x - y\sqrt{2003}}{y - z\sqrt{2003}}$$

is rational. Prove that

- (a)  $xz = y^2$ ;  
 (b) when  $y \neq 1$ , the numbers  $x^2 + y^2 + z^2$  and  $x^2 + 4z^2$  are composite.

272. Let  $ABCD$  be a parallelogram whose area is 2003 sq. cm. Several points are chosen on the sides of the parallelogram.
- (a) If there are 1000 points in addition to  $A, B, C, D$ , prove that there always exist three points among these 1004 points that are vertices of a triangle whose area is less than 2 sq. cm.
- (b) If there are 2000 points in addition to  $A, B, C, D$ , is it true that there always exist three points among these 2004 points that are vertices of a triangle whose area is less than 1 sq. cm?

273. Solve the logarithmic inequality

$$\log_4(9^x - 3^x - 1) \geq \log_2 \sqrt{5}.$$

274. The inscribed circle of an isosceles triangle  $ABC$  is tangent to the side  $AB$  at the point  $T$  and bisects the segment  $CT$ . If  $CT = 6\sqrt{2}$ , find the sides of the triangle.
275. Find all solutions of the trigonometric equation

$$\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x.$$

276. Let  $a, b, c$  be the lengths of the sides of a triangle and let  $s = \frac{1}{2}(a + b + c)$  be its semi-perimeter and  $r$  be the radius of the inscribed circle. Prove that

$$(s - a)^{-2} + (s - b)^{-2} + (s - c)^{-2} \geq r^{-2}$$

and indicate when equality holds.

277. Let  $m$  and  $n$  be positive integers for which  $m < n$ . Suppose that an arbitrary set of  $n$  integers is given and the following operation is performed: select any  $m$  of them and add 1 to each. For which pairs  $(m, n)$  is it always possible to modify the given set by performing the operation finitely often to obtain a set for which all the integers are equal?
278. (a) Show that  $4mn - m - n$  can be an integer square for infinitely many pairs  $(m, n)$  of integers. Is it possible for either  $m$  or  $n$  to be positive?
- (b) Show that there are infinitely many pairs  $(m, n)$  of positive integers for which  $4mn - m - n$  is one less than a perfect square.
279. (a) For which values of  $n$  is it possible to construct a sequence of abutting segments in the plane to form a polygon whose side lengths are  $1, 2, \dots, n$  exactly in this order, where two neighbouring segments are perpendicular?
- (b) For which values of  $n$  is it possible to construct a sequence of abutting segments in space to form a polygon whose side lengths are  $1, 2, \dots, n$  exactly in this order, where any two of three successive segments are perpendicular?
280. Consider all finite sequences of positive integers whose sum is  $n$ . Determine  $T(n, k)$ , the number of times that the positive integer  $k$  occurs in all of these sequences taken together.
281. Let  $a$  be the result of tossing a black die (a number cube whose sides are numbers from 1 to 6 inclusive), and  $b$  the result of tossing a white die. What is the probability that there exist real numbers  $x, y, z$  for which  $x + y + z = a$  and  $xy + yz + zx = b$ ?
282. Suppose that at the vertices of a pentagon five integers are specified in such a way that the sum of the integers is positive. If not all the integers are non-negative, we can perform the following operation: suppose that  $x, y, z$  are three consecutive integers for which  $y < 0$ ; we replace them respectively by the integers  $x + y, -y, z + y$ . In the event that there is more than one negative integer, there is a choice of how this operation may be performed. Given any choice of integers, and any sequence of operations, must we arrive at a set of nonnegative integers after a finite number of steps?

For example, if we start with the numbers  $(2, -3, 3, -6, 7)$  around the pentagon, we can produce  $(1, 3, 0, -6, 7)$  or  $(2, -3, -3, 6, 1)$ .

283. (a) Determine all quadruples  $(a, b, c, d)$  of positive integers for which the greatest common divisor of its elements is 1,

$$\frac{a}{b} = \frac{c}{d}$$

and  $a + b + c = d$ .

- (b) Of those quadruples found in (a), which also satisfy

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{a} ?$$

- (c) For quadruples  $(a, b, c, d)$  of positive integers, do the conditions  $a + b + c = d$  and  $(1/b) + (1/c) + (1/d) = (1/a)$  together imply that  $a/b = c/d$ ?

284. Suppose that  $ABCDEF$  is a convex hexagon for which  $\angle A + \angle C + \angle E = 360^\circ$  and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1 .$$

Prove that

$$\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1 .$$

285. (a) Solve the following system of equations:

$$(1 + 4^{2x-y})(5^{1-2x+y}) = 1 + 2^{2x-y+1} ;$$
$$y^2 + 4x = \log_2(y^2 + 2x + 1) .$$

(b) Solve for real values of  $x$ :

$$3^x \cdot 8^{x/(x+2)} = 6 .$$

Express your answers in a simple form.

286. Construct inside a triangle  $ABC$  a point  $P$  such that, if  $X, Y, Z$  are the respective feet of the perpendiculars from  $P$  to  $BC, CA, AB$ , then  $P$  is the centroid (intersection of the medians) of triangle  $XYZ$ .

287. Let  $M$  and  $N$  be the respective midpoints of the sides  $BC$  and  $AC$  of the triangle  $ABC$ . Prove that the centroid of the triangle  $ABC$  lies on the circumscribed circle of the triangle  $CMN$  if and only if

$$4 \cdot |AM| \cdot |BN| = 3 \cdot |AC| \cdot |BC| .$$

288. Suppose that  $a_1 < a_2 < \dots < a_n$ . Prove that

$$a_1 a_2^4 + a_2 a_3^4 + \dots + a_n a_1^4 \geq a_2 a_1^4 + a_3 a_2^4 + \dots + a_1 a_n^4 .$$

289. Let  $n(r)$  be the number of points with integer coordinates on the circumference of a circle of radius  $r > 1$  in the cartesian plane. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2} .$$

290. The School of Architecture in the *Olymon* University proposed two projects for the new Housing Campus of the University. In each project, the campus is designed to have several identical dormitory buildings, with the same number of one-bedroom apartments in each building. In the first project, there are 12096 apartments in total. There are eight more buildings in the second project than in the first, and each building has more apartments, which raises the total of apartments in the project to 23625. How many buildings does the second project require?

291. The  $n$ -sided polygon  $A_1, A_2, \dots, A_n$  ( $n \geq 4$ ) has the following property: The diagonals from each of its vertices divide the respective angle of the polygon into  $n - 2$  equal angles. Find all natural numbers  $n$  for which this implies that the polygon  $A_1 A_2 \dots A_n$  is regular.

292. 1200 different points are randomly chosen on the circumference of a circle with centre  $O$ . Prove that it is possible to find two points on the circumference,  $M$  and  $N$ , so that:

- $M$  and  $N$  are different from the chosen 1200 points;
- $\angle MON = 30^\circ$ ;
- there are *exactly* 100 of the 1200 points inside the angle  $MON$ .

293. Two players, Amanda and Brenda, play the following game: Given a number  $n$ , Amanda writes  $n$  different natural numbers. Then, Brenda is allowed to erase several (including none, but not all) of them, and to write either  $+$  or  $-$  in front of each of the remaining numbers, making them positive or negative, respectively. Then they calculate their sum. Brenda wins the game if the sum is a multiple of 2004. Otherwise the winner is Amanda. Determine which one of them has a winning strategy, for the different choices of  $n$ . Indicate your reasoning and describe the strategy.

294. The number  $N = 10101 \dots 0101$  is written using  $n+1$  ones and  $n$  zeros. What is the least possible value of  $n$  for which the number  $N$  is a multiple of 9999?

295. In a triangle  $ABC$ , the angle bisectors  $AM$  and  $CK$  (with  $M$  and  $K$  on  $BC$  and  $AB$  respectively) intersect at the point  $O$ . It is known that

$$|AO| \div |OM| = \frac{\sqrt{6} + \sqrt{3} + 1}{2}$$

and

$$|CO| \div |OK| = \frac{\sqrt{2}}{\sqrt{3} - 1}.$$

Find the measures of the angles in triangle  $ABC$ .

296. Solve the equation

$$5 \sin x + \frac{5}{2 \sin x} - 5 = 2 \sin^2 x + \frac{1}{2 \sin^2 x}.$$

297. The point  $P$  lies on the side  $BC$  of triangle  $ABC$  so that  $PC = 2BP$ ,  $\angle ABC = 45^\circ$  and  $\angle APC = 60^\circ$ . Determine  $\angle ACB$ .

298. Let  $O$  be a point in the interior of a quadrilateral of area  $S$ , and suppose that

$$2S = |OA|^2 + |OB|^2 + |OC|^2 + |OD|^2.$$

Prove that  $ABCD$  is a square with centre  $O$ .

299. Let  $\sigma(r)$  denote the sum of all the divisors of  $r$ , including  $r$  and 1. Prove that there are infinitely many natural numbers  $n$  for which

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k}$$

whenever  $1 \leq k \leq n$ .

300. Suppose that  $ABC$  is a right triangle with  $\angle B < \angle C < \angle A = 90^\circ$ , and let  $\mathfrak{K}$  be its circumcircle. Suppose that the tangent to  $\mathfrak{K}$  at  $A$  meets  $BC$  produced at  $D$  and that  $E$  is the reflection of  $A$  in the axis  $BC$ . Let  $X$  be the foot of the perpendicular for  $A$  to  $BE$  and  $Y$  the midpoint of  $AX$ . Suppose that  $BY$  meets  $\mathfrak{K}$  again in  $Z$ . Prove that  $BD$  is tangent to the circumcircle of triangle  $ADZ$ .

301. Let  $d = 1, 2, 3$ . Suppose that  $M_d$  consists of the positive integers that *cannot* be expressed as the sum of two or more consecutive terms of an arithmetic progression consisting of positive integers with common difference  $d$ . Prove that, if  $c \in M_3$ , then there exist integers  $a \in M_1$  and  $b \in M_2$  for which  $c = ab$ .

302. In the following,  $ABCD$  is an arbitrary convex quadrilateral. The notation  $[\cdot \cdot \cdot]$  refers to the area.

(a) Prove that  $ABCD$  is a trapezoid if and only if

$$[ABC] \cdot [ACD] = [ABD] \cdot [BCD].$$

(b) Suppose that  $F$  is an interior point of the quadrilateral  $ABCD$  such that  $ABCF$  is a parallelogram. Prove that

$$[ABC] \cdot [ACD] + [AFD] \cdot [FCD] = [ABD] \cdot [BCD].$$

303. Solve the equation

$$\tan^2 2x = 2 \tan 2x \tan 3x + 1.$$

304. Prove that, for any complex numbers  $z$  and  $w$ ,

$$(|z| + |w|) \left| \frac{z}{|z|} + \frac{w}{|w|} \right| \leq 2|z + w|.$$

305. Suppose that  $u$  and  $v$  are positive integer divisors of the positive integer  $n$  and that  $uv < n$ . Is it necessarily so that the greatest common divisor of  $n/u$  and  $n/v$  exceeds 1?
306. The circumferences of three circles of radius  $r$  meet in a common point  $O$ . They meet also, pairwise, in the points  $P$ ,  $Q$  and  $R$ . Determine the maximum and minimum values of the circumradius of triangle  $PQR$ .
307. Let  $p$  be a prime and  $m$  a positive integer for which  $m < p$  and the greatest common divisor of  $m$  and  $p$  is equal to 1. Suppose that the decimal expansion of  $m/p$  has period  $2k$  for some positive integer  $k$ , so that

$$\frac{m}{p} = .ABABABAB \dots = (10^k A + B)(10^{-2k} + 10^{-4k} + \dots)$$

where  $A$  and  $B$  are two distinct blocks of  $k$  digits. Prove that

$$A + B = 10^k - 1 .$$

(For example,  $3/7 = 0.428571 \dots$  and  $428 + 571 = 999$ .)

308. Let  $a$  be a parameter. Define the sequence  $\{f_n(x) : n = 0, 1, 2, \dots\}$  of polynomials by

$$f_0(x) \equiv 1$$

$$f_{n+1}(x) = x f_n(x) + f_n(ax)$$

for  $n \geq 0$ .

- (a) Prove that, for all  $n, x$ ,

$$f_n(x) = x^n f_n(1/x) .$$

- (b) Determine a formula for the coefficient of  $x^k$  ( $0 \leq k \leq n$ ) in  $f_n(x)$ .

309. Let  $ABCD$  be a convex quadrilateral for which all sides and diagonals have rational length and  $AC$  and  $BD$  intersect at  $P$ . Prove that  $AP$ ,  $BP$ ,  $CP$ ,  $DP$  all have rational length.

310. (a) Suppose that  $n$  is a positive integer. Prove that

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x(x+k)^{k-1} (y-k)^{n-k} .$$

- (b) Prove that

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x(x-kz)^{k-1} (y+kz)^{n-k} .$$

311. Given a square with a side length 1, let  $P$  be a point in the plane such that the sum of the distances from  $P$  to the sides of the square (or their extensions) is equal to 4. Determine the set of all such points  $P$ .
312. Given ten arbitrary natural numbers. Consider the sum, the product, and the absolute value of the difference calculated for any two of these numbers. At most how many of all these calculated numbers are odd?
313. The three medians of the triangle  $ABC$  partition it into six triangles. Given that three of these triangles have equal perimeters, prove that the triangle  $ABC$  is equilateral.
314. For the real numbers  $a$ ,  $b$  and  $c$ , it is known that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = 1 ,$$

and

$$a + b + c = 1 .$$

Find the value of the expression

$$M = \frac{1}{1 + a + ab} + \frac{1}{1 + b + bc} + \frac{1}{1 + c + ca} .$$

315. The natural numbers 3945, 4686 and 5598 have the same remainder when divided by a natural number  $x$ . What is the sum of the number  $x$  and this remainder?

316. Solve the equation

$$|x^2 - 3x + 2| + |x^2 + 2x - 3| = 11 .$$

317. Let  $P(x)$  be the polynomial

$$P(x) = x^{15} - 2004x^{14} + 2204x^{13} - \dots - 2004x^2 + 2004x ,$$

Calculate  $P(2003)$ .

318. Solve for integers  $x, y, z$  the system

$$1 = x + y + z = x^3 + y^3 + z^2 .$$

[Note that the exponent of  $z$  on the right is 2, not 3.]

319. Suppose that  $a, b, c, x$  are real numbers for which  $abc \neq 0$  and

$$\frac{xb + (1 - x)c}{a} = \frac{xc + (1 - x)a}{b} = \frac{xa + (1 - x)b}{c} .$$

Prove that  $a = b = c$ .

320. Let  $L$  and  $M$  be the respective intersections of the internal and external angle bisectors of the triangle  $ABC$  at  $C$  and the side  $AB$  produced. Suppose that  $CL = CM$  and that  $R$  is the circumradius of triangle  $ABC$ . Prove that

$$|AC|^2 + |BC|^2 = 4R^2 .$$

321. Determine all positive integers  $k$  for which  $k^{1/(k-7)}$  is an integer.

322. The real numbers  $u$  and  $v$  satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 .$$

Determine  $u + v$ .

323. Alfred, Bertha and Cedric are going from their home to the country fair, a distance of 62 km. They have a motorcycle with sidecar that together accommodates at most 2 people and that can travel at a maximum speed of 50 km/hr. Each can walk at a maximum speed of 5 km/hr. Is it possible for all three to cover the 62 km distance within 3 hours?

324. The base of a pyramid  $ABCDV$  is a rectangle  $ABCD$  with  $|AB| = a$ ,  $|BC| = b$  and  $|VA| = |VB| = |VC| = |VD| = c$ . Determine the area of the intersection of the pyramid and the plane parallel to the edge  $VA$  that contains the diagonal  $BD$ .

325. Solve for positive real values of  $x, y, t$ :

$$(x^2 + y^2)^2 + 2tx(x^2 + y^2) = t^2y^2 .$$

Are there infinitely many solutions for which the values of  $x, y, t$  are all positive integers?

*Optional rider:* What is the smallest value of  $t$  for a positive integer solution?

326. In the triangle  $ABC$  with semiperimeter  $s = \frac{1}{2}(a + b + c)$ , points  $U, V, W$  lie on the respective sides  $BC, CA, AB$ . Prove that

$$s < |AU| + |BV| + |CW| < 3s .$$

Give an example for which the sum in the middle is equal to  $2s$ .

327. Let  $A$  be a point on a circle with centre  $O$  and let  $B$  be the midpoint of  $OA$ . Let  $C$  and  $D$  be points on the circle on the same side of  $OA$  produced for which  $\angle CBO = \angle DBA$ . Let  $E$  be the midpoint of  $CD$  and let  $F$  be the point on  $EB$  produced for which  $BF = BE$ .

(a) Prove that  $F$  lies on the circle.

(b) What is the range of angle  $EAO$ ?

328. Let  $\mathfrak{C}$  be a circle with diameter  $AC$  and centre  $D$ . Suppose that  $B$  is a point on the circle for which  $BD \perp AC$ . Let  $E$  be the midpoint of  $DC$  and let  $Z$  be a point on the radius  $AD$  for which  $EZ = EB$ .

Prove that

(a) The length  $c$  of  $BZ$  is the length of the side of a regular pentagon inscribed in  $\mathfrak{C}$ .

(b) The length  $b$  of  $DZ$  is the length of the side of a regular decagon (10-gon) inscribed in  $\mathfrak{C}$ .

(c)  $c^2 = a^2 + b^2$  where  $a$  is the length of a regular hexagon inscribed in  $\mathfrak{C}$ .

(d)  $(a + b) : a = a : b$ .

329. Let  $x, y, z$  be positive real numbers. Prove that

$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geq \sqrt{x^2 + xz + z^2} .$$

330. At an international conference, there are four official languages. Any two participants can communicate in at least one of these languages. Show that at least one of the languages is spoken by at least 60% of the participants.

331. Some checkers are placed on various squares of a  $2m \times 2n$  chessboard, where  $m$  and  $n$  are odd. Any number (including zero) of checkers are placed on each square. There are an odd number of checkers in each row and in each column. Suppose that the chessboard squares are coloured alternately black and white (as usual). Prove that there are an even number of checkers on the black squares.

332. What is the minimum number of points that can be found (a) in the plane, (b) in space, such that each point in, respectively, (a) the plane, (b) space, must be at an irrational distance from at least one of them?

333. Suppose that  $a, b, c$  are the sides of triangle  $ABC$  and that  $a^2, b^2, c^2$  are in arithmetic progression.

(a) Prove that  $\cot A, \cot B, \cot C$  are also in arithmetic progression.

(b) Find an example of such a triangle where  $a, b, c$  are integers.

334. The vertices of a tetrahedron lie on the surface of a sphere of radius 2. The length of five of the edges of the tetrahedron is 3. Determine the length of the sixth edge.

335. Does the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{12}{a+b+c}$$

have infinitely many solutions in positive integers  $a, b, c$ ?

336. Let  $ABCD$  be a parallelogram with centre  $O$ . Points  $M$  and  $N$  are the respective midpoints of  $BO$  and  $CD$ . Prove that the triangles  $ABC$  and  $AMN$  are similar if and only if  $ABCD$  is a square.

337. Let  $a, b, c$  be three real numbers for which  $0 \leq c \leq b \leq a \leq 1$  and let  $w$  be a complex root of the polynomial  $z^3 + az^2 + bz + c$ . Must  $|w| \leq 1$ ?

338. A *triangular triple*  $(a, b, c)$  is a set of three positive integers for which  $T(a) + T(b) = T(c)$ . Determine the smallest triangular number of the form  $a + b + c$  where  $(a, b, c)$  is a triangular triple. (*Optional investigations*: Are there infinitely many such triangular numbers  $a + b + c$ ? Is it possible for the three numbers of a triangular triple to each be triangular?)

339. Let  $a, b, c$  be integers with  $abc \neq 0$ , and  $u, v, w$  be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0.$$

Let  $r$  be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

341. Let  $s, r, R$  respectively specify the semiperimeter, inradius and circumradius of a triangle  $ABC$ .

(a) Determine a necessary and sufficient condition on  $s, r, R$  that the sides  $a, b, c$  of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on  $s, r, R$  that the sides  $a, b, c$  of the triangle are in geometric progression.

342. Prove that there are infinitely many solutions in positive integers of the system

$$\begin{aligned} a + b + c &= x + y \\ a^3 + b^3 + c^3 &= x^3 + y^3. \end{aligned}$$

343. A sequence  $\{a_n\}$  of integers is defined by

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 2a_{n-1} + a_{n-2}$$

for  $n > 1$ . Prove that, for each nonnegative integer  $k$ ,  $2^k$  divides  $a_n$  if and only if  $2^k$  divides  $n$ .

344. A function  $f$  defined on the positive integers is given by

$$f(1) = 1, \quad f(3) = 3, \quad f(2n) = f(n),$$

$$f(4n + 1) = 2f(2n + 1) - f(n)$$

$$f(4n + 3) = 3f(2n + 1) - 2f(n),$$

for each positive integer  $n$ . Determine, with proof, the number of positive integers no exceeding 2004 for which  $f(n) = n$ .

345. Let  $\mathfrak{C}$  be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of  $\mathfrak{C}$ , keeping the new faces perpendicular to the diagonals of the cube and keeping the newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.

346. Let  $n$  be a positive integer. Determine the set of all integers that can be written in the form

$$\sum_{k=1}^n \frac{k}{a_k}$$

where  $a_1, a_2, \dots, a_n$  are all positive integers.

347. Let  $n$  be a positive integer and  $\{a_1, a_2, \dots, a_n\}$  a finite sequence of real numbers which contains at least one positive term. Let  $S$  be the set of indices  $k$  for which at least one of the numbers

$$a_k, a_k + a_{k+1}, a_k + a_{k+1} + a_{k+2}, \dots, a_k + a_{k+1} + \dots + a_n$$

is positive. Prove that

$$\sum \{a_k : k \in S\} > 0.$$

348. (a) Suppose that  $f(x)$  is a real-valued function defined for real values of  $x$ . Suppose that  $f(x) - x^3$  is an increasing function. Must  $f(x) - x - x^2$  also be increasing?

(b) Suppose that  $f(x)$  is a real-valued function defined for real values of  $x$ . Suppose that both  $f(x) - 3x$  and  $f(x) - x^3$  are increasing functions. Must  $f(x) - x - x^2$  also be increasing on all of the real numbers, or on at least the positive reals?

349. Let  $s$  be the semiperimeter of triangle  $ABC$ . Suppose that  $L$  and  $N$  are points on  $AB$  and  $CB$  produced (*i.e.*,  $B$  lies on segments  $AL$  and  $CN$ ) with  $|AL| = |CN| = s$ . Let  $K$  be the point symmetric to  $B$  with respect to the centre of the circumcircle of triangle  $ABC$ . Prove that the perpendicular from  $K$  to the line  $NL$  passes through the incentre of triangle  $ABC$ .

350. Let  $ABCDE$  be a pentagon inscribed in a circle with centre  $O$ . Suppose that its angles are given by  $\angle B = \angle C = 120^\circ$ ,  $\angle D = 130^\circ$ ,  $\angle E = 100^\circ$ . Prove that  $BD$ ,  $CE$  and  $AO$  are concurrent.

351. Let  $\{a_n\}$  be a sequence of real numbers for which  $a_1 = 1/2$  and, for  $n \geq 1$ ,

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$

Prove that, for all  $n$ ,  $a_1 + a_2 + \dots + a_n < 1$ .

352. Let  $ABCD$  be a unit square with points  $M$  and  $N$  in its interior. Suppose, further, that  $MN$  produced does not pass through any vertex of the square. Find the smallest value of  $k$  for which, given any position of  $M$  and  $N$ , at least one of the twenty triangles with vertices chosen from the set  $\{A, B, C, D, M, N\}$  has area not exceeding  $k$ .

353. The two shortest sides of a right-angled triangle,  $a$  and  $b$ , satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3.$$

Find the perimeter of this triangle.

354. Let  $ABC$  be an isosceles triangle with  $AC = BC$  for which  $|AB| = 4\sqrt{2}$  and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.
355. (a) Find all natural numbers  $k$  for which  $3^k - 1$  is a multiple of 13.  
 (b) Prove that for any natural number  $k$ ,  $3^k + 1$  is not a multiple of 13.
356. Let  $a$  and  $b$  be real parameters. One of the roots of the equation  $x^{12} - abx + a^2 = 0$  is greater than 2. Prove that  $|b| > 64$ .
357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.
358. Find all integers  $x$  which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1.$$

359. Let  $ABC$  be an acute triangle with angle bisectors  $AA_1$  and  $BB_1$ , with  $A_1$  and  $B_1$  on  $BC$  and  $AC$ , respectively. Let  $J$  be the intersection of  $AA_1$  and  $BB_1$  (the incentre),  $H$  be the orthocentre and  $O$  the circumcentre of the triangle  $ABC$ . The line  $OH$  intersects  $AC$  at  $P$  and  $BC$  at  $Q$ . Given that  $C$ ,  $A_1$ ,  $J$  and  $B_1$  are vertices of a concyclic quadrilateral, prove that  $PQ = AP + BQ$ .
360. Eliminate  $\theta$  from the two equations

$$\begin{aligned}x &= \cot \theta + \tan \theta \\y &= \sec \theta - \cos \theta,\end{aligned}$$

to get a polynomial equation satisfied by  $x$  and  $y$ .

361. Let  $ABCD$  be a square,  $M$  a point on the side  $BC$ , and  $N$  a point on the side  $CD$  for which  $BM = CN$ . Suppose that  $AM$  and  $AN$  intersect  $BD$  at  $P$  and  $Q$  respectively. Prove that a triangle can be constructed with sides of length  $|BP|$ ,  $|PQ|$ ,  $|QD|$ , one of whose angles is equal to  $60^\circ$ .
362. The triangle  $ABC$  is inscribed in a circle. The interior bisectors of the angles  $A$ ,  $B$ ,  $C$  meet the circle again at  $U$ ,  $V$ ,  $W$ , respectively. Prove that the area of triangle  $UVW$  is not less than the area of triangle  $ABC$ .
363. Suppose that  $x$  and  $y$  are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2}.$$

364. Determine necessary and sufficient conditions on the positive integers  $a$  and  $b$  such that the vulgar fraction  $a/b$  has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than  $a/b$  and that at a later time, the fraction of heads is greater than  $a/b$ ; then at some intermediate time, the fraction of heads must be exactly  $a/b$ .*
365. Let  $p(z)$  be a polynomial of degree greater than 4 with complex coefficients. Prove that  $p(z)$  must have a pair  $u, v$  of roots, not necessarily distinct, for which the real parts of both  $u/v$  and  $v/u$  are positive. Show that this does not necessarily hold for polynomials of degree 4.
366. What is the largest real number  $r$  for which

$$\frac{x^2 + y^2 + z^2 + xy + yz + zx}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \geq r$$

holds for all positive real values of  $x, y, z$  for which  $xyz = 1$ .

367. Let  $a$  and  $c$  be fixed real numbers satisfying  $a \leq 1 \leq c$ . Determine the largest value of  $b$  that is consistent with the condition

$$a + bc \leq b + ac \leq c + ab .$$

368. Let  $A, B, C$  be three distinct points of the plane for which  $AB = AC$ . Describe the locus of the point  $P$  for which  $\angle APB = \angle APC$ .

369.  $ABCD$  is a rectangle and  $APQ$  is an inscribed equilateral triangle for which  $P$  lies on  $BC$  and  $Q$  lies on  $CD$ .

(a) For which rectangles is the configuration possible?

(b) Prove that, when the configuration is possible, then the area of triangle  $CPQ$  is equal to the sum of the areas of the triangles  $ABP$  and  $ADQ$ .

370. A deck of cards has  $nk$  cards,  $n$  cards of each of the colours  $C_1, C_2, \dots, C_k$ . The deck is thoroughly shuffled and dealt into  $k$  piles of  $n$  cards each,  $P_1, P_2, \dots, P_k$ . A game of solitaire proceeds as follows: The top card is drawn from pile  $P_1$ . If it has colour  $C_i$ , it is discarded and the top card is drawn from pile  $P_i$ . If it has colour  $C_j$ , it is discarded and the top card is drawn from pile  $P_j$ . The game continues in this way, and will terminate when the  $n$ th card of colour  $C_1$  is drawn and discarded, as at this point, there are no further cards left in pile  $P_1$ . What is the probability that every card is discarded when the game terminates?

371. Let  $X$  be a point on the side  $BC$  of triangle  $ABC$  and  $Y$  the point where the line  $AX$  meets the circumcircle of triangle  $ABC$ . Prove or disprove: if the length of  $XY$  is maximum, then  $AX$  lies between the median from  $A$  and the bisector of angle  $BAC$ .

372. Let  $b_n$  be the number of integers whose digits are all 1, 3, 4 and whose digits sum to  $n$ . Prove that  $b_n$  is a perfect square when  $n$  is even.

373. For each positive integer  $n$ , define

$$a_n = 1 + 2^2 + 3^3 + \dots + n^n .$$

Prove that there are infinitely many values of  $n$  for which  $a_n$  is an odd composite number.

374. What is the maximum number of numbers that can be selected from  $\{1, 2, 3, \dots, 2005\}$  such that the difference between any pair of them is not equal to 5?

375. Prove or disprove: there is a set of concentric circles in the plane for which both of the following hold:

(i) each point with integer coordinates lies on one of the circles;

(ii) no two points with integer coefficients lie on the same circle.

376. A soldier has to find whether there are mines buried within or on the boundary of a region in the shape of an equilateral triangle. The effective range of his detector is one half of the height of the triangle. If he starts at a vertex, explain how he can select the shortest path for checking that the region is clear of mines.

377. Each side of an equilateral triangle is divided into 7 equal parts. Lines through the division points parallel to the sides divide the triangle into 49 smaller equilateral triangles whose vertices consist of a set of 36 points. These 36 points are assigned numbers satisfying both the following conditions:

(a) the number at the vertices of the original triangle are 9, 36 and 121;

(b) for each rhombus composed of two small adjacent triangles, the sum of the numbers placed on one pair of opposite vertices is equal to the sum of the numbers placed on the other pair of opposite vertices.

Determine the sum of all the numbers. Is such a choice of numbers in fact possible?

378. Let  $f(x)$  be a nonconstant polynomial that takes only integer values when  $x$  is an integer, and let  $P$  be the set of all primes that divide  $f(m)$  for at least one integer  $m$ . Prove that  $P$  is an infinite set.
379. Let  $n$  be a positive integer exceeding 1. Prove that, if a graph with  $2n + 1$  vertices has at least  $3n + 1$  edges, then the graph contains a circuit (*i.e.*, a closed non-self-intersecting chain of edges whose terminal point is its initial point) with an even number of edges. Prove that this statement does not hold if the number of edges is only  $3n$ .
380. Factor each of the following polynomials as a product of polynomials of lower degree with integer coefficients:
- $(x + y + z)^4 - (y + z)^4 - (z + x)^4 - (x + y)^4 + x^4 + y^4 + z^4$  ;
  - $x^2(y^3 - z^3) + y^2(z^3 - x^3) + z^2(x^3 - y^3)$  ;
  - $x^4 + y^4 - z^4 - 2x^2y^2 + 4xyz^2$  ;
  - $(yz + zx + xy)^3 - y^3z^3 - z^3x^3 - x^3y^3$  ;
  - $x^3y^3 + y^3z^3 + z^3x^3 - x^4yz - xy^4z - xyz^4$  ;
  - $2(x^4 + y^4 + z^4 + w^4) - (x^2 + y^2 + z^2 + w^2)^2 + 8xyzw$  ;
  - $6(x^5 + y^5 + z^5) - 5(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)$  .
381. Determine all polynomials  $f(x)$  such that, for some positive integer  $k$ ,

$$f(x^k) - x^3f(x) = 2(x^3 - 1)$$

for all values of  $x$ .

382. Given an odd number of intervals, each of unit length, on the real line, let  $S$  be the set of numbers that are in an odd number of these intervals. Show that  $S$  is a finite union of disjoint intervals of total length not less than 1.
383. Place the numbers  $1, 2, \dots, 9$  in a  $3 \times 3$  unit square so that
- the sums of numbers in each of the first two rows are equal;
  - the sum of the numbers in the third row is as large as possible;
  - the column sums are equal;
  - the numbers in the last row are in descending order.
- Prove that the solution is unique.

384. Prove that, for each positive integer  $n$ ,

$$(3 - 2\sqrt{2})(17 + 12\sqrt{2})^n + (3 + 2\sqrt{2})(17 - 12\sqrt{2})^n - 2$$

is the square of an integer.

385. Determine the minimum value of the product  $(a + 1)(b + 1)(c + 1)(d + 1)$ , given that  $a, b, c, d \geq 0$  and

$$\frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1} + \frac{1}{d + 1} = 1 .$$

386. In a round-robin tournament with at least three players, each player plays one game against each other player. The tournament is said to be *competitive* if it is impossible to partition the players into two sets, such that each player in one set beat each player in the second set. Prove that, if a tournament is not competitive, it can be made so by reversing the result of a single game.

387. Suppose that  $a, b, u, v$  are real numbers for which  $av - bu = 1$ . Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3}.$$

Give an example to show that equality is possible. (Part marks will be awarded for a result that is proven with a smaller bound on the right side.)

388. A class with at least 35 students goes on a cruise. Seven small boats are hired, each capable of carrying 300 kilograms. The combined weight of the class is 1800 kilograms. It is determined that any group of 35 students can fit into the boats without exceeding the capacity of any one of them. Prove that it is unnecessary to leave any student off the cruise.

389. Let each of  $m$  distinct points on the positive part of the  $x$ -axis be joined by line segments to  $n$  distinct points on the positive part of the  $y$ -axis. Obtain a formula for the number of intersections of these segments (exclusive of endpoints), assuming that no three of the segments are concurrent.

390. Suppose that  $n \geq 2$  and that  $x_1, x_2, \dots, x_n$  are positive integers for which  $x_1 + x_2 + \dots + x_n = 2(n+1)$ . Show that there exists an index  $r$  with  $0 \leq r \leq n-1$  for which the following  $n-1$  inequalities hold:

$$x_{r+1} \leq 3$$

$$x_{r+1} + x_{r+2} \leq 5$$

...

$$x_{r+1} + x_{r+2} + \dots + x_{r+i} \leq 2i + 1$$

...

$$x_{r+1} + x_{r+2} + \dots + x_n \leq 2(n-r) + 1$$

...

$$x_{r+1} + \dots + x_n + x_1 + \dots + x_j \leq 2(n+j-r) + 1$$

...

$$x_{r+1} + x_{r+2} + \dots + x_n + x_1 + \dots + x_{r-1} \leq 2n - 1$$

where  $1 \leq i \leq n-r$  and  $1 \leq j \leq r-1$ . Prove that, if all the inequalities are strict, then  $r$  is unique, and that, otherwise, there are exactly two such  $r$ .

391. Show that there are infinitely many nonsimilar ways that a square with integer side lengths can be partitioned into three nonoverlapping polygons with integer side lengths which are similar, but no two of which are congruent.

392. Determine necessary and sufficient conditions on the real parameter  $a, b, c$  that

$$\frac{b}{cx+a} + \frac{c}{ax+b} + \frac{a}{bx+c} = 0$$

has exactly one real solution.

393. Determine three positive rational numbers  $x, y, z$  whose sum  $s$  is rational and for which  $x - s^3, y - s^3, z - s^3$  are all cubes of rational numbers.

394. The average age of the students in Ms. Ruler's class is 17.3 years, while the average age of the boys is 17.5 years. Give a cogent argument to prove that the average age of the girls cannot also exceed 17.3 years.

395. None of the nine participants at a meeting speaks more than three languages. Two of any three speakers speak a common language. Show that there is a language spoken by at least three participants.
396. Place 32 white and 32 black checkers on a  $8 \times 8$  square chessboard. Two checkers of different colours form a *related pair* if they are placed in either the same row or the same column. Determine the maximum and the minimum number of related pairs over all possible arrangements of the 64 checkers.
397. The altitude from  $A$  of triangle  $ABC$  intersects  $BC$  in  $D$ . A circle touches  $BC$  at  $D$ , intersects  $AB$  at  $M$  and  $N$ , and intersects  $AC$  at  $P$  and  $Q$ . Prove that

$$(AM + AN) : AC = (AP + AQ) : AB .$$

398. Given three disjoint circles in the plane, construct a point in the plane so that all three circles subtend the same angle at that point.
399. Let  $n$  and  $k$  be positive integers for which  $k < n$ . Determine the number of ways of choosing  $k$  numbers from  $\{1, 2, \dots, n\}$  so that no three consecutive numbers appear in any choice.
400. Let  $a_r$  and  $b_r$  ( $1 \leq r \leq n$ ) be real numbers for which  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$  and

$$b_1 \geq a_1 , \quad b_1 b_2 \geq a_1 a_2 , \quad b_1 b_2 b_3 \geq a_1 a_2 a_3 , \quad \dots , \quad b_1 b_2 \dots b_n \geq a_1 a_2 \dots a_n .$$

Show that

$$b_1 + b_2 + \dots + b_n \geq a_1 + a_2 + \dots + a_n .$$

401. Five integers are arranged in a circle. The sum of the five integers is positive, but at least one of them is negative. The configuration is changed by the following moves: at any stage, a negative integer is selected and its sign is changed; this negative integer is added to each of its neighbours (*i.e.*, its absolute value is subtracted from each of its neighbours).

Prove that, regardless of the negative number selected for each move, the process will eventually terminate with all integers nonnegative in exactly the same number of moves with exactly the same configuration.

402. Let the sequences  $\{x_n\}$  and  $\{y_n\}$  be defined, for  $n \geq 1$ , by  $x_1 = x_2 = 10$ ,  $x_{n+2} = x_{n+1}(x_n + 1) + 1$  ( $n \geq 1$ ) and  $y_1 = y_2 = -10$ ,  $y_{n+2} = y_{n+1}(y_n + 1) + 1$  ( $n \geq 1$ ). Prove that there is no number that is a term of both sequences.

403. Let  $f(x) = |1 - 2x| - 3|x + 1|$  for real values of  $x$ .

(a) Determine all values of the real parameter  $a$  for which the equation  $f(x) = a$  has two different roots  $u$  and  $v$  that satisfy  $2 \leq |u - v| \leq 10$ .

(b) Solve the equation  $f(x) = \lfloor x/2 \rfloor$ .

404. Several points in the plane are said to be *in general position* if no three are collinear.

(a) Prove that, given 5 points in general position, there are always four of them that are vertices of a convex quadrilateral.

(b) Prove that, given 400 points in general position, there are at least 80 nonintersecting convex quadrilaterals, whose vertices are chosen from the given points. (Two quadrilaterals are nonintersecting if they do not have a common point, either in the interior or on the perimeter.)

(c) Prove that, given 20 points in general position, there are at least 969 convex quadrilaterals whose vertices are chosen from these points. (**Bonus:** Derive a formula for the number of these quadrilaterals given  $n$  points in general position.)

405. Suppose that a permutation of the numbers from 1 to 100, inclusive, is given. Consider the sums of all triples of consecutive numbers in the permutation. At most how many of these sums can be odd?

406. Let  $a, b, c$  be natural numbers such that the expression

$$\frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a}$$

is also equal to a natural number. Prove that the greatest common divisor of  $a, b$  and  $c$ ,  $\gcd(a, b, c)$ , does not exceed  $\sqrt[3]{ab+bc+ca}$ , i.e.,

$$\gcd(a, b, c) \leq \sqrt[3]{ab+bc+ca}.$$

407. Is there a pair of natural numbers,  $x$  and  $y$ , for which

(a)  $x^3 + y^4 = 2^{2003}$ ?

(b)  $x^3 + y^4 = 2^{2005}$ ?

Provide reasoning for your answers to (a) and (b).

408. Prove that a number of the form  $a000\dots0009$  (with  $n+2$  digits for which the first digit  $a$  is followed by  $n$  zeros and the units digit is 9) cannot be the square of another integer.

409. Find the number of ways of dealing  $n$  cards to two persons ( $n \geq 2$ ), where the persons may receive unequal (positive) numbers of cards. Disregard the order in which the cards are received.

410. Prove that  $\log n \geq k \log 2$ , where  $n$  is a natural number and  $k$  the number of distinct primes that divide  $n$ .

411. Let  $b$  be a positive integer. How many integers are there, each of which, when expressed to base  $b$ , is equal to the sum of the squares of its digits?

412. Let  $A$  and  $B$  be the midpoints of the sides,  $EF$  and  $ED$ , of an equilateral triangle  $DEF$ . Extend  $AB$  to meet the circumcircle of triangle  $DEF$  at  $C$ . Show that  $B$  divides  $AC$  according to the golden section. (That is, show that  $BC : AB = AB : AC$ .)

413. Let  $I$  be the incentre of triangle  $ABC$ . Let  $A', B'$  and  $C'$  denote the intersections of  $AI, BI$  and  $CI$ , respectively, with the incircle of triangle  $ABC$ . Continue the process by defining  $I'$  (the incentre of triangle  $A'B'C'$ ), then  $A''B''C''$ , etc.. Prove that the angles of triangle  $A^{(n)}B^{(n)}C^{(n)}$  get closer and closer to  $\pi/3$  as  $n$  increases.

414. Let  $f(n)$  be the greatest common divisor of the set of numbers of the form  $k^n - k$ , where  $2 \leq k$ , for  $n \geq 2$ . Evaluate  $f(n)$ . In particular, show that  $f(2n) = 2$  for each integer  $n$ .

415. Prove that

$$\cos \frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6} \left( \cos \left( \frac{1}{3} \arccos \frac{1}{2\sqrt{7}} \right) + \sqrt{3} \sin \left( \frac{1}{3} \arccos \frac{1}{2\sqrt{7}} \right) \right).$$

416. Let  $P$  be a point in the plane.

(a) Prove that there are three points  $A, B, C$  for which  $AB = BC$ ,  $\angle ABC = 90^\circ$ ,  $|PA| = 1$ ,  $|PB| = 2$  and  $|PC| = 3$ .

(b) Determine  $|AB|$  for the configuration in (a).

(c) A rotation of  $90^\circ$  about  $B$  takes  $C$  to  $A$  and  $P$  to  $Q$ . Determine  $\angle APQ$ .

417. Show that for each positive integer  $n$ , at least one of the five numbers  $17^n, 17^{n+1}, 17^{n+2}, 17^{n+3}, 17^{n+4}$  begins with 1 (at the left) when written to base 10.

418. (a) Show that, for each pair  $m, n$  of positive integers, the minimum of  $m^{1/n}$  and  $n^{1/m}$  does not exceed  $3^{1/2}$ .

(b) Show that, for each positive integer  $n$ ,

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 \geq n^{1/n} \geq 1.$$

(c) Determine an integer  $N$  for which

$$n^{1/n} \leq 1.00002005$$

whenever  $n \geq N$ . Justify your answer.

419. Solve the system of equations

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$$

for  $x, y, z$  not all equal. Determine  $xyz$ .

420. Two circles intersect at  $A$  and  $B$ . Let  $P$  be a point on one of the circles. Suppose that  $PA$  meets the second circle again at  $C$  and  $PB$  meets the second circle again at  $D$ . For what position of  $P$  is the length of the segment  $CD$  maximum?

421. Let  $ABCD$  be a tetrahedron. Prove that

$$|AB| \cdot |CD| + |AC| \cdot |BD| \geq |AD| \cdot |BC|.$$

422. Determine the smallest two positive integers  $n$  for which the numbers in the set  $\{1, 2, \dots, 3n-1, 3n\}$  can be partitioned into  $n$  disjoint triples  $\{x, y, z\}$  for which  $x + y = 3z$ .

423. Prove or disprove: if  $x$  and  $y$  are real numbers with  $y \geq 0$  and  $y(y+1) \leq (x+1)^2$ , then  $y(y-1) \leq x^2$ .

424. Simplify

$$\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2}$$

to a fraction whose numerator and denominator are of the form  $u\sqrt{v}$  with  $u$  and  $v$  each linear polynomials. For which values of  $x$  is the equation valid?

425. Let  $\{x_1, x_2, \dots, x_n, \dots\}$  be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer  $n \geq 2$ ,

$$\frac{1}{x_1 x_2} + \frac{1}{x_2 x_3} + \dots + \frac{1}{x_{n-1} x_n} = \frac{n-1}{x_1 x_n}.$$

426. (a) The following paper-folding method is proposed for trisecting an acute angle.

(1) transfer the angle to a rectangular sheet so that its vertex is at one corner  $P$  of the sheet with one ray along the edge  $PY$ ; let the angle be  $XPY$ ;

(2) fold up  $PY$  over  $QZ$  to fall on  $RW$ , so that  $PQ = QR$  and  $PY \parallel QZ \parallel RW$ , with  $QZ$  between  $PY$  and  $RW$ ;

(3) fold across a line  $AC$  with  $A$  on the sheet and  $C$  on the edge  $PY$  so that  $P$  falls on a point  $P'$  on  $QZ$  and  $R$  on a point  $R'$  on  $PX$ ;

(4) suppose that the fold  $AC$  intersects the fold  $QZ$  at  $B$  and carries  $Q$  to  $Q'$ ; make a fold along  $BQ'$ .

It is claimed that the fold  $BQ'$  passes through  $P$  and trisects angle  $XPY$ .

Explain why the fold described in (3) is possible. Does the method work? Why?

(b) What happens with a right angle?

(c) Can the method be adapted for an obtuse angle?

427. The radius of the inscribed circle and the radii of the three escribed circles of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

428.  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are three lines in space. Neither  $\mathbf{a}$  nor  $\mathbf{b}$  is perpendicular to  $\mathbf{c}$ . Points  $P$  and  $Q$  vary on  $\mathbf{a}$  and  $\mathbf{b}$ , respectively, so that  $PQ$  is perpendicular to  $\mathbf{c}$ . The plane through  $P$  perpendicular to  $\mathbf{b}$  meets  $\mathbf{c}$  at  $R$ , and the plane through  $Q$  perpendicular to  $\mathbf{a}$  meets  $\mathbf{c}$  at  $S$ . Prove that  $RS$  is of constant length.

429. Prove that

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \binom{kn}{n} = (-1)^{n+1} n^n .$$

430. Let triangle  $ABC$  be such that its excircle tangent to the segment  $AB$  is also tangent to the circle whose diameter is the segment  $BC$ . If the lengths of the sides  $BC$ ,  $CA$  and  $AB$  of the triangle form, in this order, an arithmetic sequence, find the measure of the angle  $ACB$ .

431. Prove the following trigonometric identity, for any natural number  $n$ :

$$\sin \frac{\pi}{4n+2} \cdot \sin \frac{3\pi}{4n+2} \cdot \sin \frac{5\pi}{4n+2} \cdots \sin \frac{(2n-1)\pi}{4n+2} = \frac{1}{2^n} .$$

432. Find the exact value of:

(a)

$$\sqrt{\frac{1}{6} + \frac{\sqrt{5}}{18}} - \sqrt{\frac{1}{6} - \frac{\sqrt{5}}{18}} ;$$

(b)

$$\sqrt{1 + \frac{2}{5}} \cdot \sqrt{1 + \frac{2}{6}} \cdot \sqrt{1 + \frac{2}{7}} \cdot \sqrt{1 + \frac{2}{8}} \cdots \sqrt{1 + \frac{2}{57}} \cdot \sqrt{1 + \frac{2}{58}} .$$

433. Prove that the equation

$$x^2 + 2y^2 + 98z^2 = 77777 \dots 777$$

does not have a solution in integers, where the right side has 2006 digits, all equal to 7.

434. Find all natural numbers  $n$  for which  $2^n + n^{2004}$  is equal to a prime number.

435. A circle with centre  $I$  is the incircle of the convex quadrilateral  $ABCD$ . The diagonals  $AC$  and  $BD$  intersect at the point  $E$ . Prove that, if the midpoints of the segments  $AD$ ,  $BC$  and  $IE$  are collinear, then  $AB = CD$ .

436. In the Euro-African volleyball tournament, there were nine more teams participating from Europe than from Africa. In total, the European won nine times as many points as were won by all of the African teams. In this tournament, each team played exactly once against each other team; there were no ties; the winner of a game gets 1 point, the loser 0. What is the greatest possible score of the best African team?

437. Let  $a$ ,  $b$ ,  $c$  be the side lengths and  $m_a$ ,  $m_b$ ,  $m_c$  the lengths of their respective medians, of an arbitrary triangle  $ABC$ . Show that

$$\frac{3}{4} < \frac{m_a + m_b + m_c}{a + b + c} < 1 .$$

Furthermore, show that one cannot find a smaller interval to bound the ratio.

438. Determine all sets  $(x, y, z)$  of real numbers for which

$$x + y = 2 \quad \text{and} \quad xy - z^2 = 1 .$$

439. A natural number  $n$ , less than or equal to 500, has the property that when one chooses a number  $m$  randomly among  $\{1, 2, 3, \dots, 500\}$ , the probability that  $m$  divides  $n$  (*i.e.*,  $n/m$  is an integer) is  $1/100$ . Find the largest such  $n$ .
440. You are to choose 10 distinct numbers from  $\{1, 2, 3, \dots, 2006\}$ . Show that you can choose such numbers with a sum greater than 10039 in more ways than you can choose such numbers with a sum less than 10030.
441. Prove that, no matter how 15 points are placed inside a circle of radius 2 (including the boundary), there exists a circle of radius 1 (including the boundary) containing at least 3 of the 15 points.
442. Prove that the regular tetrahedron has minimum diameter among all tetrahedra that circumscribe a given sphere. (The diameter of a tetrahedron is the length of its longest edge.)
443. For  $n \geq 3$ , show that  $n - 1$  straight lines are sufficient to go through the interior of every square of an  $n \times n$  chessboard. Are  $n - 1$  lines necessary?
444. (a) Suppose that a  $6 \times 6$  square grid of unit squares (chessboard) is tiled by  $1 \times 2$  rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.  
(b) Is the same thing true for an  $8 \times 8$  array?  
(c) Is the same thing true for a  $6 \times 8$  array?
445. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.
446. Suppose that you have a  $3 \times 3$  grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

Two players  $A$  and  $B$  play a game. They take alternate turns,  $A$  putting a 0 in any unoccupied square of the grid and  $B$  putting a 1. The first player is  $A$ , and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then  $B$  can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

- (a) What is the maximum number of legitimate moves possible in a game?  
(b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?  
(c) Which player has a winning strategy? Explain.
447. A high school student asked to solve the surd equation

$$\sqrt{3x - 2} - \sqrt{2x - 3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so  $x = 6$ . The answer is, in fact, correct.

Show that there are infinitely many real quadruples  $(a, b, c, d)$  for which this method leads to a correct solution of the surd equation

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 .$$

448. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

449. Let  $S = \{x : x > -1\}$ . Determine all functions from  $S$  to  $S$  which both

(a) satisfies the equation  $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$  for all  $x, y \in S$ , and

(b)  $f(x)/x$  is strictly increasing or strictly decreasing on each of the two intervals  $\{x : -1 < x < 0\}$  and  $\{x : x > 0\}$ .

450. The 4-sectors of an angle are the three lines through its vertex that partition the angle into four equal parts; adjacent 4-sectors of two angles that share a side consist of the 4-sector through each vertex that is closest to the other vertex.

Prove that adjacent 4-sectors of the angles of a parallelogram meet in the vertices of a square if and only if the parallelogram has four equal sides.

451. Let  $a$  and  $b$  be positive integers and let  $u = a + b$  and  $v = \text{lcm}(a, b)$ . Prove that

$$\text{gcd}(u, v) = \text{gcd}(a, b) .$$

452. (a) Let  $m$  be a positive integer. Show that there exists a positive integer  $k$  for which the set

$$\{k + 1, k + 2, \dots, 2k\}$$

contains exactly  $m$  numbers whose binary representation has exactly three digits equal to 1.

(b) Determine all integers  $m$  for which there is exactly one such integer  $k$ .

453. Let  $A, B$  be two points on a circle, and let  $AP$  and  $BQ$  be two rays of equal length that are tangent to the circle that are directed counterclockwise from their tangency points. Prove that the line  $AB$  intersects the segment  $PQ$  at its midpoint.

454. Let  $ABC$  be a non-isosceles triangle with circumcentre  $O$ , incentre  $I$  and orthocentre  $H$ . Prove that the angle  $OIH$  exceeds  $90^\circ$ .

455. Let  $ABCDE$  be a pentagon for which the position of the base  $AB$  and the lengths of the five sides are fixed. Find the locus of the point  $D$  for all such pentagons for which the angles at  $C$  and  $E$  are equal.

456. Let  $n + 1$  cups, labelled in order with the numbers  $0, 1, 2, \dots, n$ , be given. Suppose that  $n + 1$  tokens, one bearing each of the numbers  $0, 1, 2, \dots, n$  are distributed randomly into the cups, so that each cup contains exactly one token.

We perform a sequence of moves. At each move, determine the smallest number  $k$  for which the cup with label  $k$  has a token with label  $m$  not equal to  $k$ . Necessarily,  $k < m$ . Remove this token; move all the tokens in cups labelled  $k + 1, k + 2, \dots, m$  to the respective cups labelled  $k, k + 1, m - 1$ ; drop the token with label  $m$  into the cup with label  $m$ . Repeat.

Prove that the process terminates with each token in its own cup (token  $k$  in cup  $k$  for each  $k$ ) in not more than  $2^n - 1$  moves. Determine when it takes exactly  $2^n - 1$  moves.

457. Suppose that  $u_1 > u_2 > u_3 > \dots$  and that there are infinitely many indices  $n$  for which  $u_n \geq 1/n$ . Prove that there exists a positive integer  $N$  for which

$$u_1 + u_2 + u_3 + \dots + u_N > 2006 .$$

458. Let  $ABC$  be a triangle. Let  $A_1$  be the reflected image of  $A$  with axis  $BC$ ,  $B_1$  the reflected image of  $B$  with axis  $CA$  and  $C_1$  the reflected image of  $C$  with axis  $AB$ . Determine the possible sets of angles of triangle  $ABC$  for which  $A_1B_1C_1$  is equilateral.

459. At an International Conference, there were exactly 2006 participants. The organizers observed that: (1) among any three participants, there were two who spoke the same language; and (2) every participant spoke at most 5 languages. Prove that there is a group of at least 202 participants who speak the same language.

460. Given two natural numbers  $x$  and  $y$  for which

$$3x^2 + x = 4y^2 + y ,$$

prove that their positive difference is a perfect square. Determine a nontrivial solution of this equation.

461. Suppose that  $x$  and  $y$  are integers for which  $x^2 + y^2 \neq 0$ . Determine the minimum value of the function

$$f(x, y) \equiv |5x^2 + 11xy - 5y^2| .$$

462. For any positive real numbers  $a, b, c, d$ , establish the inequality

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+d}} + \sqrt{\frac{c}{d+a}} + \sqrt{\frac{d}{a+b}} > 2 .$$

463. In *Squareland*, a newly-created country in the shape of a square with side length of 1000 km, there are 51 cities. The country can afford to build at most 11000 km of roads. Is it always possible, within this limit, to design a road map that provides a connection between any two cities in the country?

464. A square is partitioned into non-overlapping rectangles. Consider the circumcircles of all the rectangles. Prove that, if the sum of the areas of all these circles is equal to the area of the circumcircle of the square, then all the rectangles must be squares, too.

465. For what positive real numbers  $a$  is

$$\sqrt[3]{2 + \sqrt{a}} + \sqrt[3]{2 - \sqrt{a}}$$

an integer?

466. For a positive integer  $m$ , let  $\overline{m}$  denote the sum of the digits of  $m$ . Find all pairs of positive integers  $(m, n)$  with  $m < n$  for which  $(\overline{m})^2 = n$  and  $(\overline{n})^2 = m$ .

467. For which positive integers  $n$  does there exist a set of  $n$  distinct positive integers such that

- (a) each member of the set divides the sum of all members of the set, and  
 (b) none of its proper subsets with two or more elements satisfies the condition in (a)?

468. Let  $a$  and  $b$  be positive real numbers satisfying  $a + b \geq (a - b)^2$ . Prove that

$$x^a(1-x)^b + x^b(1-x)^a \leq \frac{1}{2^{a+b-1}}$$

for  $0 \leq x \leq 1$ , with equality if and only if  $x = \frac{1}{2}$ .

469. Solve for  $t$  in terms of  $a, b$  in the equation

$$\sqrt{\frac{t^3 + a^3}{t + a}} + \sqrt{\frac{t^3 + b^3}{t + b}} = \sqrt{\frac{a^3 - b^3}{a - b}}$$

where  $0 < a < b$ .

470. Let  $ABC$ ,  $ACP$  and  $BCQ$  be nonoverlapping triangles in the plane with angles  $CAP$  and  $CBQ$  right. Let  $M$  be the foot of the perpendicular from  $C$  to  $AB$ . Prove that lines  $AQ$ ,  $BP$  and  $CM$  are concurrent if and only if  $\angle BCQ = \angle ACP$ .

471. Let  $I$  and  $O$  denote the incentre and the circumcentre, respectively, of triangle  $ABC$ . Assume that triangle  $ABC$  is not equilateral. Prove that  $\angle AIO \leq 90^\circ$  if and only if  $2BC \leq AB + CA$ , with equality holding only simultaneously.

472. Find all integers  $x$  for which

$$(4 - x)^{4-x} + (5 - x)^{5-x} + 10 = 4^x + 5^x .$$

473. Let  $ABCD$  be a quadrilateral; let  $M$  and  $N$  be the respective midpoint of  $AB$  and  $BC$ ; let  $P$  be the point of intersection of  $AN$  and  $BD$ , and  $Q$  be the point of intersection of  $DM$  and  $AC$ . Suppose the  $3BP = BD$  and  $3AQ = AC$ . Prove that  $ABCD$  is a parallelogram.

474. Solve the equation for positive real  $x$ :

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3 .$$

475. Let  $z_1, z_2, z_3, z_4$  be distinct complex numbers for which  $|z_1| = |z_2| = |z_3| = |z_4|$ . Suppose that there is a real number  $t \neq 1$  for which

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| .$$

Show that, in the complex plane,  $z_1, z_2, z_3, z_4$  lie at the vertices of a rectangle.

476. Let  $p$  be a positive real number and let  $|x_0| \leq 2p$ . For  $n \geq 1$ , define

$$x_n = 3x_{n-1} - \frac{1}{p^2}x_{n-1}^3 .$$

Determine  $x_n$  as a function of  $n$  and  $x_0$ .

477. Let  $S$  consist of all real numbers of the form  $a + b\sqrt{2}$ , where  $a$  and  $b$  are integers. Find all functions that map  $S$  into the set  $\mathbf{R}$  of reals such that (1)  $f$  is increasing, and (2)  $f(x + y) = f(x) + f(y)$  for all  $x, y$  in  $S$ .

478. Solve the equation

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + x}}} + \sqrt{3}\sqrt{2 - \sqrt{2 + \sqrt{2 + x}}} = 2x$$

for  $x \geq 0$

479. Let  $x, y, z$  be positive integer for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

and the greatest common divisor of  $x$  and  $z$  is 1. Prove that  $x + y$ ,  $x - z$  and  $y - z$  are all perfect squares. Give two examples of triples  $(x, y, z)$  that satisfy these conditions.

480. Let  $a$  and  $b$  be positive real numbers for which  $60^a = 3$  and  $60^b = 5$ . Without the use of a calculator or of logarithms, determine the value of

$$12^{\frac{1-a-b}{2(1-b)}} .$$

481. In a certain town of population  $2n + 1$ , one knows those to whom one is known. For any set  $A$  of  $n$  citizens, there is some person among the other  $n + 1$  who knows everyone on  $A$ . Show that some citizen of the town knows all the others.

[This problem was published as #11262 in the *American Mathematical Monthly* (113:10 (December, 2006), 940). Solvers of this problem should send their solutions to Prof. Barbeau and are invited to submit their solutions to the problems editor for the *Monthly*: Prof. Doug Hensley, Monthly Problems, Department of Mathematics, Texas A & M University, 3368 TAMU, College Station, TX 77843-3368, USA. A pdf file of the solution may be sent to [monthlyproblems@math.tamu.edu](mailto:monthlyproblems@math.tamu.edu).]

482. A trapezoid whose parallel sides have the lengths  $a$  and  $b$  is partitioned into two trapezoids of equal area by a line segment of length  $c$  parallel to these sides. Determine  $c$  as a function of  $a$  and  $b$ .
483. Let  $A$  and  $B$  be two points on the circumference of a circle, and  $E$  be the midpoint of arc  $AB$  (either arc will do). Let  $P$  be any point on the minor arc  $EB$  and  $N$  the foot of the perpendicular from  $E$  to  $AP$ . Prove that  $AN = NP + PB$ .
484.  $ABC$  is a triangle with  $\angle A = 40^\circ$  and  $\angle B = 60^\circ$ . Let  $D$  and  $E$  be respective points of  $AB$  and  $AC$  for which  $\angle DCB = 70^\circ$  and  $\angle EBC = 40^\circ$ . Furthermore, let  $F$  be the point of intersection of  $DC$  and  $EB$ . Prove that  $AF \perp BC$ .
485. From the foot of each altitude of the triangle, perpendiculars are dropped to the other two sides. Prove that the six feet of these perpendiculars lie on a circle.
486. Determine all quintuplets  $(a, b, c, d, u)$  of nonzero integers for which

$$\frac{a}{b} = \frac{c}{d} = \frac{ab + u}{cd + u} .$$

487.  $ABC$  is an isosceles triangle with  $\angle A = 100^\circ$  and  $AB = AC$ . The bisector of angle  $B$  meets  $AC$  in  $D$ . Show that  $BD + AD = BC$ .
488. A host is expecting a number of children, which is either 7 or 11. She has 77 marbles as gifts, and distributes them into  $n$  bags in such a way that whether 7 or 11 children come, each will receive a number of bags so that all 77 marbles will be shared equally among the children. What is the minimum value of  $n$ ?
489. Suppose  $n$  is a positive integer not less than 2 and that  $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$ ,

$$\sum_{i=1}^n x_i \leq 400 \quad \text{and} \quad \sum_{i=1}^n x_i^2 \geq 10^4 .$$

Prove that  $\sqrt{x_1} + \sqrt{x_2} \geq 10$ . is it possible to have equality throughout? [Bonus: Formulate and prove a generalization.]

490. (a) Let  $a, b, c$  be real numbers. Prove that

$$\min [(a - b)^2, (b - c)^2, (c - a)^2] \leq \frac{1}{2}[a^2 + b^2 + c^2] .$$

(b) Does there exist a number  $k$  for which

$$\min [(a-b)^2, (a-c)^2, (a-d)^2, (b-c)^2, (b-d)^2, (c-d)^2] \leq k[a^2 + b^2 + c^2 + d^2]$$

for any real numbers  $a, b, c, d$ ? If so, determine the smallest such  $k$ .

[*Bonus*: Determine if there is a generalization.]

491. Given that  $x$  and  $y$  are positive real numbers for which  $x + y = 1$  and that  $m$  and  $n$  are positive integers exceeding 1, prove that

$$(1 - x^m)^n + (1 - y^n)^m > 1 .$$

492. The faces of a tetrahedron are formed by four congruent triangles. If  $\alpha$  is the angle between a pair of opposite edges of the tetrahedron, show that

$$\cos \alpha = \frac{\sin(B - C)}{\sin(B + C)}$$

where  $B$  and  $C$  are the angles adjacent to one of these edges in a face of the tetrahedron.

493. Prove that there is a natural number with the following characteristics: (a) it is a multiple of 2007; (b) the first four digits in its decimal representation are 2009; (c) the last four digits in its decimal representation are 2009.

494. (a) Find all real numbers  $x$  that satisfy the equation

$$(8x - 56)\sqrt{3 - x} = 30x - x^2 - 97 .$$

(b) Find all real numbers  $x$  that satisfy the equation

$$\sqrt{x} + \sqrt[3]{x + 7} = \sqrt[4]{x + 80} .$$

495. Let  $n \geq 3$ . A regular  $n$ -gon has area  $S$ . Squares are constructed externally on its sides, and the vertices of adjacent squares that are not vertices of the polygon are connected to form a  $2n$ -sided polygon, whose area is  $T$ . Prove that  $T \leq 4(\sqrt{3} + 1)S$ . For what values of  $n$  does equality hold?

496. Is the hundreds digit of  $N = 2^{2006} + 2^{2007} + 2^{2008}$  even or odd? Justify your answer.

497. Given  $n \geq 4$  points in the plane with no three collinear, construct all segments connecting two of these points. It is known that the length of each of these segments is a positive integer. Prove that the lengths of at least  $1/6$  of the segments are multiples of 3.

498. Let  $a$  be a real parameter. Consider the simultaneous system of two equations:

$$\frac{1}{x + y} + x = a - 1 ; \tag{1}$$

$$\frac{x}{x + y} = a - 2 . \tag{2}$$

(a) For what value of the parameter  $a$  does the system have exactly one solution?

(b) Let  $2 < a < 3$ . Suppose that  $(x, y)$  satisfies the system. For which value of  $a$  in the stated range does  $(x/y) + (y/x)$  reach its maximum value?

499. The triangle  $ABC$  has all acute angles. The bisector of angle  $ACB$  intersects  $AB$  at  $L$ . Segments  $LM$  and  $LN$  with  $M \in AC$  and  $N \in BC$  are constructed, perpendicular to the sides  $AC$  and  $BC$  respectively. Suppose that  $AN$  and  $BM$  intersect at  $P$ . Prove that  $CP$  is perpendicular to  $AB$ .

500. Find all sets of distinct integers  $1 < a < b < c < d$  for which  $abcd - 1$  is divisible by  $(a - 1)(b - 1)(c - 1)(d - 1)$ .
501. Given a list of  $3n$  not necessarily distinct elements of a set  $S$ , determine necessary and sufficient conditions under which these  $3n$  elements can be divided into  $n$  triples, none of which consist of three distinct elements.
502. A set consisting of  $n$  men and  $n$  women is partitioned at random into  $n$  disjoint pairs of people. What are the expected value and variance of the number of male-female couples that result? (The *expected value*  $E$  is the average of the number  $N$  of male-female couples over all possibilities, *i.e.* the sum of the numbers of male-female couples for the possibilities divided by the number of possibilities. The *variance* is the average of the difference  $(E - N)^2$  over all possibilities, *i.e.* the sum of the values of  $(E - N)^2$  for the possibilities divided by the number of possibilities.)
503. A natural number is *perfect* if it is the sum of its proper positive divisors. Prove that no two consecutive numbers can both be perfect.
504. Find all functions  $f$  taking the real numbers into the real numbers for which the following conditions hold simultaneously:
- (a)  $f(x + f(y) + yf(x)) = y + f(x) + xf(y)$  for every real pair  $(x, y)$ ;
- (b)  $\{f(x)/x : x \neq 0\}$  is a finite set.
505. What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?
506. A two-person game is played as follows. A position consists of a pair  $(a, b)$  of positive integers. Players move alternately. A move consists of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make a move loses. (This happens, for example, when  $a = b$ .) Determine those positions  $(a, b)$  from which the first player can guarantee a win with optimal play.
507. Prove that, if  $a, b, c$  are positive reals, then

$$\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b} + \frac{3}{4} \geq \log(abc) .$$

508. Let  $a, b, c$  be integers exceeding 1 for which both  $\log_a b + \log_b a$  and  $\log_a^2 b + \log_b^2 a$  are rational. Prove that, for every positive integer  $n$ ,  $\log_a^n b + \log_b^n a$  is rational.
509. Let  $ABCD A' B' C' D'$  be a cube where the point  $O$  is the centre of the face  $ABCD$  and  $|AB| = 2a$ . Calculate the distance from the point  $B$  to the line of intersection of the planes  $A' B' O$  and  $ADD' A'$  and the distance between  $AB'$  and  $BD$ . ( $AA', BB', CC', DD'$  are edges of the cube.)
510. Solve the equation

$$\sqrt[3]{x^2 + 2} + \sqrt[3]{4x^2 + 3x - 2} = \sqrt[3]{3x^2 + x + 5} + \sqrt[3]{2x^2 + 2x - 5} .$$

511. Find the sum of the last 100 digits of the number

$$A = 1 \cdot 2 \cdot 3 \cdot \dots \cdot 2005 \cdot 2006 + 2007 .$$

512. Prove that

$$\binom{3n}{n} = \sum_{k=0}^n \binom{2n}{k} \binom{n}{k}$$

when  $n \geq 1$ .

513. Solve the equation

$$2^{1-2\sin^2 x} = 2 + \log_2(1 - \sin^2 x) .$$

514. Prove that there do not exist polynomials  $f(x)$  and  $g(x)$  with complex coefficients for which

$$\log_b x = \frac{f(x)}{g(x)}$$

where  $b$  is any base exceeding 1.

515. Let  $n$  be a fixed positive integer exceeding 1. To any choice of  $n$  real numbers  $x_i$  satisfying  $0 \leq x_i \leq 1$ , we can associate the sum

$$\sum \{|x_i - x_j| : 1 \leq i < j \leq n\} .$$

What is the maximum possible value of this sum and for which values of the  $x_i$  is it assumed?

516. Let  $n \geq 1$ . Is it true that, for any  $2n + 1$  positive real numbers  $x_1, x_2, \dots, x_{2n+1}$ , we have that

$$\frac{x_1 x_2}{x_3} + \frac{x_2 x_3}{x_4} + \dots + \frac{x_{2n+1} x_1}{x_2} \geq x_1 + x_2 + \dots + x_{2n+1} ,$$

with equality if and only if all the  $x_i$  are equal?

517. A man bought four items in a *Seven-Eleven* store. The clerk entered the four prices into a pocket calculator and *multiplied* to get a result of 7.11 dollars. When the customer objected to this procedure, the clerk realized that he should have added and redid the calculation. To his surprise, he again got the answer 7.11. What did the four items cost?

518. Let  $I$  be the incentre of triangle  $ABC$ , and let  $AI, BI, CI$ , produced, intersect the circumcircle of triangle  $ABC$  at the respective points  $D, E, F$ . Prove that  $EF \perp AD$ .

519. Let  $AB$  be a diameter of a circle and  $X$  any point other than  $A$  and  $B$  on the circumference of the circle. Let  $t_A, t_B$  and  $t_X$  be the tangents to the circle at the respective points  $A, B$  and  $X$ . Suppose that  $AX$  meets  $t_B$  at  $Z$  and  $BX$  meets  $t_A$  at  $Y$ . Show that the three lines  $YZ, t_X$  and  $AB$  are either concurrent (i.e. passing through a common point) or parallel.

520. The *diameter* of a plane figure is the largest distance between any pair of points in the figure. Given an equilateral triangle of side 1, show how, by a straight cut, one can get two pieces that can be rearranged to form a figure with maximum diameter

(a) if the resulting figure is convex (i.e. the line segment joining any two of its points must lie inside the figure);

(b) if the resulting figure is not necessarily convex, but it is connected (i.e. any two points in the figure can be connected by a curve lying inside the figure).

521. On a  $8 \times 8$  chessboard, either  $+1$  or  $-1$  is written in each square cell. Let  $A_k$  be the product of all the numbers in the  $k$ th row, and  $B_k$  the product of all the numbers in the  $k$ th column of the board ( $k = 1, 2, \dots, 8$ ). Prove that the number

$$A_1 + A_2 + \dots + A_8 + B_1 + B_2 + \dots + B_8$$

is a multiple of 4.

522. (a) Prove that, in each scalene triangle, the angle bisector from one of its vertices is always “between” the median and the altitude from the same vertex.

(b) Find the measures of the angles of a triangle if the lengths of the median, the angle bisector and the altitude from one of its vertices are in the ratio  $\sqrt{5} : \sqrt{2} : 1$ .

523. Let  $ABC$  be an isosceles triangle with  $AB = AC$ . The segments  $BC$  and  $AC$  are used as hypotenuses to construct three right triangles  $BCM$ ,  $BCN$  and  $ACP$ . Prove that, if  $\angle ACP + \angle BCM + \angle BCN = 90^\circ$ , then the triangle  $MPN$  is isosceles.

524. Solve the irrational equation

$$\frac{7}{\sqrt{x^2 - 10x + 26} + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 41}} = x^4 - 9x^3 + 16x^2 + 15x + 26 .$$

525. The circle inscribed in the triangle  $ABC$  divides the median from  $A$  into three segments of the same length. If the area of  $ABC$  is  $6\sqrt{14}$ , calculate the lengths of its sides.

526. For the non-negative numbers  $a, b, c$ , prove the inequality

$$4(a + b + c) \geq 3(a + \sqrt{ab} + \sqrt[3]{abc}) .$$

When does equality hold?

527. Consider the set  $A$  of the  $2n$ -digit natural numbers, with 1 and 2 each occurring  $n$  times as a digit, and the set  $B$  of the  $n$ -digit numbers all of whose digits are 1, 2, 3, 4 with the digits 1 and 2 occurring with equal frequency. Show that  $A$  and  $B$  contain the same number of elements (*i.e.*, have the same cardinality).